FRAGMENTATION AND DISTRIBUTION OF ASTEROIDS

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As a result of mutual inelastic collisions, frequent on a geologic time scale, the mass distribution of asteroids undergoes constant change. Using a simplified velocity distribution for asteroids, the redistribution of their masses caused by collisions can be mathematically modeled as a stochastic process and the distribution of asteroidal masses can then be obtained as the solution. This paper is a review of recent progress on this problem.

The most detailed discussion of this problem considers the influence of the following collisional processes on the asteroidal mass distribution: (1) loss of asteroids by catastrophic breakup, (2) creation of new objects from the fragments of a catastrophically disrupted one, (3) erosive reduction in the masses of individual asteroids, and (4) erosive creation of new objects (i.e., production of secondary ejecta during erosive cratering by projectiles not large enough to catastrophically disrupt the target object). The main result is that after a sufficiently long period of time the population of asteroids may reach a quasi-steady-state distribution, regardless of the initial distribution. This final distribution is a product of a slowly decreasing function of time by a power law of index 11/6 for masses smaller than the largest asteroids. For the largest asteroids, an additional factor is included that expresses the influence on the distribution of the absence of masses larger than those observed. The observed distribution of bright asteroids from the McDonald asteroidal survey and that of faint ones from the Palomar-Leiden asteroidal survey are each individually consistent with the theoretical distribution, although they differ from each other by a numerical factor.

As a result of mutual inelastic collisions, frequent on a geologic time scale, the distribution of asteroids is constantly changing. We shall, in this paper, discuss the influence these collisions have on the mass distribution of belt asteroids and compare the results with observation.

Ideally, one would consider the mass and orbital elements of each asteroid and establish their origin from precise calculations. This method has been employed by Anders (1965); making the usual assumption that the members of each Hirayama (1923, 1928) family are collisional fragments of some parent object, Anders (1965) has reconstructed the original parent objects and, subtracting the fragments, has estimated the hypothetical initial distribution of asteroids. Hartmann and Hartmann (1968) further studied this problem; they suggested that the present distribution may indeed have evolved, under the influence of collisional fragmentation, from Anders' (1965) estimated initial distribution. Alfvén (1964*a,b*, 1969), on the other hand, discussed the origin of asteroids making the alternate assumption that asteroids in Hirayama families constitute original jetstreams. (Also see Kuiper, 1953.)

In view, however, of the fact that next to nothing is known about the distribution of asteroids too faint to be observed, and much still remains to be learned about those cataloged, it appears worthwhile to employ statistical methods to improve our understanding of some of the gross properties of the population of asteroids. Ideally, one would like to combine the distribution of orbital elements for asteroids with their mass distribution in a complete statistical analysis. This difficult problem can be simplified by two methods:

(1) Studying the distribution of the masses of asteroids using an assumed spatial (and velocity) distribution
(2) Studying the asteroidal population by using precise spatial and velocity distributions combined with an assumed mass distribution (Wetherill, 1967).

This second method has its basis on Öpik's (1951, 1963, 1966) statistical treatment of the dispersal of stray objects by planetary (gravitational) perturbations, and in its most highly developed form has been applied to asteroids by Wetherill (1967).

In this paper we shall limit our attention to method (1). Method (2) is, however, complementary to method (1), because a complete analysis would employ a combination of both methods; i.e., a joint mass, velocity, and position distribution.

Method (1) is the physical and mathematical modeling of a population of objects that undergo mutual inelastic collisions. Such collisions take place with an assumed mean encounter velocity, and the larger of the colliding masses may completely shatter (catastrophic collision) or it may lose a modest fraction of its mass (erosive collision) depending on the relative size of the other colliding object.

The result is a process by which the masses of individual objects in the population decrease with time because of erosion and by which some objects are violently destroyed from time to time. Redistribution of the comminuted debris produced during erosive and catastrophic collisions constitute a particle creation mechanism. A correct modeling of these processes would enable one to describe the evolution of the distribution of these colliding masses.

Piotrowski (1953) has derived a mathematical expression for the rate at which asteroids disappear because of catastrophic collisions and the rate at which the number of asteroids in any given mass range changes because of the erosive reduction of their masses caused by the cratering collisions with relatively small objects. He did not include the particle creation resulting from fragmentation during collisions and his analysis therefore is restricted to cases in which the replenishment (i.e., feedback) of the population by comminuted fragments is insignificant.

Jones (1968) has studied the evolution of the mass distribution of asteroids using a more detailed model; the contribution of fragmentation was considered but later discarded because the size of the fragments produced during collisions was taken to be insignificantly small.

Dohnanyi (1969) (see also Dohnanyi, 1967*a,b,c;* 1970*a*) has discussed a model that includes the influence on the distribution of asteroidal masses of the following collisional processes:

(1) Disappearance of asteroids because of catastrophic breakup
(2) Reappearance of new asteroids from the fragments of catastrophically disrupted ones
(3) Progressive change in the number of asteroids in any given mass range caused by the gradual reduction of asteroidal masses by erosive cratering of small projectile particles
(4) Reappearance, as tiny asteroids, of secondary ejecta produced during erosive cratering

Numerical values for all parameters were taken from experiment and observation, wherever possible; and a particular solution of a simple power-law type was obtained, under the provision that the distribution could be assumed stationary.

The study was continued (Dohnanyi, 1970*b*), and it was found that the mass distribution of asteroids may indeed approach a stationary form, regardless of initial conditions, after a sufficiently long time period has elapsed. The uniqueness of the solution obtained in Dohnanyi (1969) was considered, and it was found to be the only analytic solution that can be expanded into a power series in m, for masses m far from the limiting masses of the distribution. An approximate solution for large asteroids was also obtained.

Hellyer (1970) has also examined this problem; he considered large asteroids and small ones separately. For small asteroids he studied the influence on the mass distribution of fragmentation and his treatment is comparable to that in Dohnanyi (1969, 1970*b*) except that it is less detailed but mathematically much shorter.

Because of their completeness compared with earlier work, we shall, in what follows, give a review of these studies (Dohnanyi, 1969, 1970*b*). Most of the earlier work can readily be discussed by comparing it with special cases of these studies.

OBSERVATIONAL EVIDENCE

McDonald Asteroidal Survey[1]

In their survey of asteroids at the McDonald Observatory (the McDonald survey (MDS)), Kuiper et al. (1958) obtained statistical data for the brighter

[1] Currently under revision; see van Houten in the "Discussion" following this paper.

asteroids up to a limiting apparent magnitude of 16. The observation covered the asteroid belt over all longitudes and a 40° width in latitude. The absolute photographic magnitudes of 1554 asteroids were obtained in half-magnitude intervals together with correction factors for estimating the true number of asteroids in each magnitude interval, based on the completeness of the survey.

To estimate the masses of asteroids, we assume a geometric albedo of $0.2 \times 3^{\pm 1}$ and material density of 3.5×10^3 kg/m^3. The upper limit on the geometric albedo represents a completely white smooth surface and the lower limit corresponds to basalt. The nominal value of 0.2 is the mean of the estimated geometric albedos of the asteroids Ceres, Pallas, Juno, and Vesta. (See, e.g., Sharonov, 1964.) The result is

$$\log_{10} m = 22.67 \pm 0.72 - 0.6g \tag{1}$$

where m is the mass, in kilograms, of a spherical asteroid with absolute photographic magnitude g (i.e., relative photographic magnitude at a distance of 1 AU from both Earth and the Sun). A measure of the uncertainty due to albedo is indicated.

The observational material of MDS is presented in figure 1. Plotted in this figure are the cumulative number of observed asteroids (solid histogram) as well as the probable true number of asteroids (dashed line histogram) versus absolute photographic magnitude g, as given by MDS. The curve is complete up to $g = 9.5$; i.e., the observed number of these objects is believed to equal the true number. Above $g \gtrsim 9.5$ the difference between the true and the observed number of asteroids, based on the completeness of the survey, has been tabulated in MDS (also see Kiang, 1962); the dashed line histogram in figure 1 is their mean value.

The solid curve in figure 1 is the cumulative number $N(m)$ of asteroids larger than m

$$N(m) = \int_m^{M_\infty} f(M)\, dM \tag{2}$$

as a function of mass m (or g) obtained in Dohnanyi (1969). In that study, we took $M_\infty = 1.86 \times 10^{20}$ kg corresponding to $g = 4$ and

$$f(m) = 2.59 \times 10^{16}\, m^{-1.837} \tag{3}$$

where the numerical (normalization) factor is empirical and the numerical value of the exponent was theoretically obtained for relatively small (faint) asteroids. It can be seen that there is close agreement between theory and the statistical results of MDS.

