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Some Conditions for Decay of Convolution Powers and Heat Kernels on Groups

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Abstract. Let K be a function on a unimodular locally compact group G, and denote by $K_n = K * K * \cdots * K$ the n-th convolution power of K. Assuming that K satisfies certain operator estimates in $L^2(G)$, we give estimates of the norms $\|K_n\|_2$ and $\|K_n\|_\infty$ for large n. In contrast to previous methods for estimating $\|K_n\|_\infty$, we do not need to assume that the function K is a probability density or non-negative. Our results also adapt for continuous time semigroups on G. Various applications are given, for example, to estimates of the behaviour of heat kernels on Lie groups.

1 Introduction

Given a suitable function K on a locally compact group G, many authors have considered the problem of estimating the L^∞ norm $\|K_n\|_\infty$ for large n, where

$$K_n = K * K * \cdots * K$$

denotes the n-th convolution power of K for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. If G is a Lie group, this problem occurs in describing the large time behaviour of the heat kernel corresponding to a group invariant differential operator on G. For relevant background in these areas, see [8, 21, 22, 25] and references therein.

If K is a probability density, $K \geq 0$ and $\int_G K = 1$, then there are various strategies for estimating K_n. These include probabilistic methods, but also some general functional-analytic methods, using Nash or Sobolev inequalities, for estimating semigroups which are uniformly bounded in L^p for all $1 \leq p \leq \infty$ (see, for example, [2, 3, 21, 25]). But if the function K takes negative or complex values then these methods are not readily applicable, and it is not clear how to proceed. Similar difficulties occur in studying the heat kernels of differential operators which are not second order or which have complex coefficients (see the survey [5]). Large time estimates of such kernels on Lie groups are considered in [1,4,6,8–10,13,14] for example, but these works rely on the structure theory of specific classes of Lie groups.

In this paper, we offer a new method for estimating K_n, which does not assume $K \geq 0$ and which applies in great generality to (unimodular) locally compact groups. We derive estimates on $\|K_n\|_\infty$ from certain semigroup estimates in L^2; no assumptions are made about semigroup behaviour in L^1. Our work yields a surprisingly direct connection between L^2 operator estimates and L^∞ kernel estimates on groups, which has many applications to estimates of group invariant heat kernels. Indeed, in some specific situations we can derive new estimates, while in other situations we

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can give much simpler proofs of known results (for examples, see Sections 3 and 4 below).

An original aspect of this paper is the use of certain "convolution" Nash inequalities (see Lemma 2.2 below), which are a modified version of standard Nash type inequalities. A more systematic treatment of convolution Nash inequalities and their relationship with estimates for convolution semigroups will be given elsewhere.

To state our basic results, we fix some notation. Throughout, G will be a second countable, unimodular locally compact group, with identity element e. We fix a (bi-invariant) Haar measure dg for G, and consider the spaces $L^p = L^p(G; dg)$, $1 \le p \le \infty$, of complex-valued functions. The norm of a bounded operator $T\colon L^p \to L^q$ is written $\|T\|_{p\to q}$; when $p = q = 2$ we simply write $\|T\|$. The identity operator is I. In general, c, c', b and so on, denote positive constants whose value may change from line to line when convenient.

Let $L = L_G$ be the left regular representation of G, so that $(L(g)f)(h) = f(g^{-1}h)$, $g, h \in G$, for a function $f\colon G \to \mathbb{C}$. Given any locally integrable function f_1, let $L(f_1)$ denote the right invariant convolution operator given by

$$(L(f_1)f_2)(g) = (f_1 * f_2)(g) = \int_G dh\, f_1(h) f_2(h^{-1}g)$$

for $g \in G$ and suitable functions f_2. This is at least well defined when $f_2 \in C_c(G)$, the continuous, compactly supported functions on G. We can thus consider the (possibly infinite) norms

$$\|L(f_1)\|_{p\to q} = \sup\{\|L(f_1)f_2\|_q \colon f_2 \in C_c(G), \|f_2\|_p \le 1\}$$

when $1 \le p < \infty$.

In our basic results, we assume that $K \in L^2$, and that $T = L(K)$ is a contraction in L^2, $\|T\| \le 1$. Then the n-th convolution powers $K_n = K * K * \cdots * K = T^{n-1}K$ exist in L^2, and

$$n \mapsto \|K_n\|_2$$

is a non-increasing function of $n \in \mathbb{N}$, because $K_{n+1} = TK_n$.

The following terminology will be useful. A Borel measurable function $\rho\colon G \to [0, \infty\rangle$ is said to be a *modulus function* on G if $\rho(g) > 0$ for all $g \neq e$ and $B_\rho(r) := \{g \in G : \rho(g) \le r\}$ is a relatively compact subset of G for each $r > 0$. We usually denote by $V_\rho(r) = dg(B_\rho(r))$ the (finite) Haar measure of $B_\rho(r)$.

Heuristically, we think of $\rho(g)$ as a distance between g and the identity e, so that $B_\rho(r)$ is a ball of radius r.

Our basic theorem is the following.

Theorem 1.1 *Suppose $K \in L^2$ such that $T = L(K)$ is a contraction in L^2. Let $\rho\colon G \to [0, \infty\rangle$ be a modulus function, and suppose there are $\alpha > 0$, $\nu > 0$, such that*

$$\|(I - L(g))T^n\| \le c(\rho(g)n^{-\alpha})^\nu \tag{1}$$

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for all $n \in \mathbb{N}$ and $g \in G$. Assume there exist $a > 0$, $D > 0$ with $V_\rho(r) \geq ar^D$ for all $r \geq 1$. Then there exists $c' > 0$ such that

$$\|K_n\|_2 \leq c' n^{-D\alpha/2}$$

for all $n \in \mathbb{N}$ and

$$\|K_n\|_\infty \leq c' n^{-D\alpha}$$

for all $n \in \mathbb{N}$ with $n \geq 2$.

We make a number of remarks about Theorem 1.1.

• The proof will show that the constant c' in the conclusion depends only on α, ν, c, a, D and $\|K\|_2$.

• It is also of interest to consider cases of faster growth, for example, where $V_\rho(r) \geq ae^{ar}$ for large r. See Theorem 2.7 in Section 2 below for an extension of Theorem 1.1 for such situations.

• The hypothesis $\|T\| \leq 1$ in Theorem 1.1 can easily be replaced by the weaker condition $\sup_{n\in\mathbb{N}} \|T^n\| = c_1 < \infty$. But $\|T\| \leq 1$ in all of the applications of the theorem that we know of. Note that Theorem 1.1 can sometimes also be applied in cases where $\|T^n\| \leq c_1\theta^n$, $n \in \mathbb{N}$, for some $c_1, \theta > 0$, by considering the operator $\widetilde{T} = \theta^{-1}T$.

• It is a useful technical remark that (1) holds automatically if $\rho(g)n^{-\alpha} \geq 1$, because one has $\|(I - L(g))T^n\| \leq 2\|T^n\| \leq 2$ for all $g \in G$. Thus when verifying (1) it suffices to consider the case where $\rho(g) \leq n^\alpha$.

• Theorem 1.1 is actually equivalent to the special case where $\nu = 1$. Indeed, for a general $\nu > 0$ we can apply that case to the modulus $\rho_2 := \rho^\nu$. It is, however, of some interest to state the theorem for general ν, because for a fixed ρ, the condition (1) is weakened if ν is decreased. We can roughly think of ν as a Hölder exponent.

• In our applications of Theorem 1.1, the modulus function ρ will also be *subadditive* and *symmetric*, that is,

$$\rho(gh) \leq \rho(g) + \rho(h), \quad \rho(g) = \rho(g^{-1})$$

for all $g, h \in G$, but these properties are not assumed in general. There is, however, a natural connection between these properties and the hypothesis (1). Namely, if $\nu = 1$ and $\tilde{\rho}$ is the smallest function such that (1) holds with $c = 1$, that is,

$$\tilde{\rho}(g) = \sup_{n\in\mathbb{N}} n^\alpha \|(I - L(g))T^n\|,$$

then it is easy to check that $\tilde{\rho}$ is subadditive and symmetric.