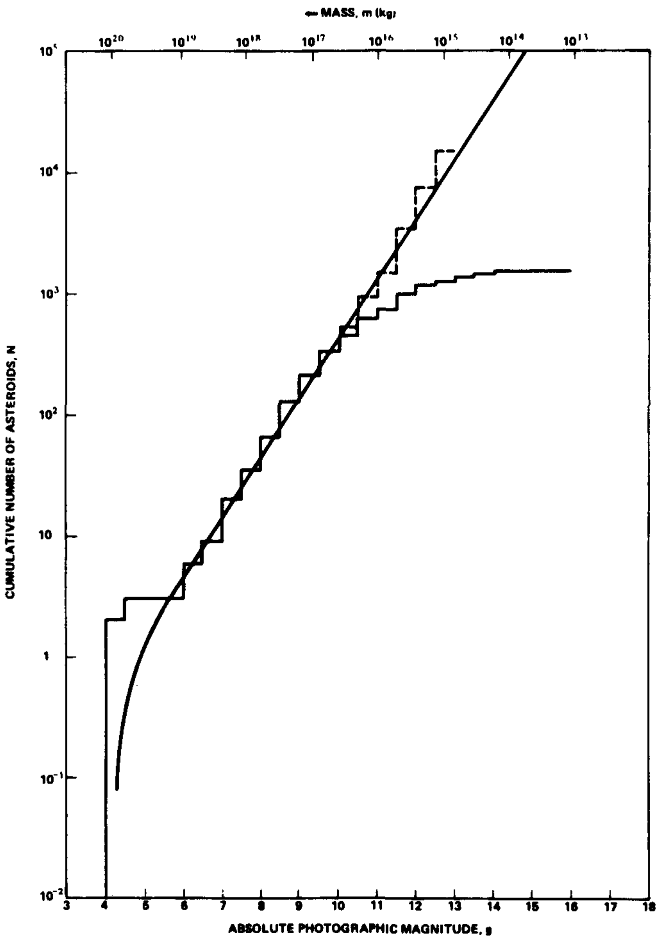

Figure 1.—Cumulative number of asteroids having an absolute photographic magnitude g or smaller (i.e., mass m or greater), obtained by the MDS. Observed value = solid line histogram; probable value = dashed line histogram; earlier theory (Dohnanyi, 1969) = solid curve.

Palomar-Leiden Survey

A series of observations of faint asteroids with limiting apparent magnitudes of less than 20 was made by van Houten et al. (1970) at Hale Observatories (Mount Palomar) (the Palomar-Leiden survey (PLS)). The angular area covered was only 18° by 12° and a compilation of the estimated number of faint asteroids as a function of absolute magnitude, in the range $11 \leqslant g \leqslant 17$, was prepared. Whereas in the MDS results, the number of asteroids found is believed to be complete up to an absolute magnitude of about $g = 9.5$, in the case of the PLS results, the number of observed asteroids needs to be corrected for completeness for all values of g because of the smaller area covered (about 1 percent of the MDS area). Thus, to estimate the total number of faint asteroids in the entire asteroid belt as a function of absolute magnitude g, the PLS data have to be extrapolated over the large regions not covered by the survey.

The result is displayed in figure 2, a plot of the cumulative number of asteroids having an absolute magnitude g or greater (per half-magnitude intervals) obtained by MDS and PLS, as indicated. It can be seen that the two curves display the same trend, i.e., the shapes of the two distributions are identical, but that the MDS results are almost an order of magnitude higher than corresponding PLS results, and likewise for their respective extrapolations. It was pointed out in the PLS report that this discrepancy may be due to the method of estimating completeness factors in MDS. Because the true cause for this discrepancy has not yet been given, we shall avoid combining the results of MDS with those of PLS and will consider them separately.

In figure 3, we plot the cumulative number of asteroids from PLS as a function of absolute magnitude g and seek to represent the results by an empirical formula of the form

$$N(m) = Am^{-\alpha+1} \tag{4}$$

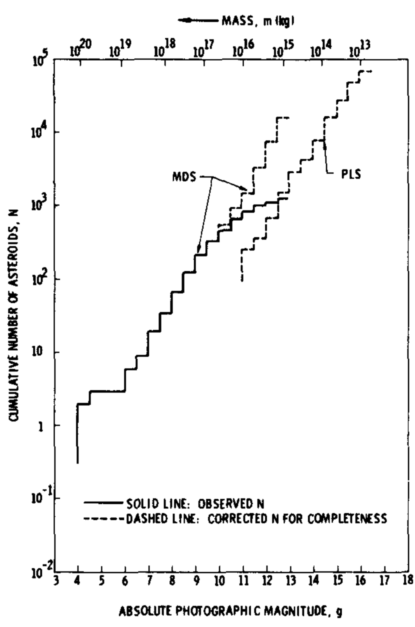

Figure 2.–Cumulative number of asteroids obtained by the MDS and the PLS. Solid line is the observed number; dashed line is the corrected number for completeness.

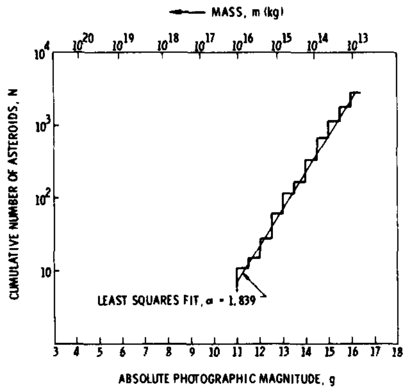

Figure 3.—Cumulative number N of the PLS asteroids with a least-squares fit to N.

where $N(m)$ is the cumulative number of asteroids having masses of magnitude m or greater and A is a constant. A least-squares fit to the data of equation (4) gives

$$\alpha = 1.839 \tag{5}$$

which, in view of uncertainties, can be regarded as identical to the theoretical result (eq. (3)) of $\alpha = 1.837$ obtained in Dohnanyi (1969) and found to represent very well the MDS results. (If the five objects too bright for measurement in the iris photometer employed by PLS are included, one obtains $\alpha = 1.815$; i.e., an insignificant difference of 0.024 for magnitudes $g \geqslant 11$.)

Kessler (1969) has studied the joint distribution of magnitudes, radial distance from the Sun, and heliocentric longitudes of the cataloged asteroids. It appears, from his results, that equations (4) and (5) are good representations of his overall results (NASA SP-8038, 1970).

Recent work by Roosen (1970) indicates that the counterglow may be caused, almost entirely, by particles in the asteroid belt. We may therefore have direct evidence that the distribution of minor planets extends to the size range of micrometeoroids. (See Dohnanyi, 1971.)

We shall, in the remainder of this paper, discuss the manner in which power-law distributions of the types in equations (3), (4), and (5) arise.

IMPACT MECHANICS

Mean Impact Velocity

When two asteroidal objects collide, the damage done to the colliding bodies depends on, besides other factors, the magnitude of the relative velocity of the two colliding objects. A statistical treatment of asteroidal collisions should, therefore, include the velocity distribution function as well as the mass distribution of the colliding masses. We shall, however, confine our attention to the influence of collisions on the mass distribution, using a mean encounter velocity. Such a simplified approach leads to a model that is mathematically tractable, as we shall see later. An alternate approach, in which the velocity distribution is modeled using Monte Carlo techniques but using an assumed mass distribution, has been given elsewhere. (See Wetherill, 1967, for a review and references.)

Consider two asteroidal objects with masses M_1 and M_2. Using a simple molecules-in-a-box approach, kinetic theory tells us that the expected number of times these two objects collide per unit time is

$$\pi(R_1 + R_2)^2 \frac{\bar{\nu}}{V_0} \tag{6}$$

where R_1 and R_2 are the effective radii of the two objects, $\bar{\nu}$ is the mean encounter velocity, and V_0 is the effective volume of the asteroid belt.

Using the distribution of the inclinations and eccentricities for known asteroids, I have estimated (Dohnanyi, 1969) the rms encounter velocity with the estimated dispersion as

$$\sqrt{\overline{\nu^2}} \approx 5 \pm 5 \qquad \text{km/s} \tag{7}$$

in agreement with Piotrowski's (1953) estimate of 5 km/s. The distribution of encounter velocities appears to be rather broad and individual encounter velocities may vary considerably as suggested by equation (7).

Comminution Law

Collisions at impact velocities of several kilometers per second are inelastic and result in fragmentation. Gault et al. (1963) have fired projectiles into effectively semi-infinite basalt targets at very high velocities over a range not exceeding 10 km/s and over a range of projectile kinetic energies from 10 to 10^4 J. The result of the impact was the production of a crater and the ejection of crushed material. The total ejected mass M_e was found to be proportional to the projectile kinetic energy and the size distribution of the ejecta could be approximated by a power-law distribution.

We, therefore, choose (Dohnanyi, 1969) a comminution law of the form

$$g(m; M_1, M_2)\, dm = C(M_1, M_2) m^{-\eta}\, dm \tag{8}$$

where $g(m; M_1, M_2)\, dm$ is the number of fragments in the mass range m to $m + dm$ created when a projectile object M_1 strikes a larger target object of mass M_2. The factor $C(M_1, M_2)$ is a function of the colliding masses and η is a constant,

$$\eta \approx 1.8 \tag{9}$$

for semi-infinite targets. (See Hartmann, 1969, for a survey.)

Using the fact that mass is conserved during impact, it is readily shown that

$$C(M_1, M_2) = (2-\eta) M_e M_b{}^{\eta-2} \tag{10}$$

where M_e is the total ejected mass and M_b is the upper limit to the mass of the largest fragment.