• In case G is compactly generated, there is a natural modulus function on G. We briefly discuss this modulus, which occurs in many important applications of Theorem 1.1. To recall the definition, pick a compact neighborhood U of e with

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$U = U^{-1}$ and $G = \bigcup_{n=1}^{\infty} U^n$, where $U^n = \{u_1 \dots u_n : u_1, \dots, u_n \in U\}$. One defines (as in [25, p. 77]) a symmetric, subadditive modulus $\rho_U : G \to \mathbb{N}$ by

$$\rho_U(g) = \inf\{n \in \mathbb{N} : g \in U^n\}.$$

Note that $U^n = \{g \in G : \rho_U(g) \le n\}$ and $V_{\rho_U}(n) = dg(U^n)$ for $n \in \mathbb{N}$.

It is well known that the behaviour of ρ_U is essentially independent of U and depends only on G. For if U' is another such neighborhood, then there exists $k \in \mathbb{N}$ with $U^n \subseteq (U')^{kn}$, $(U')^n \subseteq U^{kn}$ for all $n \in \mathbb{N}$, and $k^{-1}\rho_U \le \rho_{U'} \le k\rho_U$. In the sequel, we will often write $V_G(n)$ instead of $V_{\rho_U}(n)$ without specifying a particular choice of U.

Recall that the compactly generated group G is said to have polynomial growth of order $D \ge 0$ if one has an estimate $c^{-1}n^D \le V_G(n) \le cn^D$ for all $n \in \mathbb{N}$.

For applications of our results to Lie groups, it is useful to recall (see [15]) that any connected, non-compact Lie group either has polynomial growth of some order $D \ge 1$, or has exponential growth in the sense that $V_G(n) \ge ae^{an}$, $n \in \mathbb{N}$, for some constant $a > 0$.

It is interesting to point out that ρ_U is essentially the largest subadditive function on a compactly generated group G. More precisely, if $\sigma : G \to [0, \infty\rangle$ is a subadditive function which is bounded over compact sets, then an easy argument shows that $\sigma \le c_1\, \rho_U$ where $c_1 := \sup\{\sigma(g) : g \in U\} < \infty$.

• There are also interesting applications of Theorem 1.1 where the modulus function behaves differently from the canonical modulus ρ_U. In Section 4, we will present such an application using certain "weighted" modulus functions on Lie groups.

Our next result is derived from Theorem 1.1 by choosing $\rho = \rho_U$ (see Section 2).

Corollary 1.2 *Suppose G is compactly generated. Let $K \in L^2$ such that $T = L(K)$ is a contraction in L^2, and suppose there exist $\alpha > 0$ and a compact generating neighborhood U' of e with*

$$\|(I - L(u))T^n\| \le cn^{-\alpha}$$

for all $n \in \mathbb{N}$ and $u \in U'$. If $D > 0$ with $V_G(n) \ge an^D$, $n \in \mathbb{N}$, then the conclusion of Theorem 1.1 holds, that is, there is a $c > 0$ with $\|K_n\|_2 \le cn^{-D\alpha/2}$, $n \in \mathbb{N}$, and $\|K_n\|_\infty \le cn^{-D\alpha}$, $n \ge 2$.

In case G is a Lie group, it is natural to try to rewrite our results by replacing the difference operators $I - L(g)$ with group-invariant vector fields. We next state one such result, which is derivable from Corollary 1.2 (see Section 2).

Suppose G is a connected Lie group, with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. To each element $x \in \mathfrak{g}$ we associate a right invariant vector field $X = dL_G(x)$: as an operator, $dL_G(x) = \lim_{t\to 0} t^{-1}(L(\exp tx) - I)$. By a *generating list* we mean a finite list $a_1, \dots, a_{d'} \in \mathfrak{g}$ of elements which algebraically generate the Lie algebra $\mathfrak{g}$.

Corollary 1.3 *Let G be a connected unimodular Lie group and $D \ge 1$ with $V_G(n) \ge an^D$, $n \in \mathbb{N}$. Suppose $K \in L^2$ such that $T = L(K)$ is a contraction in L^2, and assume* *there exist* $\alpha > 0$ *and a generating list* $a_1, \dots, a_{d'}$ *for* $\mathfrak{g}$ *such that, with* $A_i = dL_G(a_i)$,

$$\|A_i T^n\| \le cn^{-\alpha}$$

for all $n \in \mathbb{N}$ *and* $i \in \{1, \dots, d'\}$. *Then the conclusion of Theorem* 1.1 *holds.*

Let us mention a typical application of the preceding results; further examples will be given in subsequent sections. On a Lie group G, consider a subelliptic operator of the form

$$H = (-1)^{M/2} \sum_{i=1}^{d'} A_i^M,$$

where $M \ge 2$ is an even positive integer, $A_i = dL_G(a_i)$, and $a_1, \dots, a_{d'}$ generate $\mathfrak{g}$. It is known (see [11, 16]) that the semigroup e^{-tH} generated by the non-negative self-adjoint operator H is given by $e^{-tH} = L(K_t)$, $t > 0$, where the kernel $K_t \in L^1 \cap L^\infty$ is smooth (and satisfies Gaussian estimates for $0 < t \le 1$.) Moreover, $K_{t+s} = K_t * K_s$ for all $s, t > 0$.

We will obtain the following result from Corollary 1.3 with $\alpha = 1/M$ (see Section 3).

Theorem 1.4 *Let G be a connected unimodular Lie group and consider H as above. If* $D \ge 1$ *with* $V_G(n) \ge an^D$, $n \in \mathbb{N}$, *then the kernel* K_t *satisfies estimates*

$$\|K_t\|_2 \le ct^{-D/(2M)}, \quad \|K_t\|_\infty \le ct^{-D/M}$$

for all $t \ge 1$.

The estimate of Theorem 1.4 is new in general, though it was well known in case $M = 2$ (for example, [25]), or for general M on a nilpotent Lie group (see [14]). Note that for certain solvable groups of polynomial growth and $M \ge 4$, detailed analyses of [7, 12] show that the exponent D/M is not necessarily optimal, depending on the choice of generators $a_1, \dots, a_{d'}$. The methods of the current paper probably could be used to generalize results of [7, 12], but we will not pursue this here.

In case the unimodular Lie group G has exponential growth, Theorem 1.4 shows that $\lim_{t\to\infty} t^\mu \|K_t\|_\infty = 0$, for any $\mu > 0$. In fact, in Section 3 we will obtain the more precise bound $\|K_t\|_\infty \le c\exp(-bt^{1/(M+1)})$ for $t \ge 1$. This bound is well known for the case $M = 2$ (see [25]) but is quite new for $M \ge 4$.

We caution that the results of this paper do not give optimal estimates for all classes of groups or all convolution powers. For example, if G is a non-compact semisimple Lie group, then the heat kernel of a sublaplacian $H = -\sum_i A_i^2$ on G satisfies a bound $\|K_t\|_\infty \le c\exp(-\lambda_0 t)$, $t \ge 1$, where $\lambda_0 > 0$ is the spectral gap of H. This is sharper than the estimate $\|K_t\|_\infty \le c\exp(-bt^{1/3})$ provided by our methods, and shows that other factors besides the growth of $V_\rho(r)$ influence the decay of $\|K_t\|_\infty$ (see [24] for estimates of sublaplacian heat kernels on arbitrary Lie groups using an algebraic classification of the Lie algebras). Nevertheless, our results seem to provide good estimates for interesting examples on large classes of groups, for example, on amenable unimodular Lie groups.

We have focussed on convolution powers in this introduction, but our methods also apply for semigroups of operators $\{T_t\}$ acting in $L^2(G)$, provided T_t is group invariant. In this semigroup setting, it is remarkable that we can deduce the existence of a bounded convolution kernel for T_t, assuming only that T_t satisfies certain operator estimates in L^2. See Theorem 2.9 for a typical result of this kind.

2 Proofs of the Basic Results

The first aim of this section is to prove the basic results Theorem 1.1 and Corollaries 1.2 and 1.3. We then give an extension of these results for situations of "superpolynomial" growth, and a local version of Theorem 1.1 for continuous time semigroups.

We begin with two essential lemmas. The second lemma is a convolution Nash inequality; it differs essentially from more standard versions of the Nash inequality (see for example [2,21]) by replacing the L^1 norm $\|f\|_1$ with the convolution operator norm $\|L(f)\|$. This is an improvement since $\|L(f)\| \leq \|f\|_1$, and is just what is needed to avoid the use of L^1 estimates.

Recall that the locally compact group G is assumed to be second countable and unimodular.

Lemma 2.1 *One has*

$$\|f_1 * f_2\|_2 \leq \|L(f_1)\| \|f_2\|_2, \quad \|f_2 * f_1\|_2 \leq \|L(f_1)\| \|f_2\|_2 \tag{2}$$

for all locally integrable functions f_1, f_2 on G satisfying $\|L(f_1)\| < \infty$ and $f_2 \in L^2$.

Proof By density, it is enough to check inequalities (2) when $f_2 \in C_c(G)$. The first inequality is immediate from the definition of $\|L(f_1)\|$. To prove the second inequality, given a locally integrable function f, define

$$\tilde{f}(g) = \overline{f(g^{-1})} \tag{3}$$

for $g \in G$. Then $\|\tilde{f}\|_p = \|f\|_p$ for all $p \in [1, \infty]$ because G is unimodular. Also $L(\tilde{f})$ is formally adjoint to $L(f)$ in L^2, so that $\|L(\tilde{f})\| = \|L(f)\|$. Notice that $(f_2 * f_1)\tilde{} = \tilde{f}_1 * \tilde{f}_2$. Therefore, we obtain

$$\|f_2 * f_1\|_2 = \|(f_2 * f_1)\tilde{}\|_2 = \|\tilde{f}_1 * \tilde{f}_2\|_2 \leq \|L(f_1)\| \|f_2\|_2,$$

as desired.

There is an alternative proof of the second inequality in (2), as follows. Observe that, for any $f \in C_c(G)$ with $\|f\|_2 \leq 1$,

$$\|L(f_2 * f_1) f\|_\infty = \|f_2 * f_1 * f\|_\infty \leq \|L(f_2)\|_{2\to\infty} \|L(f_1)\|.$$

Then take supremums over all such f, and apply the standard equality

$$\|L(f_3)\|_{2\to\infty} = \|f_3\|_2. \qquad \blacksquare$$

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Let us fix a modulus function ρ on G. If $f \in L^2$, define a "Hölder" seminorm with parameter $\nu > 0$ by

$$[f]_\nu = \sup_{\substack{g\in G\\ g\neq e}} \rho(g)^{-\nu}\|(I - L(g))f\|_2 \in [0,\infty],$$

recalling that $\rho(g) > 0$ for $g \neq e$. We now give some convolution Nash inequalities associated with these seminorms.

Lemma 2.2 *Let ρ be a modulus function on G.*

(i) *One has*

$$\|f\|_2 \leq \sup_{\substack{g\in G\\ \rho(g)\leq r}} \|(I - L(g))f\|_2 + V_\rho(r)^{-1/2}\|L(f)\|$$
$$\leq r^\nu [f]_\nu + V_\rho(r)^{-1/2}\|L(f)\|$$

for all $f \in L^2$, $r > 0$ and $\nu > 0$, where the right sides of these inequalities are permitted to be infinite.

(ii) *Suppose there exist $a > 0$, $D > 0$ with $V_\rho(r) \geq ar^D$ for all $r \geq 1$. Given $\delta > 0$, $\nu > 0$, then there exists $c = c(\delta,\nu,a,D) > 0$ such that*

$$\|f\|_2^{1+(2\nu/D)} \leq c[f]_\nu$$

for all $f \in L^2$ satisfying $\|f\|_2 \leq \delta$ and $\|L(f)\| \leq 1$.

Remark 2.3 When $\nu = 1$, we can intuitively think of $[f]_1$ as the L^2 norm of the "gradient" of f. Indeed, if G is a Lie group and ρ is chosen to be the Caratheodory modulus associated with a list of generators of the Lie algebra (see [21, 25]), then $[f]_1$ is equivalent to the L^2 norm of a subelliptic gradient of f; one direction of this equivalence follows from the standard inequality (8) below.

Thus, in the Lie group case we can obtain versions of the above convolution Nash inequalities with a subelliptic gradient replacing $[f]_1$ (these inequalities resemble Nash inequalities of [21], but with the important difference that $\|L(f)\|$ replaces $\|f\|_1$). But for greater generality, we prefer to work directly with the inequalities of Lemma 2.2, which are not restricted to the Lie group setting.

Note that part (ii) of the lemma is essentially an alternative form of part (i), in the case where $V_\rho(r) \geq ar^D$ for large r.

Proof of Lemma 2.2 To prove part (i), let $r > 0$ be given. If $V_\rho(r) = 0$ then the desired inequalities hold trivially, because we interpret $V_\rho(r)^{-1/2} = \infty$. Assume therefore that $V_\rho(r) > 0$.

Following an idea of Robinson [21, Proposition IV.2.4], consider the function $\chi = V_\rho(r)^{-1}1_{B_\rho(r)}$, where 1_E denotes the characteristic function of a subset $E \subseteq G$ and we recall that $B_\rho(r) = \{g \in G : \rho(g) \leq r\}$. Since $\int_G \chi = 1$, one has

$$f = \chi * f + \int_G dg\, \chi(g)(I - L(g))f.$$

Now take L^2 norms on both sides. Using (2) we find that

$$\|\chi * f\|_2 \le \|\chi\|_2 \|L(f)\| \le V_\rho(r)^{-1/2} \|L(f)\|.$$

Also,

$$\int_G dg \chi(g) \|(I - L(g)) f\|_2 \le \sup_{\rho(g) \le r} \|(I - L(g)) f\|_2 \le r^\nu [f]_\nu,$$

and we obtain part (i) of the lemma.

To prove part (ii), suppose $V_\rho(r) \ge a r^D$ for all $r \ge 1$, and let $f \in L^2$ with $0 < \|f\|_2 \le 2a^{-1/2}$ and $\|L(f)\| \le 1$. By applying part (i) with

$$r = (2^{-1} a^{1/2} \|f\|_2)^{-2/D} \ge 1,$$

and noting that $V_\rho(r)^{-1/2} \le 2^{-1} \|f\|_2$, we get

$$\|f\|_2 \le r^\nu [f]_\nu + 2^{-1} \|f\|_2.$$

By the choice of r, after a rearrangement this becomes

$$\|f\|_2^{1+(2\nu/D)} \le 2(2^{2/D} a^{-1/D})^\nu \, [f]_\nu$$

for all f such that $\|f\|_2 \le 2a^{-1/2}$ and $\|L(f)\| \le 1$. Part (ii) follows, since given $\delta > 0$, we can choose a small enough so that $2a^{-1/2} \ge \delta$. ■

Proof of Theorem 1.1 Assume that $K \ne 0$ and put $\delta = \|K\|_2 = \|K_1\|_2 > 0$. Since $n \mapsto \|K_n\|_2$ is non-increasing, then $\|K_n\|_2 \le \delta$ for all $n \in \mathbb{N}$. Because $\|L(K_n)\| = \|T^n\| \le 1$, part (ii) of Lemma 2.2 gives an estimate

$$\|K_n\|_2^{1+(2\nu/D)} \le c[K_n]_\nu$$

for all $n \in \mathbb{N}$. The identity $(I - L(g))K_{n+m} = (I - L(g))T^n(K_m)$ and the hypothesis (1) imply that

$$\|(I - L(g))K_{n+m}\|_2 \le \|(I - L(g))T^n\| \|K_m\|_2 \le c(\rho(g) n^{-\alpha})^\nu \|K_m\|_2, \tag{4}$$

or in other words,

$$[K_{n+m}]_\nu \le c n^{-\alpha\nu} \|K_m\|_2$$

for all $n, m \in \mathbb{N}$. Therefore,

$$\|K_{n+m}\|_2^{1+(2\nu/D)} \le c' n^{-\alpha\nu} \|K_m\|_2 \tag{5}$$

for all $n, m \in \mathbb{N}$. For $k \in \mathbb{N}$ set

$$\theta(k) = \sup_{1 \le n \le k} n^{D\alpha/2} \|K_n\|_2.$$

Choosing $m = n$ or $m = n + 1$ in (5) yields

$$\big(n^{D\alpha/2}\max\{\|K_{2n}\|_2, \|K_{2n+1}\|_2\}\big)^{1+(2\nu/D)} \le c'' n^{D\alpha/2}\|K_n\|_2$$

for all $n \in \mathbb{N}$. From this estimate, together with the trivial inequality

$$\|K_1\|_2^{1+(2\nu/D)} \le \delta^{2\nu/D}\|K_1\|_2 \le \delta^{2\nu/D}\,\theta(k),$$

we see that there is a $c_1 > 0$ with

$$\theta(k)^{1+(2\nu/D)} \le c_1\theta(k)$$

for all $k \in \mathbb{N}$. Thus $\theta(k) \le c_1^{D/(2\nu)} = c_2$ for all k, which means that $\|K_n\|_2 \le c_2 n^{-D\alpha/2}$ for all $n \in \mathbb{N}$.