Erosive and Catastrophic Collisions

We shall presently distinguish between two different types of collisions depending on the mass M_1 of the projectile compared with the mass M_2 of the target. For

$$M_1 \ll M_2 \tag{11}$$

the target mass is effectively infinite and Gault's (Gault et al., 1963) results apply. These collisions we shall denote as erosive; clearly, during erosive collisions the projectile craters out a relatively minor amount of mass, leaving the large target mass otherwise intact.

For these collisions, M_e is proportional to the projectile mass M_1 (Gault et al., 1963) and we write (Dohnanyi, 1969), for basalt targets,

$$M_e = \Gamma M_1 \qquad \Gamma \approx 5v^2 \tag{12}$$

with the impact speed v expressed in kilometers per second. (See Marcus, 1969, for a detailed discussion.)

The upper limit to the mass of the largest fragment is given by

$$M_b \simeq \frac{M_e}{\lambda} \qquad \lambda \simeq 10 \tag{13}$$

If the target mass M_2 is not effectively infinite, then some projectile masses will be sufficiently large to catastrophically disrupt the target. Not much is known about the precise relationship between the target mass M_2 and the smallest projectile mass M_1 necessary for catastrophic disruption of M_2 or about the precise nature of the catastrophic failure mode of colliding objects with arbitrary sizes, shapes, and physical composition.

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Experiments (Moore and Gault, 1965) with basalt targets conducted at relatively low impact velocities in the range of 1.4 to 2 km/s imply that a target mass M_2 about 50Γ times the projectile mass or smaller will be catastrophically disrupted. The failure mode of the spherical target consists in the separation of a spherical shell of debris leaving an approximately spherical core behind as the largest fragment.

More recent experiments (Gault and Wedekind, 1969) on finite glass targets indicate a failure mode in which, in addition to a crater having a size determined by equation (12) for semi-infinite targets, a spall fragment on the surface of the spherical target opposite the point of impact will be produced. Both glass and basalt targets are seen to have comparable failure modes; the difference is that the basalt target fails by the production of a spall engulfing most of the spherical surface of a spherical target M_2, whereas the glass sphere target fails by the formation of a spall opposite the impact.

In both cases the distribution of fragments can be represented reasonably well by a formula of the form of equation (8). The total ejected mass is now given by

$$M_e = M_1 + M_2 \tag{14}$$

for catastrophic collisions, and the largest target mass M_2 catastrophically disrupted by M_1 will be taken as $M_2 = \Gamma' M_1$. Thus,

$$M_2 \leqslant \Gamma' M_1 \tag{15}$$

for catastrophic collisions, and

$$M_2 \geqslant \Gamma' M_1 \tag{16}$$

for erosive collisions.

The quantity Γ' is difficult to estimate precisely; combining results by Gault et al. (1963), Moore and Gault (1965), and Gault and Wedekind (1969), we may write

$$\begin{aligned} \Gamma' &\approx 50\Gamma \qquad \text{for basalt} \\ \Gamma' &\approx 10^3\Gamma \qquad \text{for glass} \end{aligned} \tag{17}$$

The large difference in these numbers is due mainly to the differences in the catastrophic failure modes between basalt and glass. Less energy is needed to detach a spall from a glass sphere than to detach a spherical shell of fragments from a basalt sphere.

The limit of the mass of the largest fragment for catastrophic collisions can be taken as

$$M_b = \frac{M_2}{\lambda'} \tag{18}$$

This formula is an idealization because for catastrophic collisions the size of the largest fragment should be approximately inversely proportional to the collisional kinetic energy. This relation defines the expected size of the largest fragment during an average catastrophic collision. For a more detailed definition of M_b we can take

$$M_b = \frac{M_2{}^2}{\lambda_0 M_1}$$

where M_b is inversely proportional to M_1 and λ_0 is a constant. The main effect of this refinement on the subsequent analysis (unpublished) is to add further detail without, however, altering the main conclusions. We therefore choose to retain the mathematically simpler but physically less correct definition of M_b (eq. (18)). Collecting formulas, we have

$$g(m; M_1, M_2)\, dm = (2 - \eta)\Gamma^{\eta-1}\lambda^{2-\eta}M_1{}^{\eta-1}m^{-\eta}\, dm \qquad M_2 \geqslant \Gamma' M_1 \quad (19)$$

for erosive collisions, and

$$g(m; M_1, M_2) = (2 - \eta)(M_1 + M_2)M_2{}^{\eta-2}(\lambda')^{2-\eta}m^{-\eta}\, dm \quad M_2 < \Gamma' M_1 \quad (20)$$

for catastrophic collisions.

COLLISION EQUATION

Collisions between asteroids must undoubtedly affect their mass distribution. To gain insight into this problem, we give a precise mathematical model of the evaluation of the asteroidal mass distribution under the influence of mutual inelastic collisions.

Let $f(m, t)\, dm$ be the number density per unit volume of asteroids in the mass range m to $m + dm$ at time t. Clearly, $f(m, t)\, dm$ will change in time because of (1) erosion, (2) removal by catastrophic collisions of objects in this mass range, and (3) creation of fragments into this mass range by the erosive or catastrophic collisions of larger objects.

Assuming a uniform spatial distribution throughout the asteroid belt, one can write a continuity equation for the number density $f(m, t)$:

$$\frac{\partial f(m,t)}{\partial t}\, dm = dm \left.\frac{\partial f}{\partial t}\right|_{\text{erosion}} + dm \left.\frac{\partial f}{\partial t}\right|_{\text{catastrophic collisions}}$$
$$+ dm \left.\frac{\partial f}{\partial t}\right|_{\text{catastrophic creation}} + dm \left.\frac{\partial f}{\partial t}\right|_{\text{erosive creation}} \qquad (21)$$

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Here $[\partial f(m, t)/\partial t]\, dm$ is the time rate of change of the number density per unit volume of asteroids in the mass range m to $m + dm$ because of all the collisional processes listed on the right-hand side of the equation. The individual terms on the right-hand side of equation (21) are discussed below.

Erosion

The first term on the right-hand side of equation (21) is the contribution of the erosive reduction in the particle masses; i.e., the reduction in the number of particles with given mass because much smaller erosive projectiles crater out minor amounts of mass from these particles.

It has been shown in Dohnanyi (1969) that

$$\left.\frac{\partial f}{\partial t}\right|_{\text{erosion}} = -\frac{\partial}{\partial m}\left[f(m, t)\,\frac{dm}{dt}\right] \tag{22}$$

where

$$\frac{dm}{dt} = -\Gamma K \int_{\mu}^{m/\Gamma'} Mf(M, t)(M^{1/3} + m^{1/3})^2\, dM \tag{23}$$

is the mass lost per unit time by an object having a mass m that is being "sandblasted" by erosive collisions, and where

$$K = \left(\frac{3\pi^{1/2}}{4\rho}\right)^{2/3} \bar{\nu} \tag{24}$$

The parameter μ is the smallest mass permitted to be present by radiation pressure. Although objects may be present that are smaller than the limiting small mass blown away by radiation pressure as determined by geometric optics, we shall not concern ourselves with this problem. We shall assume that masses smaller than μ are either absent or simply do not participate in the collisional processes considered here.

The expression for dm/dt in equation (23) can be seen to be correct because the amount of mass per unit time lost by m because of erosive collisions with particles in the mass range M to $M + dM$ is, using equations (6) and (12),

$$-\Gamma M[R(M) + R(m)]^2 f(M, t)\, dM = -K\Gamma M(M^{1/3} + m^{1/3})^2 f(M, t)\, dM \tag{25}$$

and the right-hand side of equation (23) is just the contribution to $\dot{m}$ of all erosive projectiles; i.e., all projectiles with masses smaller than m/Γ' (cf. eq. (16)).

The contribution of this erosive reduction of the masses to the distribution $\partial f/\partial t|_{\text{erosion}}$ is then seen to be correctly given by equation (22), because $\dot{m}f(m, t)$ is the one-dimensional flux of particles in "mass space" and the right-hand side of equation (22) is the negative divergence of the flux in mass space. The net contribution of $\partial f/\partial t|_{\text{erosion}}$ may be positive or negative, depending on whether more masses are eroded into the range m to $m + dm$ than are eroded out of this range per unit time, or vice versa. (See Dohnanyi, 1967*b*, for a detailed derivation.)

Catastrophic Collisions

The second term on the right-hand side of equation (21) is the contribution of catastrophic collisions to the evolution of the population. It is (Dohnanyi, 1969),

$$dm \left.\frac{\partial f(m, t)}{\partial t}\right|_{\text{catastrophic collisions}}$$

$$= - Kf(m, t)\, dm \int_{m/\Gamma'}^{M_\infty} f(M, t)(M^{1/3} + m^{1/3})^2\, dM \quad (26)$$

where M_∞ is the largest mass present.

This equation is readily derived because the number of collisions per unit volume of space and unit time $\delta^2 n$ between spherical particles with masses in the range M_1 to $M_1 + dM_1$ and M_2 to $M_2 + dM_2$ is (cf. eqs. (6) and (25))

$$\delta^2 n = K(M_1{}^{1/3} + M_2{}^{1/3})^2 f(M_1, t) f(M_2, t)\, dM_1 dM_2 \quad (27)$$

The total number per unit volume and unit time of catastrophic collisions that objects with masses in the range m to $m + dm$ experience is then given by the integral $\delta^2 n$ over the permissible limits, which is just equation (26). The range of values for the dummy integration variable M, $m/\Gamma' \leqslant M \leqslant M_\infty$ is seen to include all mass values that would completely disrupt m during an inelastic collision (cf. eq. (15)).

Creation by Catastrophic Collisions

We shall presently derive an expression for the creation per unit volume and unit time of objects in the mass range m to $m + dm$ by the catastrophic disruption of larger objects.

We first note that the number of fragments in the mass range m to $m + dm$ created by the catastrophic disruption of two objects having masses M_1 and M_2

is given by equation (20). The number of collisions $\delta^2 n$ per unit volume of space and unit time between two spherical objects with masses in the range M_1 to $M_1 + dM_1$ and M_2 to $M_2 + dM_2$ is given by equation (27). Hence, combining these, we obtain the number of fragments in a mass range m to $m + dm$ created per unit time and volume by catastrophic collisions between masses in the range M_1 to $M_1 + dM_1$ and M_2 to $M_2 + dM_2$ (with $M_2 > M_1$):

$$g(m; M_1, M_2)\, dm\, \delta^2 n = m^{-\eta}\, dm(2-\eta)(\lambda')^{2-\eta} M_2{}^{\eta-2}(M_1 + M_2)$$
$$\times K(M_1{}^{1/3} + M_2{}^{1/3})^2 f(M_1, t) f(M_2, t)\, dM_1\, dM_2 \quad (28)$$

This expression is valid for

$$m \leqslant \frac{M_2}{\lambda'} \quad (29)$$

because m cannot exceed the mass of the largest fragment produced by the catastrophic collision of M_1 with M_2 (cf. eq. (18)).

Integrating expression (28) over all permissible masses M_2 and M_1, we obtain the contribution of this creation process to equation (21):

$$\left.\frac{\partial f(m, t)}{\partial t}\right|_{\text{catastrophic creation}} = K(2-\eta)(\lambda')^{2-\eta} m^{-\eta} \int_{\lambda' m}^{M_\infty} dM_2$$
$$\times \int_{M_2/\Gamma'}^{M_2} dM_1\, M_2{}^{\eta-2}(M_1 + M_2)(M_1{}^{1/3} + M_2{}^{1/3})^2 f(M_1, t) f(M_2, t) \quad (30)$$

which is the desired expression.

Creation by Erosive Collisions

Using the same reasoning as the one employed in the derivation of equation (30), we can obtain the corresponding expression for the erosive creation of objects into the mass range m to $m + dm$. Combining the comminution law for erosive collisions, equation (19), with the differential frequency of these collisions $\delta^2 n$ and integrating, over all permissible masses M_1 and M_2, we obtain

$$\left.\frac{\partial f(m, t)}{\partial t}\right|_{\text{erosive creation}} = K(2-\eta)\Gamma^{\eta-1}\lambda^{2-\eta} m^{-\eta} \int_{\lambda m/\Gamma}^{M_\infty/\Gamma'} dM_1$$
$$\times \int_{\Gamma' M_1}^{M_\infty} dM_2\, M_1{}^{\eta-1}(M_1{}^{1/3} + M_2{}^{1/3})^2 f(M_1, t) f(M_2, t) \quad (31)$$

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This completes the derivation of the explicit form of $\partial f(m, t)/\partial t$, equation (21).

SOLUTION FOR SMALL MASSES

Asymptotic Solution

The general solution of the collision equation (eq. (21)) is difficult to obtain. We shall, however, seek an asymptotic solution valid after a long period of time of the creation of the asteroids.

Specifically, we seek a solution of the form

$$f(m, t) \simeq a_0(m) + \frac{a_1(M)}{t} + \frac{a_2(m)}{t^2} + \ldots \tag{32}$$

valid when t becomes very large. We substitute equation (32) into equation (21) and equate the coefficients of like powers of t to zero.

Using equations (21), (22), (26), (30), and (31) we get, for $a_0(m)$,

$$\begin{aligned} 0 = K\Gamma \frac{\partial}{\partial m} &\left[a_0(m) \int_{\mu}^{m/\Gamma'} M a_0(M)(M^{1/3} + M^{1/3})\, dM\right] \\ &- K a_0(m) \int_{m/\Gamma'}^{M_\infty} a_0(M)(M^{1/3} + M^{1/3})^2\, dM + K(2-\eta)(\lambda')^{2-\eta} m^{-\eta} \\ &\times \int_{\lambda' m}^{M_\infty} dM_2 \int_{M_2/\Gamma'}^{M_2} dM_1\, M_2^{\eta-2}(M_1 + M_2)(M_1^{1/3} + M_2^{1/3})^2 a_0(M_1) a_0(M_2) \\ &+ K(2-\eta)\Gamma^{\eta-1}\lambda^{2-\eta} m^{-\eta} \int_{\lambda m/\Gamma}^{M_\infty/\Gamma'} dM_1 \\ &\times \int_{\Gamma' M_1}^{M_\infty} dM_2\, M_1^{\eta-1}(M_1^{1/3} + M_2^{1/3})^2 a_0(M_1) a_0(M_2) \end{aligned} \tag{33}$$

which is the equation for the steady-state solution (cf. Dohnanyi, 1969; 1970*b*).

A time-independent solution of equation (21) is not valid because

$$\lim f(m, t) = 0 \qquad t \to \infty \tag{34}$$

in the absence of sources. We, therefore, take $a_0(m)$ to be a slowly varying function of time, satisfying equation (34), and approximately satisfying equation (33). This argument requires that the creation and destruction terms on the right-hand side of equation (21) balance each other. It is, however, clear that for masses

$$m > \frac{M_\infty}{\lambda'} \tag{35}$$

no particle creation is possible because the upper limit of the largest fragment during a catastrophic collision involving M_∞ is smaller than m. Equation (32) can therefore be valid only if

$$a_0(m) = 0 \qquad \frac{M_\infty}{\lambda'} \leqslant m \leqslant M_\infty \tag{36}$$

and we have, therefore, a different solution for the distribution of large masses.

Solution in a Power Series of m for Small Masses

The power series solution to the leading terms of the steady-state equation, equation (33), is (Dohnanyi, 1969; 1970*b*)

$$a_0\,(m) = Am^{-\alpha} \tag{37}$$

where A is a constant and the population index α is

$$\alpha = \frac{11}{6} \tag{38}$$

The leading terms in equation (33) are those describing particle creation and destruction by catastrophic collisions caused by the impact of projectile particles whose masses and geometric cross sections are negligibly small compared with the target objects.

The contribution of erosion to the steady-state process, equation (33), is a minor one. The contribution of $\partial f/\partial t|_{\text{erosion}}$ (cf. eqs. (21) and (22)) is negative; i.e., the number of objects in the mass range m to $m + dm$ will decrease because of the erosive reduction of particle masses when the distribution is given by $\alpha = 11/6$ (eq. (38)). This happens because for a power-law-type distribution, $\partial f/\partial t|_{\text{erosion}}$ (eq. (22)) is positive for $\alpha > 4/3$.

It is interesting to note that when $\alpha = 11/6$, the leading terms of $\partial f/\partial t|_{\text{erosion}}$ and $\partial f/\partial t|_{\text{erosive creation}}$ cancel each other out.

If the comparatively small contribution of terms associated with the mass and size of the projectile during catastrophic collisions as well as the contribution of erosive processes are included in equation (33), the value of α is not appreciably different from 11/6. It was found (Dohnanyi, 1969) that at mean impact velocities ranging from 1 to 20 km/s and values for the comminution index η (eq. (8)) ranging from 1.7 to 1.9, numerical solutions for α ranged from 1.841 at 1 km/s to 1.835 at 20 km/s mean impact velocity.

It therefore follows that the steady-state solution is rather insensitive to changes in the physical parameters. This solution represents a population whose evolution is mainly controlled by the catastrophic destruction of objects in a given mass range, and by the creation of fragments in this same mass range by catastrophic disruption of larger masses. These two competing processes cancel each other in a steady-state population described by the solution of equation (33).

SOLUTION FOR LARGE MASSES

Very Large Masses

For the largest masses, creation by fragmentation cannot be very effective and the number of these asteroids decreases with time. The collision equation (21) becomes correspondingly simplified.

Specifically, for masses in the range

$$\frac{M_\infty}{\lambda'} \leqslant m \leqslant M_\infty \tag{39}$$

i.e., for masses greater than the largest fragment when M_∞ is disrupted, no creation by fragmentation is possible. If the number density of asteroids in this mass range is denoted by $F(m, t)$ and the number density for masses $m \leqslant M_\infty/\lambda'$ is denoted by $f(m, t)$, we have, using equations (21) and (26)

$$\frac{\partial F(m, t)}{\partial t} = -KF(m, t) \left[\int_{m/\Gamma'}^{M_\infty/\lambda'} f(M, t)(M^{1/3} + m^{1/3})^2 \, dM \right.$$

$$\left. + \int_{M_\infty/\lambda'}^{M_\infty} F(M, t)(M^{1/3} + M^{1/3})^2 \, dM \right] \tag{40}$$

where

$$\lambda' < \Gamma' \tag{41}$$

The contribution of erosion has been dropped because the largest asteroids have a sufficiently strong gravitational field to retain most of the secondary ejecta produced during erosive cratering (Marcus, 1969; Hartmann, 1968).