An inspection of this proof shows that c_2 depends only on the parameters α, ν, c, a, D and $\|K\|_2$ occurring in the statement of Theorem 1.1.

Finally, observe that $\|K_{n+m}\|_\infty = \|K_n * K_m\|_\infty \le \|K_n\|_2\|K_m\|_2$ for all $n, m \in \mathbb{N}$. With $m = n$ or $m = n + 1$, we deduce that $\|K_k\|_\infty \le ck^{-D\alpha}$ for all $k \in \mathbb{N}$ with $k \ge 2$. The proof of Theorem 1.1 is complete. ■

Let us also record the following corollary.

Corollary 2.4 *Assume the hypotheses of Theorem* 1.1*. Then there exists* $c' > 0$ *such that*

$$\|(I - L(g))K_n\|_2 \le c'(\rho(g)n^{-\alpha})^\nu n^{-D\alpha/2}$$

for all $n \in \mathbb{N}$ *with* $n \ge 2$*, and*

$$\|(I - L(g))K_n\|_\infty \le c'(\rho(g)n^{-\alpha})^\nu n^{-D\alpha}$$

for all $n \in \mathbb{N}$ *with* $n \ge 3$.

Proof The first estimate follows through (4), and the second estimate then follows by observing that $\|(I - L(g))K_{m+n}\|_\infty \le \|(I - L(g))K_m\|_2\|K_n\|_2$. ■

Remark 2.5 Let us sketch an alternative version of the proof of Theorem 1.1, which uses part (i) of Lemma 2.2 instead of part (ii). In the sequel, we will prefer this alternative approach when proving extensions of Theorem 1.1 (for example, see Theorem 2.9 below).

Define $\beta_n = n^{D\alpha/2}\|K_n\|_2$ for $n \in \mathbb{N}$, and put $f = K_{2n}$ in part (i) of Lemma 2.2. Because $[K_{2n}]_\nu \le cn^{-\alpha\nu}\|K_n\|_2$, one finds that there is a $c_1 > 0$ such that

$$\beta_{2n} \le c_1(rn^{-\alpha})^\nu\beta_n + c_1(rn^{-\alpha})^{-D/2}$$

for all $n \in \mathbb{N}$ and $r \ge 1$. Now fix $\varepsilon > 0$ small enough so that $c_1\varepsilon^\nu < 2^{-1}$, and fix $k_0 \in \mathbb{N}$ large enough so that $2^{k_0} > \varepsilon^{-1/\alpha}$. Putting $r = \varepsilon n^\alpha$, we obtain an inequality

$$\beta_{2n} \le 2^{-1}\beta_n + c_2 \le \max\{\beta_n, 2c_2\} \tag{6}$$

for all $n \in \mathbb{N}$ satisfying $n \geq 2^{k_0}$; this condition on n ensures that $r \geq 1$. It follows by induction from (6) that

$$\beta_{2^k} \leq \max\{\beta_{2^{k_0}}, 2c_2\} = c_3$$

for all integers $k \geq k_0$. Since $n \mapsto \|K_n\|_2$ is non-increasing, it is then easy to deduce the required estimate $\|K_n\|_2 \leq cn^{-D\alpha/2}$ for all $n \in \mathbb{N}$.

Proofs of Corollary 1.2 and Corollary 1.3 Let U' be a compact generating neighborhood of e as in the hypothesis. Set $U = U' \cup (U')^{-1}$, so that $U = U^{-1}$. Let ρ_U be the canonical modulus function associated with U as in Section 1. One has the basic estimate

$$\|(I - L(g))f\|_2 \leq \rho_U(g) \sup_{u\in U} \|(I - L(u))f\|_2 = \rho_U(g) \sup_{u\in U'} \|(I - L(u))f\|_2 \tag{7}$$

for all $g \in G$ and $f \in L^2$. This is easily verified by noting the inequality $\|(I - L(u_1u_2\cdots u_n))f\|_2 \leq \sum_{i=1}^n \|(I - L(u_i))f\|_2$ for any $u_1, \dots, u_n \in U$.

Replacing f by $T^n f$ in (7), and using the hypothesis of the corollary, we see that

$$\|(I - L(g))T^n\| \leq c\, \rho_U(g) n^{-\alpha}$$

for all $n \in \mathbb{N}$ and $g \in G$. Therefore, the corollary follows from Theorem 1.1 with $\rho = \rho_U$.

To prove Corollary 1.3, we consider the Caratheodory modulus $\rho_{\mathbf{A}} \colon G \to [0, \infty\rangle$ associated with the vector fields $A_1, \dots, A_{d'}$ (see [21,25]). One has the basic inequality

$$\|(I - L(g))f\|_2 \leq \rho_{\mathbf{A}}(g) \Big(\sum_{i=1}^{d'} \|A_i f\|_2^2 \Big)^{1/2} \tag{8}$$

for all $f \in L^2$ with $A_i f \in L^2$, $i \in \{1, \dots, d'\}$ (see [21, p. 268]). Let us fix a compact neighborhood U' of e. Note that $\sup\{\rho_{\mathbf{A}}(u) : u \in U'\}$ is finite, because $\rho_{\mathbf{A}}$ is a continuous function. Therefore, replacing f with $T^n f$ in (8) yields an inequality

$$\sup_{u\in U'} \|(I - L(u))T^n\| \leq c \sum_{i=1}^{d'} \|A_i T^n\|$$

for all $n \in \mathbb{N}$. Then Corollary 1.3 is a consequence of Corollary 1.2. ∎

Remark 2.6 Alternatively, Corollary 1.3 can be deduced directly from Theorem 1.1 by taking $\rho = \rho_{\mathbf{A}}$. With this alternative approach, one uses (8), and the standard fact (see [25, Section III.4]) that $\rho_{\mathbf{A}}$ is equivalent at infinity to ρ_U at infinity, or more precisely,

$$c^{-1}\rho_U \leq \rho_{\mathbf{A}} + 1 \leq c\rho_U$$

for some $c > 1$. Thus the bound $V_G(n) \geq an^D$, $n \in \mathbb{N}$, is equivalent to $V_{\rho_{\mathbf{A}}}(r) \geq a'r^D$, $r \geq 1$.

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The next result improves on Theorem 1.1 in cases where $V_\rho(r)$ grows as fast as a function $\exp(r^\sigma)$ for large r.

Theorem 2.7 *Let $K \in L^2$ such that $T = L(K)$ is a contraction in L^2. Let ρ be a modulus function and suppose $\alpha > 0$ such that*

$$\|(I - L(g))T^n\| \le c\rho(g)n^{-\alpha}$$

for all $n \in \mathbb{N}$ and $g \in G$. Suppose there exist $a > 0$ and $\sigma > 0$ with

$$V_\rho(r) \ge ae^{ar^\sigma}$$

for all $r \ge 1$. Then, setting $\gamma = \sigma\alpha/(1 + \sigma\alpha)$, there exist $c', b > 0$ such that

$$\|K_n\|_2 \le c'e^{-bn^\gamma}$$

for all $n \in \mathbb{N}$ and

$$\|K_n\|_\infty \le c'e^{-bn^\gamma}$$

for all $n \in \mathbb{N}$ with $n \ge 2$.