The most important feature of equation (40) is the strong coupling between the solutions $F(M, t)$ and $f(m, t)$ for $\lambda' \ll \Gamma'$. Because Γ' is of the order of 10^3 to 10^5, depending on whether we assume asteroids to be more similar to basalt spheres or to glass spheres and because λ' is about 1, we conclude that, for asteroids,

$$\lambda' \ll \Gamma' \tag{42}$$

If, however, we make the opposite assumption and let

$$\lambda' \to \frac{M_\infty}{\mu} \tag{43}$$

i.e., all collisional fragments somehow just go away as do masses smaller than μ, we obtain the equation

$$\frac{\partial F(m, t)}{\partial t} = - KF(m, t) \int_{M_\infty/\Gamma'}^{M_\infty} F(M, t)(M^{1/3} + m^{1/3})^2 \, dM \tag{44}$$

This equation, first posed for asteroids by Piotrowski (1953), has received attention by a number of authors. Piotrowski found that equation (44) can be solved approximately if we separate variables and let

$$F(m, t) = \rho(m)T(t) \tag{45}$$

The result is

$$F(m, t) \approx T(t)m^{-5/3} \tag{46}$$

and has the property that the total cross-sectional area of asteroids having masses in the range m_1 and m_2 is proportional to ln (m_1/m_2) and therefore independent of m_1 or m_2.

The stability of the solution, equation (46), has been discussed by Piotrowski (1953) and in greater detail by Marcus (1965); they conclude that once the population reaches a distribution of the form of equation (46), it is stable.

Jones (1968) examined the problem when Γ' is small and λ' is large; i.e., a case similar to the one defined by equation (43). He also obtained an approximate solution of the form of equation (46).

More recently, Hellyer (1970), in an effort to obtain separate solutions for large and small asteroids, discussed Piotrowski's equation (44) and again verified the approximate solution, equation (46).

Unfortunately, there are difficulties associated with the application of equation (44) and its particular solution, equation (46), for asteroids as has also been pointed out by Hartmann and Hartmann (1968). First, equation (44) is incorrect, unless it is assumed that in a collision the colliding objects are virtually atomized (eq. (43)) and hence their fragments do not contribute to the population of smaller objects (cf. eq. (40)). This assumption is in contrast with results of experiments on laboratory-sized objects and there appears to be evidence (Anders, 1965) that large asteroids break up into a spectrum of debris that significantly contributes to the population of observed asteroids. It is indeed probable that most asteroids are collisional fragments (Anders, 1965; Dohnanyi, 1969). Thus it appears that equation (44) is not a good mathematical model for asteroidal collisions.

Even if the physical applicability of equation (44) could somehow be maintained, it is difficult to interpret the significance of the approximate particular solution in equation (46). Because Piotrowski's equation (eq. (44)) is a partial differential equation, its solution must include an arbitrary function. Physically, this is an obvious consequence of the fact that one should be able to prepare a fairly arbitrary initial distribution that should satisfy equation (44) at some point of time. No evidence has yet been advanced for the existence of an initial distribution, other than the solution equation (46) itself, which would approach the $m^{-5/3}$ power-law distribution. In short, there appears to be no evidence that equation (46) is indeed an asymptotic solution valid after some long period of time has elapsed since creation.

Asymptotic Solution for Long Times

In this section, we shall derive an asymptotic form for the distribution of large asteroids valid after some long period of time has elapsed since their creation.

We shall take

$$\lambda' = 1 \tag{47}$$

which means that the mass of the target object becomes the upper limit to the mass of the largest fragment; i.e., the "threshold" of the failure mode is included. (The *expected* size of the largest fragment is naturally smaller than the target object.)

Using this relation (eq. (47)) in the continuity equation for the largest asteroids, equation (40), we see that the second integral on the right-hand side vanishes with $\lambda' \cong 1$. We also must include the contribution $\partial F(m, t)/\partial t|_{\text{catastrophic creation}}$ (eq. (30)) to $\partial F(m, t)/\partial t$ (eq. (40)), because the largest fragment of a catastrophic process involving a large asteroid may still be within the size range of the largest asteroids. We may therefore write

$$\begin{aligned}\frac{\partial F(m, t)}{\partial t} = &- KF(m, t)\int_{m/\Gamma'}^{M_\infty} F_p(M, t)(M^{1/3} + m^{1/3})^2\, dM + K(2 - \eta)m^{-\eta}\\ &\times \int_m^{M_\infty} dM_2 \int_{M_2/\Gamma'}^{M_2} dM_1\, M_2^{\eta-2}(M_1 + M_2)(M_1{}^{1/3} + M_1{}^{1/3})^2\\ &\times F(M_1, t)F_p(M_2, t)\end{aligned} \tag{48}$$

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where the subscript p has been attached to $F(m, t)$ to denote the number density of the smaller (i.e., projectile) asteroids.

Each of the two asteroidal surveys, MDS and PLS, suggests a power-law-type population

$$f(m) \approx A m^{-11/6} \tag{49}$$

For such populations, the dominant contribution to catastrophic collisions (both the creation and removal term) is caused by the collision of projectile objects having masses of the order of M/Γ', with target objects having masses M. (See Dohnanyi, 1969.)

Because Γ' is a large number of several orders of magnitude, we shall take as a first approximation to equation (48):

$$F_p(m, t) \approx a_0(m, t) = A(t) m^{-\alpha} \qquad \alpha = \frac{11}{6} \tag{50}$$

where a_0 is the solution for the steady-state distribution of small objects, equation (37). Furthermore, a_0 is taken here as a function of t that varies slowly compared with $a_1/t + a_2/t^2 + \ldots$ in equation (32), which are here treated as transients.

Substitution of equation (50) for $F_p(m, t)$ in equation (48) yields a linear equation for $F(m, t)$. Retaining the leading terms in this linearized equation, one can solve it for $F(m, t)$ and thereby obtain a first approximation for the distribution of large asteroids. This was done in Dohnanyi (1970*b*) with the result

$$F(m, t) = A(t) m^{-11/6} \left[1 - \left(\frac{m}{M_\infty} \right)^{1/6} \right]^{6(2-\eta)-1} \tag{51}$$

where

$$A(t) = A_0 \left[1 + M_\infty{}^{-\alpha+5/3} K (\Gamma')^{\alpha-1} \frac{A_0 (t - t_0)}{\alpha - 1} \right]^{-1} \tag{52}$$

A_0 is the value of A when the time parameter t equals t_0.

We now let t_0 denote the present time and A_0 the present value of $A(t)$; we have (cf. Dohnanyi, 1969),

$$A(t) = A_0 \left[\frac{1 + (t - t_0)}{\tau_\infty} \right]^{-1} \tag{53}$$

where τ_∞ is the mean time between catastrophic collisions of the largest objects M_∞(cf. Dohnanyi, 1969), if the latter could survive these collisions. It was estimated (Dohnanyi, 1969) that

$$\tau_\infty \sim 10^9 \text{ yr} \tag{54}$$

Combining equations (53) and (54), we see that, according to the present model, it will take on the order of 3×10^9 yr for the number of asteroids to decrease to one-half its present value. $A(t)$ cannot, of course, be extrapolated backward over long periods of time because it is a first approximation to the distribution of asteroids a long time after their creation.

The first approximation of $F(m, t)$ given by equation (51) has the property that it goes over into $a_0(m, t)$; i.e., into $A(t)m^{-11/6}$ for sufficiently small m. Thus

$$F(m, t) \approx A(t)m^{-11/6} \qquad m \ll M_\infty \tag{55}$$

as can readily be seen from equation (51).

DISCUSSION OF RESULTS

Physical Significance of the Stationary Solution ($\alpha \approx 11/6$)

It is difficult to give a simple physical argument that would demonstrate, from first principles only, that $\alpha \approx 11/6$ is the obvious solution to the collision equation (eq. (21)). It has, however, been shown in Dohnanyi (1969) that the total amount of mass $\dot{M}_{12}$ crushed catastrophically per unit time by projectile masses in any finite range m_1 to m_2 is

$$\dot{M}_{12} \approx \int_{m_1}^{m_2} AM^{-\alpha}\, dM \int_M^{\Gamma' M} M_2 K M_2^{2/3} A M_2^{-\alpha}\, dM_2$$

$$= \frac{KA^2(\Gamma')^{-\alpha+8/3}}{(-\alpha + 8/3)(-2\alpha + 11/3)} \left(m_2^{-2\alpha+11/3} - m_1^{-2\alpha+11/3}\right) \tag{56}$$

when $\alpha \neq 11/6$ and $m_2 \ll M_\infty/\Gamma'$, and

$$\dot{M}_{12} = \frac{KA^2(\Gamma')^{-\alpha+8/3}}{-\alpha + 8/3} \ln \frac{m_2}{m_1} \tag{57}$$

when $\alpha = 11/6$ and $m_2 \ll M_\infty/\Gamma'$.