Proof We apply part (i) of Lemma 2.2 to $f = K_{n+m}$, noting that

$$[K_{n+m}]_1 \le cm^{-\alpha}\|K_n\|_2$$

and choosing $r = \varepsilon m^\alpha \ge 1$ for a fixed $\varepsilon > 0$. Provided ε is chosen sufficiently small, it follows that there are constants $c_1, a' > 0$ and $m_0 \in \mathbb{N}$ such that

$$\|K_{n+m}\|_2 \le 4^{-1}\|K_n\|_2 + c_1e^{-a'm^{\sigma\alpha}}$$

for all $n, m \in \mathbb{N}$ with $m \ge m_0$ (here m_0 is chosen large enough so that $\varepsilon m_0^\alpha \ge 1$). Let us set

$$\beta_n = e^{bn^\gamma}\|K_n\|_2$$

for $n \in \mathbb{N}$, where $\gamma = \sigma\alpha/(1+\sigma\alpha) \in \langle 0, 1\rangle$ and b is a positive constant to be chosen. Using the elementary estimate

$$(n + m)^\gamma - n^\gamma \le \gamma mn^{\gamma-1},$$

we obtain an inequality

$$\beta_{n+m} \le 4^{-1}e^{b\gamma mn^{\gamma-1}}\beta_n + c_1e^{-a'm^{\sigma\alpha}+b(n+m)^\gamma}$$

for all $n, m \in \mathbb{N}$ with $m \ge m_0$. If m lies in the interval $[2^{-1}n^{1-\gamma}, n^{1-\gamma}]$, then since $\gamma = \sigma\alpha(1 - \gamma)$ we have

$$-a'm^{\sigma\alpha} + b(n + m)^\gamma \le (-a'2^{-\sigma\alpha} + b2^\gamma)n^\gamma.$$

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It follows that we may fix $b > 0$ sufficiently small, so that the inequality

$$\beta_{n+m} \le 2^{-1}\beta_n + c_1 \le \max\{\beta_n, 2c_1\} \tag{9}$$

holds for all $m, n \in \mathbb{N}$ such that $2^{-1}n^{1-\gamma} \le m \le n^{1-\gamma}$ and $m \ge m_0$.

It is possible to argue from (9) that the sequence $\{\beta_n\}_{n=1}^{\infty}$ is bounded. However, we can alternatively argue as follows. Define an increasing sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 2$ and, for $k \ge 1$, $n(k+1)$ is the greatest integer less than or equal to $n(k) + n(k)^{1-\gamma}$. For all sufficiently large k, we may apply (9) with $n = n(k)$ and $m = n(k+1) - n(k)$, to conclude that

$$\sup_{k\in\mathbb{N}} \beta_{n(k)} < \infty.$$

Since $n \mapsto \|K_n\|_2$ is non-increasing, one easily deduces a bound of the desired form $\|K_n\|_2 \le c' \exp(-b'n^{\gamma})$ for all $n \in \mathbb{N}$. Finally, the estimate in L^∞ follows since $\|K_{n+m}\|_\infty \le \|K_n\|_2\|K_m\|_2$ for all $n, m \in \mathbb{N}$. ■

Remark 2.8 Let us mention some corollaries of Theorem 2.7. By combining Theorem 2.7 with the proof of Corollary 1.2, we obtain the following result. Let G be compactly generated, and assume the hypotheses of Corollary 1.2, but now assume there are constants $a > 0$, $\sigma \in \langle 0, 1]$ with

$$V_G(n) \ge a e^{an^{\sigma}} \tag{10}$$

for all $n \in \mathbb{N}$. Then the conclusion of Theorem 2.7 holds.

Note that for a compactly generated group G, there always exists $c > 0$ with $V_G(n) \le ce^{nc}$, $n \in \mathbb{N}$, so that it is not possible to have $\sigma > 1$ in (10) (see [15] for instance).

There is also an extension of Corollary 1.3 for a Lie group. Suppose G is a unimodular Lie group of exponential volume growth, that is, (10) holds with $\sigma = 1$. Then the hypotheses of Corollary 1.3 imply the conclusion of Theorem 2.7, with $\gamma = \alpha/(1+\alpha)$. This result is a consequence of the above remarks and the proof of Corollary 1.3.

We next give an analogue of Theorem 1.1 for a semigroup of operators $\{T_t\}$ defined for small times t. A crucial assumption is that the operators T_t are right invariant. A typical application of this result is to estimate the small time behaviour of heat kernels on Lie groups (see Section 3 for examples).

The result differs from Theorem 1.1 in that the existence of a convolution kernel K is not assumed and is instead part of the conclusion.

Theorem 2.9 *Let $t_0 \in \langle 0, \infty\rangle$ and suppose $\{T_t\}_{0<t<t_0}$ is a family of right invariant, bounded operators in L^2, with $\|T_t\| \le c$ and $T_{s+t} = T_tT_s$ for all $s, t, s+t \in \langle 0, t_0\rangle$. Suppose there exists a modulus function $\rho\colon G \to [0, \infty\rangle$ and $\alpha, \nu > 0$, such that*

$$\|(I - L(g))T_t\| \le c(\rho(g)t^{-\alpha})^{\nu}$$

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for all $t \in \langle 0, t_0 \rangle$ and $g \in G$. Assume there are $a > 0$, $D > 0$ with $V_\rho(r) \geq ar^D$ for all $r \in \langle 0, 1 \rangle$.

Then there exist continuous functions $K_t \colon G \to \mathbb{C}$ such that $T_t f = K_t * f$ for all $f \in L^2$, $K_{t+s} = K_t * K_s$ when $s, t, s+t \in \langle 0, t_0 \rangle$, and

$$\|K_t\|_2 \leq c' t^{-D\alpha/2}, \quad \|K_t\|_\infty \leq c' t^{-D\alpha}$$

for all $t \in \langle 0, t_0 \rangle$.

Proof Let $f \in C_c(G)$ with $\|f\|_1 \leq 1$; the constants in this proof will be independent of f. The right invariance of T_t implies that $L(T_t f) f_2 = T_t L(f) f_2$ for all $f_2 \in C_c(G)$. Consequently,

$$\|L(T_t f)\| \leq \|T_t\| \|L(f)\| \leq c\|L(f)\| \leq c\|f\|_1 \leq c$$

for all $t \in \langle 0, t_0 \rangle$. Observe that

$$\|(I - L(g))T_t f\|_2 \leq \|(I - L(g))T_{t/2}\| \|T_{t/2} f\|_2 \leq c'(\rho(g)t^{-\alpha})^\nu \|T_{t/2} f\|_2$$

for $t \in \langle 0, t_0 \rangle$. Set $\beta_t = t^{D\alpha/2} \|T_t f\|_2$. By applying Lemma 2.2(i) to the function $T_t f$, and using the above observations, we get an inequality

$$\beta_t \leq c_1 (rt^{-\alpha})^\nu \beta_{t/2} + c_1 (rt^{-\alpha})^{-D/2}$$

for all $t \in \langle 0, t_0 \rangle$ and $r \in \langle 0, 1 \rangle$. Choosing $r = \varepsilon t^\alpha$, where $\varepsilon > 0$ is a sufficiently small fixed number, we find that

$$\beta_t \leq 2^{-1} \beta_{t/2} + c_2 \leq \max\{\beta_{t/2}, 2c_2\} \tag{11}$$

for all $t \in \langle 0, t_0 \rangle$. Now

$$\lim_{t \to 0} \beta_t \leq \lim_{t \to 0} ct^{D\alpha/2} \|f\|_2 = 0,$$

so that $\beta_t \leq 2c_2$ whenever t is sufficiently close to 0. Then by (11), it easily follows that $\beta_t \leq c_3 = 2c_2$ for all $t \in \langle 0, t_0 \rangle$. This estimate is independent of f, and hence T_t extends to a bounded operator from L^1 to L^2 with

$$\|T_t\|_{1 \to 2} \leq c_3 t^{-D\alpha/2}$$

for all $t \in \langle 0, t_0 \rangle$. By duality, $\|T_t^*\|_{2 \to \infty} \leq c_3 t^{-D\alpha/2}$. Then since T_t^* is right invariant, by a standard result there exists $J_t \in L^2$ such that $T_t^* = L(J_t)$ and $\|J_t\|_2 = \|T_t^*\|_{2 \to \infty}$. Setting $K_t = \tilde{J}_t$ (see (3)), it easily follows that $T_t = L(K_t)$ and that $\|K_t\|_2$, $\|K_t\|_\infty$ satisfy the desired estimates.

Finally, let us show that K_t may be redefined to be continuous on G. The semigroup property $T_{t+s} = T_t T_s$ easily implies that K_{t+s} equals $K_t * K_s$ almost everywhere on G, when $s, t, s+t \in \langle 0, t_0 \rangle$. Since the convolution of two L^2 functions is continuous, for each $t \in \langle 0, t_0 \rangle$ we can replace K_t by the continuous function $K_t' := K_{t/2} * K_{t/2}$. ∎

3 Examples

In this section we present several examples, or classes of examples, which illustrate the application of our results. In particular, in Example 3.3 below we give the proof of Theorem 1.4.

As usual, the locally compact group G is assumed to be second countable and unimodular.

Example 3.1 Suppose that G is compactly generated, and let $0 \le K \in L^1 \cap L^\infty$ be a bounded probability density on G. Then $T = L(K)$ is a contraction in L^p for all $1 \le p \le \infty$. We assume that K is symmetric ($K(g) = K(g^{-1})$), and that there exists a compact generating neighborhood U of e with $U = U^{-1}$ and

$$\inf\{K(u) \colon u \in U\} > 0. \tag{12}$$

A basic theorem, due essentially to Varopoulos [23], states that if $D > 0$ with $V_G(n) \ge an^D$ for all $n \in \mathbb{N}$, then $\|K_n\|_\infty \le cn^{-D/2}$ for all n. Other proofs of this result have been given [17, 25].

Let us verify that this result is contained in Corollary 1.2. We require an inequality (see [25, pp. 97–98])

$$\|(I - L(u))f\|_2^2 \le c \int_{U^3} dh \|(I - L(h))f\|_2^2$$

for all $u \in U$ and $f \in C_c(G)$. One sees straightforwardly from (12) that, for some $n_0 \in \mathbb{N}$, $\inf\{K_{n_0}(g) : g \in U^3\} > 0$. Let us write $K^{(0)} = K_{n_0}$ and $T_0 = L(K^{(0)})$. Then there is $c' > 0$ such that

$$\begin{aligned} \sup_{u \in U} \|(I - L(u))f\|_2^2 &\le c' \int_G dh\, K^{(0)}(h) \|(I - L(h))f\|_2^2 \\ &= 2c' \,((I - T_0)f, f) = 2c' \,\|(I - T_0)^{1/2} f\|_2^2 \end{aligned}$$

for all $f \in C_c(G)$, where the second line follows by a standard calculation (see, for example, [25, p. 97]). Using this inequality, and the spectral theorem for the non-negative self-adjoint contraction T_0, we obtain an estimate

$$\|(I - L(u))T_0^n\| \le c\|(I - T_0)^{1/2} T_0^n\| \le c' n^{-1/2}$$

for all $u \in U$ and $n \in \mathbb{N}$. Now $T_0 = T^{n_0}$ where $T = L(K)$, and since $n \mapsto \|(I - L(u))T^n\|$ is a non-increasing function of n, we see that $\|(I - L(u))T^n\| \le c'' n^{-1/2}$ for all $u \in U$ and $n \in \mathbb{N}$.

Thus the desired bound $\|K_n\|_\infty \le cn^{-D/2}$ follows from Corollary 1.2 with $\alpha = 1/2$.

Note also that if one assumes a bound of the form (10), then Remark 2.8 yields an estimate

$$\|K_n\|_\infty \le ce^{-bn^{\sigma/(2+\sigma)}}$$

for $n \in \mathbb{N}$. This was previously proved by different methods (see [17, 25] and references therein).

Example 3.2 In this example, we assume that G is a connected unimodular Lie group, and $a_1, \dots, a_{d'}$ is a generating list for the Lie algebra $\mathfrak{g}$ of G. Consider, as in [8], a second order, right invariant differential operator

$$H = -\sum_{i,j=1}^{d'} c_{ij} A_i A_j.$$

where $A_i = dL_G(a_i)$ and the c_{ij} are complex constants which satisfy an ellipticity condition

$$\operatorname{Re} \sum_{i,j} c_{ij} \xi_i \overline{\xi_j} \geq \mu |\xi|^2$$

for some $\mu > 0$ and all $\xi \in \mathbb{C}^{d'}$. The standard case is $c_{ij} = \delta_{ij}$, for which $H = -\sum_i A_i^2$ is called a sublaplacian.

Note that H can be precisely defined using the theory of sectorial quadratic forms (see [18]), as the sectorial operator in L^2 associated with the quadratic form $Q(f) = \sum_{i,j=1}^{d'} \int_G c_{ij} A_j f A_i \overline{f}$, for $f \in L^2$ with $A_i f \in L^2$. Then standard arguments yield that H generates a holomorphic contraction semigroup e^{-tH} in L^2, with $\|He^{-tH}\| \leq ct^{-1}$ for all $t > 0$. It then follows, thanks to the Gårding inequality $\operatorname{Re} Q(f) \geq \mu \sum_i \|A_i f\|_2^2$, that

$$\|A_i e^{-tH}\| \leq c' t^{-1/2} \tag{13}$$

for all $t > 0$ and $i \in \{1, \dots, d'\}$ (see, for example, [8, Section II.2]).

The existence of a smooth kernel $K_t \in L^1 \cap L^\infty$, with $e^{-tH} = L(K_t)$ for all $t > 0$, follows from local results: see, for example, [11]. Note that $K_{t+s} = K_t * K_s = e^{-tH}(K_s)$ for all $s, t > 0$.

To estimate K_t for $t \geq 1$, we apply Corollary 1.3 with $K = K_1$, $T = L(K) = e^{-H}$, and $\alpha = 1/2$. Assuming that $D \geq 1$ with $V_G(n) \geq an^D$, $n \in \mathbb{N}$, we get estimates

$$\|K_t\|_2 \leq ct^{-D/4}, \quad \|K_t\|_\infty \leq ct^{-D/2} \tag{14}$$

for all $t \in \mathbb{N} = \{1, 2, 3, \dots\}$. Because e^{-tH} is a contraction in L^2, it is clear that $\|K_t\|_2$ is a non-increasing function of $t > 0$, so that the bounds (14) are valid for all $t \geq 1$.

If the Lie group G has polynomial volume growth of order D, then the bounds (14) are contained in [13] or [8] (where Gaussian estimates are also obtained). Our proof is simpler than the proofs of [8, 13], which rely on methods of homogenization theory and on the detailed structure theory of Lie groups of polynomial growth.

In case G has exponential volume growth, then given any $D > 0$ one has an estimate (14), and therefore $\lim_{t\to\infty} t^\mu \|K_t\|_\infty = 0$ for any $\mu > 0$. This result can be improved using Remark 2.8 above, which yields a more precise bound

$$\|K_t\|_\infty \leq ce^{-bt^{1/3}}$$

for $t \geq 1$. The latter estimate is well known in the case that H is a sublaplacian (see [21, 25]), but is new for complex coefficients c_{ij}.

Example 3.3 In this example, we show how to prove Theorem 1.4. Adopt the hypotheses and notation of that theorem; in particular, G is a connected unimodular Lie group and M is a fixed positive even integer.

Using a quadratic form procedure as in Example 3.2 above, one can define $H = (-1)^{M/2} \sum_{i=1}^{d'} A_i^M$ as a non-negative self-adjoint operator in L^2 which generates a holomorphic, self-adjoint contraction semigroup e^{-tH}. Again, local results [11, 16] yield the existence of a smooth kernel $K_t \in L^1 \cap L^\infty$ with $e^{-tH} = L(K_t)$, $t > 0$, and $K_{t+s} = K_t * K_s$ for all $s, t > 0$. The function $t \mapsto \|K_t\|_2$ is non-increasing.

To deduce Theorem 1.4 from Corollary 1.3, it is enough to show that

$$\|A_i e^{-tH}\| \leq ct^{-1/M} \tag{15}$$

for all $t > 0$ and $i \in \{1, \ldots, d'\}$. But it is straightforward to deduce from the Gårding inequality $(Hf, f) \geq \sum_i \|A_i^{M/2} f\|_2^2$ that $\|A_i^{M/2} e^{-tH}\| \leq ct^{-1/2}$, $t > 0$. Then (15) follows by a standard interpolation, and Theorem 1.4 is proved.

In case G has exponential volume growth, we get a more precise result by applying Remark 2.8 with $\alpha = 1/M$. Namely, we obtain

$$\|K_t\|_\infty \leq ce^{-bt^{1/(M+1)}}$$

for all $t \geq 1$. For $M \geq 4$, this estimate is apparently new.

Note that our results do not depend on the behaviour of the L^1 norm $\|K_t\|_1$. For $M \geq 4$, it is apparently not known whether $\sup_{t \geq 1} \|K_t\|_1$ is finite, except for special classes of G, for example, nilpotent groups [14].

Example 3.4 In Examples 3.2 and 3.3, we appealed to previous studies [11, 16] for the existence of the kernel K_t. But alternatively, the existence of K_t in $L^2 \cap L^\infty$ (though not in L^1) can easily be deduced from Theorem 2.9.