In these equations only the leading terms have been retained, treating projectiles as point particles and disregarding grazing collisions.

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We now consider equations (56) and (57) in more detail. It can readily be seen that if $\alpha > 11/6$, then $\dot{M}_{12}$ will mainly depend on m_1 if the logarithmic interval m_2/m_1 is sufficiently large; i.e., $\dot{M}_{12}$ depends on the particular value of m_1 but is insensitive to m_2. The converse is true for $\alpha < 11/6$; for sufficiently large m_2/m_1, $\dot{M}_{12}$ depends on the particular value of m_2 and not on m_1. Thus, for a sufficiently large logarithmic interval m_2/m_1, practically all mass is crushed by the smallest projectile objects in the interval for $\alpha > 11/6$ and practically all mass is crushed by the biggest projectile masses in the interval for $\alpha < 11/6$. For $\alpha = 11/6$, however, $\dot{M}_{12}$ does not depend on the particular value of either m_1 or m_2 but only on their ratio m_2/m_1. Therefore, the total mass crushed per unit time in the asteroidal belt depends, mainly, on the particular value of the limiting masses of the distribution, μ or M_∞, depending on whether $\alpha < 11/6$ or $\alpha > 11/6$, respectively; whereas for $\alpha = 11/6$ the mass production is constant for fixed logarithmic intervals of projectile masses m_2/m_1 and is independent of the limiting masses μ and M_∞ in a first order of approximation.

Relative Importance of the Various Collisional Processes

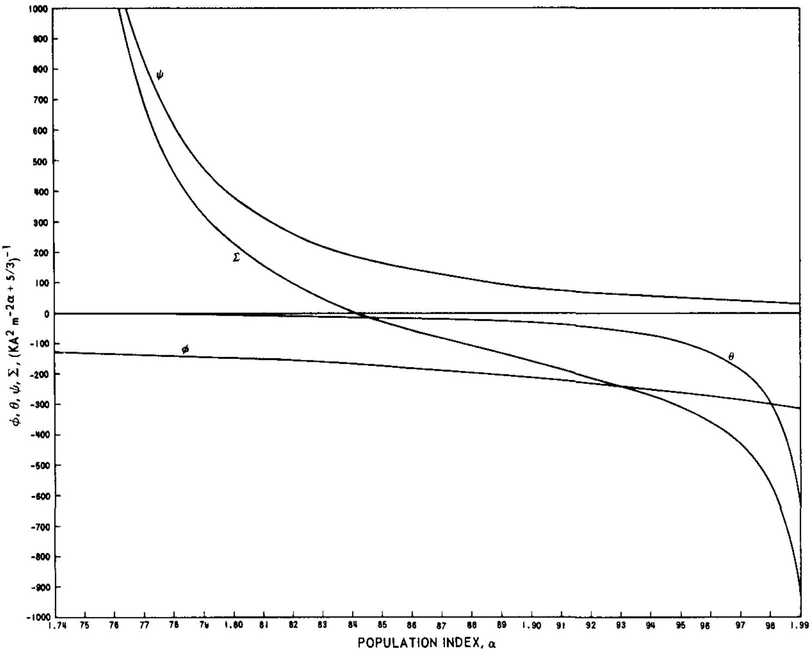
The result $\alpha = 11/6$ is valid when only the leading terms of equation (21) are retained. A more detailed treatment has to consider the influence of higher order terms, as well. This was done in Dohnanyi (1969) for a distribution of this type

$$f(m) = Am^{-\alpha} \tag{58}$$

and for masses m that are far from the limiting masses μ and M_∞.

Figures 4 and 5 are plots, in units of $(KA^2m^{-2\alpha+5/3})^{-1}$, of the number of particles per unit mass, volume, and time removed (or created) by the individual collisional processes and their sum for two different average collisional velocities, as indicated. The population index of the crushed fragments during each collision η is taken to be the experimental value 1.8. The value of α at which the curve representing the sum of all processes crosses the horizontal axis (i.e., the value of α at which the individual processes add up to zero) is the solution for α of equations (33) and (37).

It can be seen, from figures 4 and 5, that the particle creation term is significant only for values of α lower than about 1.9 and that the influence of the erosive reduction of the masses dominates for higher values of α. The individual processes and their sums exhibit remarkably similar trends; the values of α at which steady state is reached is $\alpha = 1.841$ in figure 4 and 1.835 in figure 5. It can be seen, from these figures, that the steady-state distribution is determined by the balance of the catastrophic creation and collision processes. Because, however, it is readily shown that the contribution of erosive creation is at most on the order of $\Gamma/\Gamma' \approx 1/50$ times the similar contribution of catastrophic processes, we may conclude that erosion has only a minor effect on the steady-state distribution.



ψ = rate of change in the number of particles because of particle creation by fragmentation of larger objects
ϕ = rate of change in the number of particles because of catastrophic collisions
θ = rate of change in the number of particles because of erosion
Σ = $\theta + \phi + \psi$

Figure 4.–Rate of change of the number of particles in units of $(KA^2 m^{-2\alpha+5/3})^{-1}$ per unit time and unit mass range as a function of the population index α. $\eta = 1.8$; $V = 1$ km/s; and $\Gamma = 5$.

Because the material parameter Γ is greater by a factor of 400 in figure 5 than in figure 4, we conclude that the value of α at which steady state is reached and the relative trends of the individual collisional processes are insensitive to the material parameters. The same holds for η, because a modest variation in η was found in Dohnanyi (1969) to produce no significant departures.

Distribution of the Largest Asteroids

The influence of some of the higher order terms on the solution of equation (21) was studied in Dohnanyi (1969). (Also see the previous section of this paper.) We now consider the influence of the limiting largest mass M_∞ on the solution of equation (21). This was discussed in Dohnanyi (1970*b*) and the section entitled "Solution for Large Masses" in this present paper.

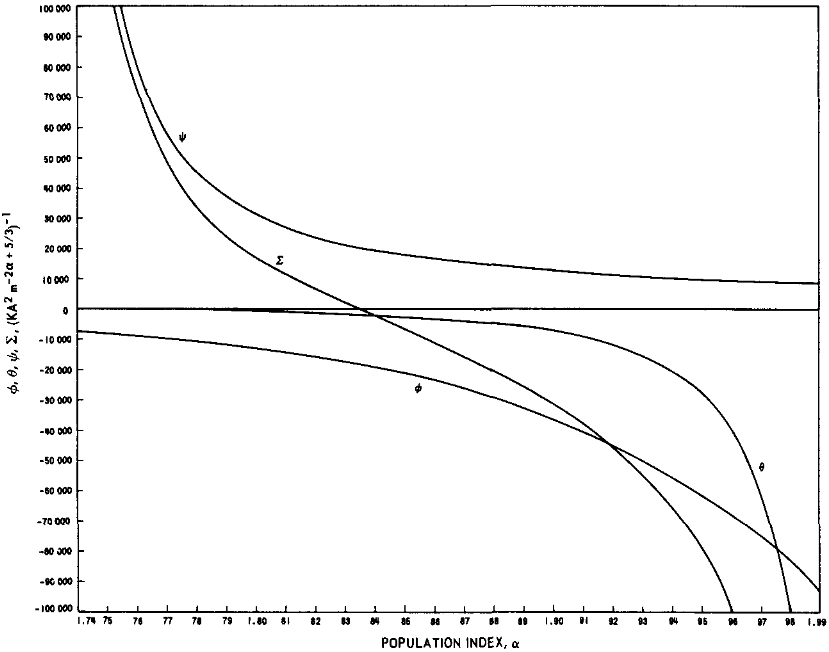


Figure 5.–Rate of change of the number of particles in units of $(KA^2m^{-2\alpha+5/3})^{-1}$ per unit time and unit mass range as a function of the population index α. η = 1.8; V = 20 km/s; and Γ = 2000.

The main result is that in the neighborhood of the limiting largest mass M_∞, the number density of asteroids is approximately

$$f(m,t) \approx A(t)m^{-11/6}\left[1-\left(\frac{m}{M_\infty}\right)^{1/6}\right]^{6(2-\eta)-1} \tag{59}$$

where $A(t)$ is given by equation (52). Equation (59) is valid only if a long period of time has elapsed since the creation of the asteroids and if any indication of the initial distribution has been lost.

Figure 6 is a plot of the cumulative number of the MDS asteroids from figure 1 together with the theoretical value

$$N(m,t) = \int_m^{M_\infty} f(M,t)\,dM \tag{60}$$

where f is given by equation (59). Plots of N for several different values for η are included; M_∞ is taken to be 1.86×10^{20} kg ($g = 4$) and A has been so chosen that $N(m, t)$ is made to coincide with observations at $g = 9$.