Indeed, with the operator H of Example 3.2, it follows from (13) and (8) that

$$\|(I - L(g))e^{-tH}\| \leq c\rho_{\mathbf{A}}(g)t^{-1/2}$$

for all $t > 0$ and $g \in G$. One has a standard local estimate, for some integer $D' \geq 1$,

$$c^{-1} r^{D'} \leq V_{\rho_{\mathbf{A}}}(r) \leq cr^{D'}$$

for all $r \in \langle 0, 1 \rangle$ (see [25, Chapter V] for instance). Thus we may apply Theorem 2.9 with $T_t = e^{-tH}$, $\alpha = 1/2$, $\rho = \rho_{\mathbf{A}}$, and any finite $t_0 > 0$. This yields the existence of a kernel $K_t \in L^2 \cap L^\infty$ with $e^{-tH} = L(K_t)$ for all $t > 0$, and

$$\|K_t\|_2 \leq ct^{-D'/4}, \quad \|K_t\|_\infty \leq ct^{-D'/2}$$

for all $t \in \langle 0, 1]$. Similar arguments apply for Example 3.3, where one should take $\alpha = 1/M$.

It is interesting that we can derive the existence of a bounded convolution kernel K_t without using any detailed regularity theory for subelliptic operators, and without needing any information about the action of the semigroup in L^1.

Example 3.5 The following example is an adaption of Example 3.2 for discrete groups.

Let G be a finitely generated discrete group and fix a list $u_1, \dots, u_{d'}$ of generators of G. Consider an operator

$$H = \sum_{i,j=1}^{d'} c_{ij}(I - L(u_i))^*(I - L(u_j)) = \sum_{i,j} c_{ij}(I - L(u_i^{-1}))(I - L(u_j))$$

acting in L^2, where c_{ij} are complex constants satisfying the ellipticity condition of Example 3.2 above. Then a formal variation of the reasoning of Example 3.2, with the operators $I - L(u_i)$ replacing A_i, establishes that e^{-tH} is a holomorphic contraction semigroup in L^2 and

$$\|(I - L(u_i))e^{-tH}\| \le ct^{-1/2}$$

for all $t > 0$ and $i \in \{1, \dots, d'\}$. The kernel $K_t \in L^2$ with $e^{-tH} = L(K_t)$ is given by $K_t = e^{-tH}(\delta_e)$, where $\delta_e \colon G \to \mathbb{R}$ is the function defined by $\delta_e(e) = 1$, $\delta_e(g) = 0$ for $g \neq e$.

If we assume that $D \ge 1$ with $V_G(n) \ge an^D$, $n \in \mathbb{N}$, then Corollary 1.2 yields estimates of the form (14) for all $t \ge 1$.

Moreover, if we assume that $V_G(n)$ satisfies a bound (10), then Remark 2.8 yields $\|K_t\|_\infty \le c\exp(-bt^\gamma)$ for all $t \ge 1$, where $\gamma = \sigma/(2 + \sigma)$. Such estimates are new for complex coefficients.

4 Weighted Modulus Functions

In this section, we suppose that G is a connected unimodular Lie group, and $a_1, \dots, a_{d'} \in \mathfrak{g}$ is a generating list for the Lie algebra $\mathfrak{g}$ of G. Set $A_i = dL_G(a_i)$. Our aim is to obtain a generalization of Corollary 1.3, in which the hypothesis $\|A_i T^n\| \le cn^{-\alpha}$, $n \in \mathbb{N}$, is replaced by

$$\|A_i T^n\| \le cn^{-\alpha_i}$$

for some positive constants α_i, $i \in \{1, \dots, d'\}$. In this situation one could of course apply Corollary 1.3 with $\alpha = \min\{\alpha_1, \dots, \alpha_{d'}\}$, but the resulting estimates of K_n may not be optimal.

To obtain more precise estimates we will apply Theorem 1.1 with respect to a "weighted" modulus function ρ which takes into account different weights in the directions A_i. In particular, the theory of this section yields interesting examples where the modulus function of Theorem 1.1 is not equivalent to the canonical modulus ρ_U associated with a compact neighborhood of e (or to the Caratheodory modulus $\rho_{\mathbf{A}}$, see Remark 2.6).

Let us fix "weights" $w_1, \dots, w_{d'} \in [1, \infty\rangle$ corresponding to $a_1, \dots, a_{d'}$. Following [11, Section 6] (see also [20]), we define a modulus $\rho = \rho_{\mathbf{A},\mathbf{w}}$ depending on the a_i and w_i. For each $r > 0$, let $C(r)$ be the set of absolutely continuous paths $\varphi \colon [0, 1] \to G$ which satisfy

$$\dot\varphi(t) = \sum_{i=1}^{d'} \sigma_i(t)\, A_i|_{\varphi(t)}$$

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almost everywhere, where σ_i are functions such that $|\sigma_i(t)| < r^{w_i}$ for all $t \in [0,1]$ and $i \in \{1,\ldots,d'\}$. Set

$$\rho(g) = \inf\{r > 0 : \text{ there exists } \varphi \in C(r) \text{ with } \varphi(0) = e, \varphi(1) = g\}.$$

It is not difficult to see that $\rho(gh) \le \rho(g) + \rho(h)$ and $\rho(g^{-1}) = \rho(g)$ for all $g, h \in G$. Moreover, it follows from a detailed local analysis (see [11, 20, 25]) that $\rho(g) < \infty$ and that ρ is a modulus function in our sense. In the "unweighted" case where $w_1 = \cdots = w_{d'} = 1$, ρ is equivalent to the Caratheodory modulus $\rho_{\mathbf{A}}$ associated with $A_1, \ldots, A_{d'}$.

The following is the main result of this section.

Theorem 4.1 *Let G be a connected unimodular Lie group and consider as above the generators $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$, the weights $w_1, \ldots, w_{d'} \in [1, \infty\rangle$, and the modulus $\rho = \rho_{\mathbf{A},\mathbf{w}}$.*

Let $K \in L^2$ such that $T = L(K)$ is a contraction in L^2. Suppose $\alpha > 0$ with

$$\|A_i T^n\| \le c n^{-w_i \alpha}$$

for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, d'\}$. If $D > 0$ with $V_\rho(r) \ge a r^D$ for all $r \ge 1$, then $\|K_n\|_2 \le c' n^{-D\alpha/2}$, $n \in \mathbb{N}$, and $\|K_n\|_\infty \le c' n^{-D\alpha}$ for $n \in \mathbb{N}$ with $n \ge 2$.

Proof The main step is to prove an inequality (compare (8))

$$\|(I - L(g))f\|_2 \le \sum_{i=1}^{d'} \rho(g)^{w_i} \|A_i f\|_2 \tag{16}$$

for all $g \in G$ and $f \in L^2$ such that $A_i f \in L^2$, $i \in \{1, \ldots, d'\}$. Given $g \in G$, $r > 0$ and $\varphi \in C(r)$ with $\varphi(0) = e$, $\varphi(1) = g^{-1}$, one calculates

$$(I - L(g))f = -\int_0^1 ds \frac{d}{ds} L(\varphi(s)^{-1}) f = -\int_0^1 ds \sum_{i=1}^{d'} \sigma_i(s) \big(L(\varphi(s)^{-1}) A_i f \big).$$

Then

$$\|(I - L(g))f\|_2 \le \sum_{i=1}^{d'} r^{w_i} \|A_i f\|_2,$$

and (16) follows from the definition of ρ and the equality $\rho(g) = \rho(g^{-1})$.

From (16) and the hypothesis of the theorem, when $\rho(g) \le n^\alpha$ we have

$$\|(I - L(g))T^n\| \le \sum_{i=1}^{d'} \rho(g)^{w_i} \|A_i T^n\| \le \sum_{i=1}^{d'} c(\rho(g) n^{-\alpha})^{w_i} \le d' c \rho(g) n^{-\alpha},$$

where the last step used $w_i \ge 1$. The theorem now follows from Theorem 1.1 (recall that (1) is trivial for $\rho(g) \ge n^\alpha$). ■

Estimates of $V_\rho(r)$ as $r \to 0$ are given in [11], but these are not relevant for Theorem 4.1. In the case of a simply connected nilpotent group, we can estimate $V_\rho(r)$, $r \geq 1$, as follows.

Example 4.2 Let the Lie group G be simply connected and nilpotent, and consider $\rho = \rho_{\mathbf{A},\mathbf{w}}$ as above, where for simplicity we assume that $w_1, \dots, w_{d'} \in \mathbb{N}$.

Following [19, Section 3], we consider the "dimension at infinity" defined by

$$D = \sum_{j \in \mathbb{N}} j(\dim(\mathfrak{g}_{(j)}) - \dim(\mathfrak{g}_{(j+1)})),$$

where $\mathfrak{g}_{(j)}$ denotes the linear subspace of $\mathfrak{g}$ spanned by all commutators in $a_1, \dots, a_{d'}$ of weighted length at least j for $j \in \mathbb{N}$. Here, a commutator

$$[a_{i_1}, [\cdots [a_{i_{n-1}}, a_{i_n}] \cdots]] \in \mathfrak{g}$$

is said to have weighted length $w_{i_1} + \cdots + w_{i_n}$. The nilpotency of $\mathfrak{g}$ ensures that $\mathfrak{g}_{(j)} = \{0\}$ for all sufficiently large j, so D is well defined. We have the following result, whose proof is described in the Appendix below.

Proposition 4.3 *Choose a vector space basis $b_1, \dots, b_d$ for $\mathfrak{g}$ and $v_1, \dots, v_d \in \mathbb{N}$, with the property that $\mathfrak{g}_{(j)} = \operatorname{span}\{b_k : v_k \geq j\}$ for all $j \in \mathbb{N}$. Define $N \colon \mathfrak{g} \to [0, \infty\rangle$ by*

$$N(x) = \sum_{k=1}^{d} |\xi_k|^{1/v_k}$$

for $x = \sum_k \xi_k b_k \in \mathfrak{g}$. Then there exists $c > 1$ such that $c^{-1}(N(x)+1) \leq \rho(\exp x)+1 \leq c(N(x) + 1)$ for all $x \in \mathfrak{g}$, and

$$c^{-1} r^D \leq V_\rho(r) \leq c r^D$$

for all $r \geq 1$.

Note that Proposition 4.3 is well known in case $w_1 = \cdots = w_{d'} = 1$ (see [25, Section IV.5]).

We give an example illustrating the use of Theorem 4.1 and Example 4.2.

Example 4.4 On a simply connected nilpotent Lie group G, consider an operator

$$H = \sum_{i=1}^{d'} (-1)^{M_i/2} A_i^{M_i},$$

where $M_1, \dots, M_{d'}$ are even positive integers. Then H generates a holomorphic contraction semigroup in L^2, and it is easy to obtain estimates (compare Example 3.3 of Section 3)

$$\|A_i e^{-tH}\| \leq c t^{-1/M_i}$$

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for all $t > 0$ and $i \in \{1, \dots, d'\}$. The existence of a kernel K_t with $e^{-tH} = L(K_t)$ is known from [11, 16], but could alternatively be deduced using Theorem 2.9.

Let M be the lowest common multiple of $M_1, \dots, M_{d'}$, and set $w_i = M/M_i$ for $i \in \{1, \dots, d'\}$. Let D be defined as in Example 4.2, relative to a_i and w_i. Applying Theorem 4.1 with $\alpha = 1/M$, we obtain that $\|K_t\|_\infty \leq ct^{-D/M}$ for $t \geq 1$. This result is known (see [9] for instance), but the proof just given is new.

A Appendix

In this appendix, we outline the proof of Proposition 4.3. Let the basis $b_1, \dots, b_d$ of $\mathfrak{g}$, and $v_1, \dots, v_d \in \mathbb{N}$, be chosen such that $\mathfrak{g}_{(j)} = \operatorname{span}\{b_k : v_k \geq j\}$ for all j. A general element in $\mathfrak{g}$ can be written as $x = \sum_{k=1}^{d} \xi_k b_k$, and we shall use the $\xi = \{\xi_k\}$ as coordinates of $\mathfrak{g}$.

Because G is simply connected and nilpotent, the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism which maps Lebesgue measure on the vector space $\mathfrak{g}$ to Haar measure dg. But it is easy to see that $\{x \in \mathfrak{g} : N(x) \leq r\}$ has Lebesgue measure cr^D for all $r > 0$. Thus, to prove the proposition it suffices to obtain the upper and lower estimates on $\rho(\exp x) + 1$.

To show that $\rho(\exp x) \leq c(N(x) + 1)$ is an easy variation of the argument of [25, Proposition IV.5.6], which we only sketch. We may choose the basis $b_1, \dots, b_d$ with the additional property that each b_k equals a commutator in a_i's of weighted length v_k. If $x = \sum_k \xi_k b_k \in \mathfrak{g}$, then following the argument of [25] one writes $\exp x$ as a product of elements of G each of form

$$\exp(\beta|\xi_k|^{w_i/v_k} a_i),$$

where β are constants independent of x, and $w_i \leq v_k$. Moreover, the number of elements in the product is independent of x. Since $|\xi_k|^{w_i/v_k} \leq N(x)^{w_i}$, one easily obtains the desired estimate.

To prove the estimate $N(x) + 1 \leq c(\rho(\exp x) + 1)$, let us suppose that $r > 1$ and $\varphi \in C(r)$ with $\varphi(0) = e$, and put $\gamma = \exp^{-1} \circ \varphi\colon [0, 1] \to \mathfrak{g}$. It is convenient to define a smooth version $N_1\colon \mathfrak{g} \to [1, \infty\rangle$ of N by setting

$$N_1\Big(\sum_{k=1}^{d} \xi_k b_k\Big) = d^{-1} \sum_{k=1}^{d} (1 + \xi_k^2)^{1/(2v_k)}$$

(the constant d^{-1} ensures that $N_1(0) = 1$). Then the desired estimate reduces to showing that

$$N_1(\gamma(1)) \leq cr, \tag{17}$$

where $c > 1$ is a constant independent of $r > 1$ and of $\varphi \in C(r)$.

Consider the vector fields on $\mathfrak{g}$ defined by $\widehat{A}_i = (\exp^{-1})_*(A_i)$, $i \in \{1, \dots, d'\}$, and let $\partial_k = \partial/(\partial \xi_k)$ be the vector fields on $\mathfrak{g}$ corresponding to the coordinates ξ_k. Calculations of [19, Section 4] show that $\widehat{A}_i$ can be written as

$$\widehat{A}_i|_x = \sum_{v_k \geq w_i} P_{i,k}(x) \partial_k,$$

where the sum is over $k \in \{1, \dots, d\}$ such that $\nu_k \geq w_i$, and the $P_{i,k}$ are polynomial functions on $\mathfrak{g}$ satisfying a growth estimate

$$|P_{i,k}(x)| \leq cN_1(x)^{\nu_k - w_i}$$

for all $x \in \mathfrak{g}$. Because N_1 satisfies estimates $|(\partial_k N_1)(x)| \leq c\, N_1(x)^{1-\nu_k}$, we deduce that

$$|(\widehat{A}_i N_1)(x)| \leq c \sum_{\nu_k \geq w_i} N_1(x)^{\nu_k - w_i} N_1(x)^{1-\nu_k} = c' N_1(x)^{1-w_i} \tag{18}$$

for all $x \in \mathfrak{g}$. Since $\varphi \in C(r)$ and $\gamma = \exp^{-1} \circ \varphi$, we may write

$$\gamma'(t) = \sum_{i=1}^{d'} \sigma_i(t) \widehat{A}_i|_{\gamma(t)}$$

where $|\sigma_i(t)| < r^{w_i}$ for all $t \in [0, 1]$. Now differentiate the function $J(t) := N_1(\gamma(t))$ and use (18) to obtain an estimate

$$|J'(t)| \leq \sum_{i=1}^{d'} |\sigma_i(t)||(\widehat{A}_i N_1)(\gamma(t))| \leq c \sum_{i=1}^{d'} r^{w_i} J(t)^{1-w_i}. \tag{19}$$

If $J(1) \leq r$ then (17) holds, so let us assume that $J(1) > r > 1$. Then, since $J(0) = N_1(0) = 1$, there exists a $t_0 \in \langle 0, 1\rangle$ with $J(t_0) = r$ and $J(t) > r$ for all $t \in \langle t_0, 1]$.

But when $J(t) > r$, it follows from (19) that $|J'(t)| \leq c'r$ (here we use $w_i \geq 1$). Therefore,

$$J(1) \leq J(t_0) + \int_{t_0}^{1} dt\, |J'(t)| \leq (1 + c')r,$$

and the proof of (17) is complete. ■

We remark that the derivative estimates (18) are the key point on which the preceding proof hinges.

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