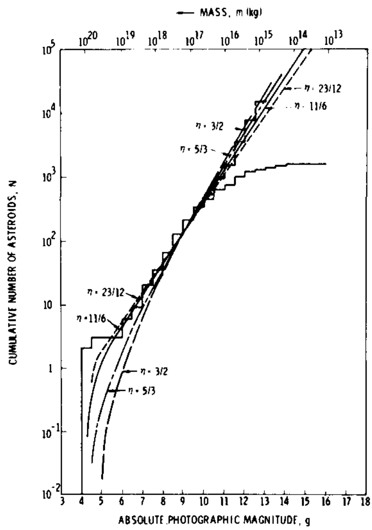


Figure 6.—Cumulative number of asteroids having an absolute photographic magnitude g or smaller (i.e., mass m or greater). Observed value = solid line histogram (MDS); probable value = dashed line histogram (MDS); theoretical value for different values of the fragmentation parameter, as indicated.

It can be seen from figure 6 that the higher values of η (11/6 and 23/12) provide the best agreement between theory and observation. The curve for $\eta = 5/3$ is still reasonably good, but for $\eta = 3/2$ the agreement with observation begins to deteriorate. For values of η less than 5/3, the number of large asteroids is underestimated by theory.

Erosion Rates

The rate $\dot{R}$ at which the effective radius of an asteroid decreases with time because of erosive collisions has been estimated by Dohnanyi (1969). The result is plotted in figure 7 and a systematic error of about a half order of magnitude may be present because of the uncertainties in the albedo alone. Because gravitational attraction has not been considered, $\dot{R}$ is an overestimate for large asteroids that retain much of the secondary ejecta produced during erosive cratering (Marcus, 1969; Hartmann, 1968).

The most conspicuous feature of the plot in figure 7 is that $\dot{R}$ is not a constant but a function of the mass of the asteroid undergoing erosion, because erosion is not due alone to collisions with minute particles but also to collisions with all masses up to m/Γ', where m is the mass of the target object being eroded. Because the population index α is here less than 2, the total mass eroded away from a given object by collisions with microparticles is much less than the mass eroded away by larger objects. Hence we expect that asteroidal surfaces are not smooth but are pock marked by relatively large craters.

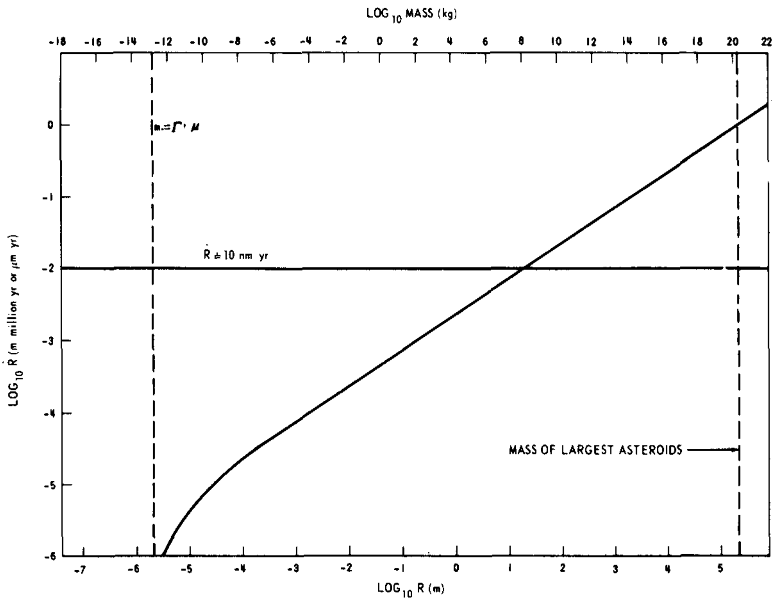


Figure 7.—Statistical rate of change because of erosion of the particle radius in meters per million years (or micrometers per year) as a function of particle mass (or particle radius). The horizontal line corresponds to a linear erosion rate of 10 nm/yr.

Values for $\dot{R}$ in figure 7 for small masses are not realistic because the influence of collisions with cometary meteoroids and spallation by cosmic rays has not been included. These processes have been estimated by Whipple (1967) to give rise to an erosion rate not exceeding about 10 nm/yr for stones. This upper limit is indicated in figure 7 as a horizontal line. Although Whipple's estimate applied to objects with orbits intersecting Earth's orbit, his upper limit is still meaningful for particles in the asteroidal belt if the erosive effect of cometary meteoroids in the asteroidal belt is taken to be comparable to, or lower than, the effect near Earth.

Lifetimes

Lifetimes of asteroids as a function of their masses and effective radii have been estimated in Dohnanyi (1969) and are plotted here in figure 8.

The lifetime with respect to catastrophic collisions is taken as the mean time between collisions of an object with mass m and other objects with masses greater than m/Γ':

$$\tau_{cc} = \left[K \int_{m/\Gamma'}^{m_\infty} AM^{-\alpha}(m^{1/3} + M^{1/3})^2 \, dM \right]^{-1} \tag{61}$$

An uncertainty due to albedo of about half an order of magnitude is present, in addition to other uncertainties.

The value of τ_{cc} for the largest asteroids is on the order of 10^9 yr (fig. 8). It can be seen, from figure 8, that the lifetime of the six largest asteroids with masses $m \geqslant 10^{19}$ kg is about 4×10^9 yr or longer and therefore these may have survived since the time of their creation. The other asteroids have shorter lifetimes τ_{cc} and may therefore be collisional fragments.

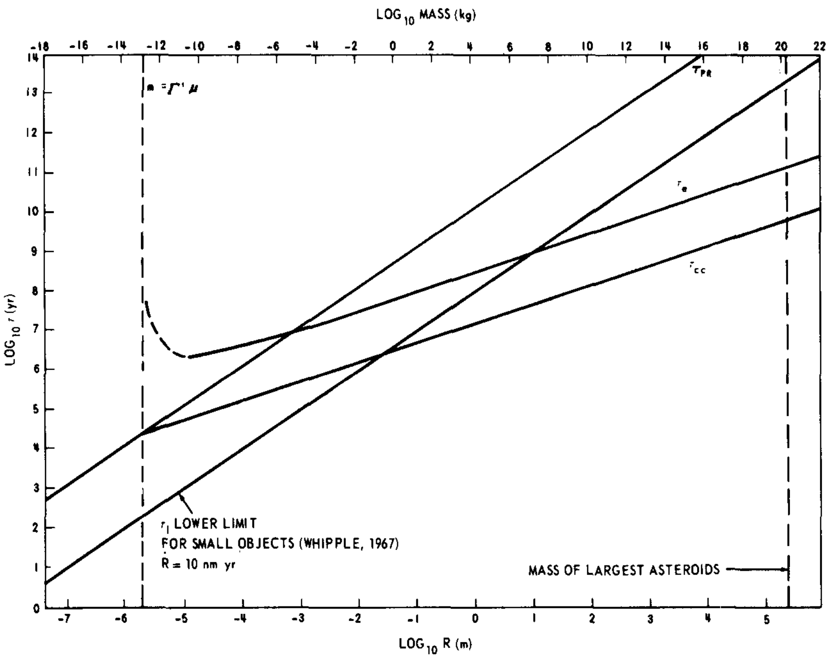


Figure 8.—Double logarithmic plot of particle lifetimes in years as a function of particle masses in kilograms (or particle radii in meters).

Using a more detailed spatial and velocity distribution, Wetherill (1967) has calculated collisional probabilities and obtained values comparable to but smaller than the values that a randomly distributed asteroid population (particle-in-a-box) would imply. He also estimated τ_{cc} for a 1 m diameter object for a number of assumed mass distributions. He considered (Wetherill, 1967, table 7) population indexes in the range $\alpha = 5/3$ to $\alpha = 1.8$; corresponding values of τ_{cc} were then computed for $\Gamma' = 10^2$, 10^3, and 10^4. Because these population indexes are lower than the steady-state value of $\alpha \approx 11/6$, the values in Dohnanyi (1969) for τ_{cc} are correspondingly shorter. The difference is about an order of magnitude in τ_{cc}.

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The lifetime with respect to erosion (i.e., erosive reduction of the particle mass) can be obtained when the expression for $\dot{m}$ (eq. (23)) is integrated. The result is, from Dohnanyi (1969),

$$\tau_e = \frac{(\Gamma')^{1/6}}{\Gamma KA} \left[\left(\sqrt{2} - 1\right) m^{1/6} + (\Gamma'\mu)^{1/6} \ln \frac{\sqrt{2}\, m^{1/6} - (\Gamma'\mu)^{1/6}}{m^{1/6} - (\Gamma'\mu)^{1/6}} \right] \qquad (62)$$

where the erosive lifetime τ_e of an object was taken to be the time required to erode it to one-half its initial radius and where $\alpha = 11/6$ was used. The logarithmic term is significant for masses approaching the value $\Gamma'\mu$, as can be seen from figure 8, τ_e becoming infinitely long for masses $m \leqslant \Gamma'\mu$. This happens because erosion stops for these small particles and all collisions they experience are catastrophic.

We also plot, in figure 8, the particle lifetimes with respect to the Poynting-Robertson effect (Robertson, 1936) τ_{PR} and the lower limit of the lifetime of small objects τ_l due to the influence of cometary meteoroids and cosmic rays estimated by Whipple (1967). Here the definition of τ_l is similar to that of τ_e; i.e., it is the time for erosion of an object to one-half its radius. τ_{PR} is the time required for an object to traverse radially one-half of the asteroidal belt, because of the Poynting-Robertson effect. It can be seen, from the figure, that catastrophic collisions dominate the lifetime of the particles greater than about 10^{-5} kg (or 1 mm in radius). Smaller particles may be subject to erosion by cometary particles to an extent that this mechanism dominates.

CONCLUSIONS

Using a stochastic model of asteroidal collisions, their mass distribution has been estimated. The results individually agree with the observed distribution of bright asteroids (MDS) and faint asteroids (PLS). After correction for completeness, the MDS and PLS distributions are similar in form but differ from each other by a numerical factor. Until this difficulty is resolved, some uncertainty remains in the precise form of the distribution of bright asteroids. Subject to this reservation, we may conclude that the mass distribution of most asteroids has reached (i.e., relaxed into) a stationary form that is independent of the original distribution and is a power-law function with index $\sim 11/6$ for faint asteroids.

The influence of catastrophic collisions dominates the evolution of the population; erosion plays a minor part. The influence of the Poynting-Robertson effect becomes dominant, however, for particles with masses of 10^{-10} g or smaller.

Whereas the particle lifetimes, erosion rates, collision probabilities, and other derived quantities of physical interest are expected to be self-consistent, uncertainties in the albedo of asteroids and in other parameters introduce an appreciable systematic error; the numerical values of these quantities should therefore be regarded as order of magnitude approximations.

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DISCUSSION

VAN HOUTEN: I wish to comment on figure 2 of Dohnanyi's paper. In this figure, the cumulative number of asteroids, as a function of absolute magnitude, is shown for MDS and PLS. In the overlapping part, the MDS values are approximately 10 times as large as the PLS values. This discrepancy could be traced to the following causes:

(1) The correction factors for incompleteness in table 15 of MDS in group III ($3.0 < a < 3.5$) are incorrect; the correct values are given in table D-I.

(2) Dohnanyi apparently used table 5 of PLS for the computation of his cumulative numbers of PLS asteroids. But to this table should be added the objects that were too bright for measurement in the iris photometer; these are five in total.

TABLE D-I.–*MDS Correction Factors*

g	$3.0 < a < 3.5$			$2.0 < a < 3.5$		
	N_{obs}	N_{min}	N_{max}	N_{obs}	N_{min}	N_{max}
9.75	39	39	40	114	114	115
10.25	64	65	69	150	151	155
10.75	93	121	143	180	208	231
11.25	78	137	169	184	246	288
11.75	77	220	320	168	334	453
12.25	45	250	450	150	418	661
12.75	17	255	472	138	568	913

After these corrections, and using the average of N_{max} and N_{min} for the MDS value as Dohnanyi did, the comparison between MDS and PLS becomes as given in table D-II.

The MDS values are still about twice as large as the PLS values, after these corrections. But the comparison is based on only 12 objects in the PLS. The statistical uncertainty of this number is such that maybe not too much importance should be attached to this difference.

TABLE D-II.—*Comparison Between MDS and PLS*

$g <$ –	Number of asteroids	
	MDS	PLS
11.25	961	505
11.75	1355	600
12.25	1895	912
12.75	2635	1704

DOHNANYI: If the five objects, too bright for photometry in PLS, are included, the resulting change in figure 2 is not significant for the purposes of my present study. A least-squares fit for asteroids with $g \geqslant 11$ gives a new population index $\alpha = 1.815$, which does not significantly differ from the previous value of $\alpha = 1.839$ that I have obtained earlier.

The maximum and minimum probable number of asteroids was estimated in MDS. No such quantitative estimate is given in PLS even though large correction factors affecting every asteroid observed in PLS were employed in extrapolating the relatively small sample of PLS to the rest of the asteroid belt. Large nonlinear corrections that have been applied are particularly visible when comparing figures 9 and 11 in PLS. The maximum in distribution of inclinations (fig. 9, PLS) is near 3° whereas the average inclination for cataloged asteroids is about 10° (Watson, 1956). The simple method based on several assumptions for estimating the completeness factors due to the inclination cutoff in PLS may be especially vulnerable to systematic error. The large correction factors employed for the small number of high-inclination orbits may be subject to noise because the number of these asteroids probably fluctuates in time. (See Nairn, 1966.)

Assuming that the PLS results are free from the type of difficulties that led van Houten to revise the MDS data, uncertainties in PLS data do still exist. Without a quantitative estimate of these uncertainties it might be arbitrary and misleading to connect the PLS results with those of the MDS without further comment at this time.

VAN HOUTEN: N_{max} and N_{min} in table 15 of MDS should not be regarded as the maximum and minimum probable number of asteroids. They are simply numbers derived from two different extrapolations of the log $N(m_0)$ relation. They should be compared with the PLS to see whether any of them approximates the real numbers. This comparison yields the results given in table D-III (cf. table 14 of MDS). It is seen that at $p_0 > 18$ even the use of equation (7) results in values that are too small. Fortunately the incompleteness corrections of the MDS were not extended to such faint magnitudes. Their lower limit is $p_0 = 17$, and here the PLS value is indeed between equations (6) and (7) (from which N_{min} and N_{max} were derived, respectively).

A check is possible on the correctness of the correction factors for the declination cutoff, used in the PLS. The corrected values should reproduce the same distribution of asteroids with respect to the ecliptic as was found in the MDS. (See fig. D-1.) It was shown in the PLS that one strip of the MDS yielded 1.90 times as many asteroids as a PLS strip for the same limiting magnitude; moreover it was found in the MDS that 10 percent of the

TABLE D-III.–*Comparison of MDS Extrapolations and PLS values*

p_0	Equation (6)	Equation (7)	PLS
17..........	1990	3700	3240
18..........	3240	8300	11500
19..........	4940	18600	28200

NOTE.–The mean photographic opposition magnitude is defined in the MDS.

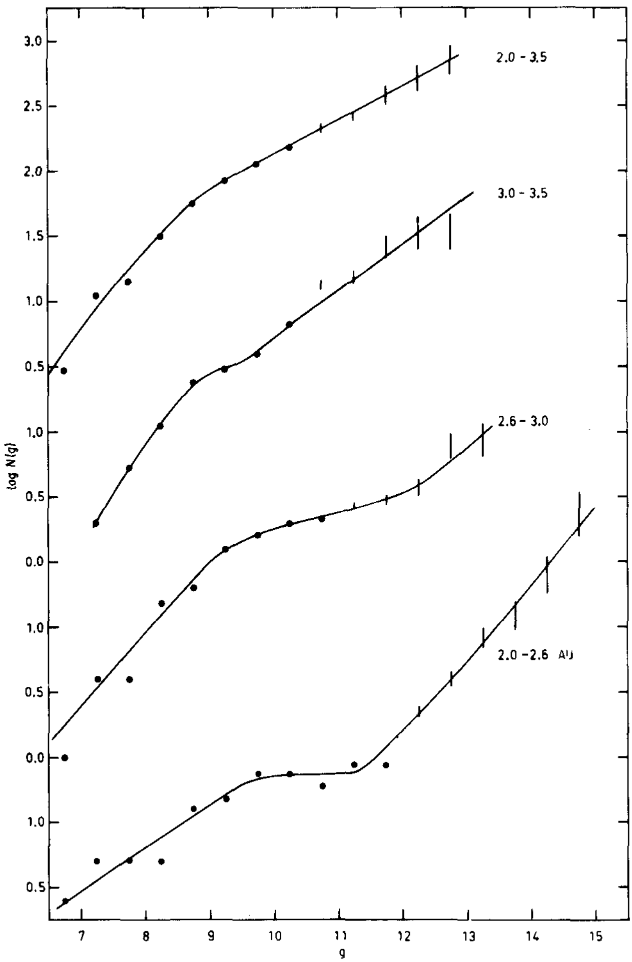

Figure D-1.–MDS frequency distribution of absolute magnitudes *g* for three distance zones and their sum. This was originally figure 5 of the MDS; revised by I. van Houten-Groeneveld.

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asteroids at a given opposition fall outside the MDS region. Therefore the comparison between MDS and PLS indicates that the correction factor for the inclination cutoff, integrated over the three distance groups, should be 1.10 times 1.90, or 2.09.

Using the approximation of circular orbits, the correction factors for the inclination cutoff for the three distance intervals separately were found to be 1.94, 2.42, and 2.45, respectively. The numbers of objects in the three distance groups are 52, 29, and 19 percent of the total (first-class orbits used only). This results in an integrated correction factor for the inclination cutoff of 2.18. The two numbers differ by only 4 percent. This shows that the correction factors for the inclination cutoff, as used in the PLS, are completely satisfactory.

The correction factor used to extend the PLS field to the whole sky depends on the size of the PLS field, which is accurately known. This correction factor cannot give rise to any inaccuracy.

In short, I do not see any reason to suppose that systematic errors are present in the PLS results. The accuracy of the PLS material is probably better than the MDS because the photometric material was larger, every asteroid being measured about six times, and because accurate reductions to absolute magnitude were available. Therefore I do not share Dohnanyi's reluctance to combine the two surveys. According to me, such a combination is completely justified.[2]

DISCUSSION REFERENCES

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[2]For additional information on the PLS, the reader is referred to van Houten's paper, p. 183.