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CLASSES OF OPERATOR-SMOOTH FUNCTIONS. I OPERATOR-LIPSCHITZ FUNCTIONS

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Abstract In this paper we study the spaces of operator-Lipschitz functions and the spaces of functions closed to them: commutator bounded. Apart from the standard operator norm on $B(H)$, we consider a rich variety of symmetric operator norms and spaces of operator-Lipschitz functions with respect to these norms. Our approach is aimed at the investigation of the interrelation and hierarchy of these spaces and of the intrinsic properties of operator-Lipschitz functions.

Keywords: operator Lipschitz functions; symmetrically normed ideals

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1. Introduction

This paper studies the spaces of operator-Lipschitz functions and some functional spaces close to them.

Let $(B(H), \|\cdot\|)$ be the algebra of all bounded operators on a Hilbert space H. Any bounded, Borel function g on a set $\alpha \subseteq \mathbb{R}$ defines, via the Spectral theorem, a map $A \to g(A)$ from the set of all self-adjoint operators with spectrum in α into $B(H)$. Various smoothness conditions when imposed on this map characterize interesting and important classes of operator-smooth functions. For example, the condition that the map $A \to g(A)$ is Gateaux differentiable defines the class of Gateaux operator-differentiable functions; the condition that the map is Lipschitzian defines the class of operator-Lipschitz functions. If, apart from the standard norm on $B(H)$, one considers other unitarily invariant operator norms and the classes of operator-differentiable and operator-Lipschitz functions with respect to these norms, then a rich variety of functional spaces arises. New interesting features of the theory also arise if functions are considered on subsets α of $\mathbb{C}$ and applied to normal operators with spectrum in α. Thus the operator theory suggests its own scale of smoothness of functions and defines naturally new functional spaces.

Much work has been done to relate the ‘operator’ smoothness of functions to the traditional ‘scalar’ smoothness conditions. Following the paper of Daletskii and Krein [**6**], there have been significant articles by Birman and Solomyak [**2**, **3**], Davies [**7**], Farforovskaya [**8**,**9**] and others in which the authors investigated the smoothness of operator-differentiable and operator-Lipschitz functions. This work culminated in the result of Peller [**20**] which placed the class of operator-differentiable functions on $[a, b]$ between two Besov spaces $B^1_{\infty 1}(a, b)$ and $B^1_{11}(a, b)$. Later Arazy *et al.* [**1**] constructed another functional space, wider than $B^1_{\infty 1}(a, b)$, contained in the class of all operator-differentiable functions.

Substantial and intriguing similarities between various properties of Gateaux operator-differentiable functions, of operator-Lipschitz functions and of the functions acting on differentiable operator algebras (on the domains of unbounded derivations of C^*-algebras first of all) point to a close and deep link between these classes of functions (see [**1**, **4**, **5**, **12**, **15**, **18**, **19**]). In the course of trying to understand these similarities, we have come to realize the necessity for a more systematic investigation of various spaces of ‘operator’ smooth functions, not only in the sense of their relation to the classical function spaces (in this respect the results of [**1**] and [**20**] could hardly be improved upon) but in the sense of their intrinsic properties, interrelation and hierarchy. In [**15**] we investigated properties of operator-differentiable functions and their link with operator-Lipschitz functions. In this paper we study operator-Lipschitz functions and functions close to them: commutator bounded. In [**16**] we continue to study various properties of operator-Lipschitz and commutator-bounded functions and consider also another class of operator-smooth functions: operator-stable functions.

We proceed now with a description of the results of the paper. Let $C(H)$ be the ideal of all compact operators and let $\mathcal{F}$ be the ideal of all finite-rank operators in $B(H)$. A two-sided ideal J of $B(H)$ is *symmetrically normed* (see [**10**]) if it is a Banach space with respect to a norm $\|\cdot\|_J$,

$$\|AXB\|_J \leqslant \|A\|\,\|X\|_J\|B\| \quad \text{for } A, B \in B(H) \text{ and } X \in J, \tag{1.1}$$

and $\|X\|_J = \|X\|$ for rank-one operators. It is a $*$-ideal and, by the Calkin theorem, $\mathcal{F} \subseteq J \subseteq C(H)$. If $X \in J$ and U is an isometry ($U^*U = \mathbf{1}$), then (see [**10**])

$$\|X\| \leqslant \|X\|_J = \|X^*\|_J \quad \text{and} \quad \|UX\|_J = \|XU^*\|_J = \|X\|_J. \tag{1.2}$$

We denote $B(H)$ by $\mathfrak{S}^b$.

Throughout the paper we denote by α a compact set in $\mathbb{C}$ and by $J_{\mathrm{nor}}(\alpha)$ the set of all normal operators in J with spectrum in α:

$$J_{\mathrm{nor}}(\alpha) = \{A \in J : A \text{ is normal, } \operatorname{Sp}(A) \subseteq \alpha\}.$$

A continuous function g on α *acts* on an symmetric norming (s.n.) ideal J if

$$g(A) \in J \quad \text{for all } A \in J_{\mathrm{nor}}(\alpha). \tag{1.3}$$

Definition 1.1. Let J be an s.n. ideal or $\mathfrak{S}^b$. A continuous function g on α is operator J-Lipschitzian on α if the map $A \to g(A)$ is Lipschitzian on $J_{\rm nor}(\alpha)$: there is $D > 0$ such that, for all $A, B \in J_{\rm nor}(\alpha)$,

$$g(A) - g(B) \in J \quad \text{and} \quad \|g(A) - g(B)\|_J \leqslant D\|A - B\|_J. \tag{1.4}$$

We denote by $J\text{-}\operatorname{Lip}(\alpha)$ the space of all J-Lipschitz functions on α.

More briefly, we will call such functions J-Lipschitzian and write *operator Lipschitzian* instead of $\mathfrak{S}^b$-Lipschitzian.

The following notion of commutator J-bounded functions has no 'scalar' analogues; its relation to the notion of J-Lipschitz functions is one of the central topics of this paper.

Definition 1.2. A continuous function g is called commutator J-bounded on α if there is $D > 0$ such that, for $A \in J_{\rm nor}(\alpha)$ and $X \in B(H)$,

$$g(A)X - Xg(A) \in J \quad \text{and} \quad \|g(A)X - Xg(A)\|_J \leqslant D\|AX - XA\|_J. \tag{1.5}$$

We denote by $J\text{-}\operatorname{CB}(\alpha)$ the space of all commutator J-bounded functions on α.

In Theorem 3.5 we establish that, for separable ideals and for their duals, a function is J-Lipschitzian if (1.4) holds for all finite-rank operators. Theorem 3.5 also shows that the space $J\text{-}\operatorname{Lip}(\alpha)$ always contains the space $J\text{-}\operatorname{CB}(\alpha)$. For some sets α (we call them *J-Fuglede*) these spaces coincide. For example, all compact sets in $\mathbb{R}$ are J-Fuglede for all ideals J (for Schatten ideals this was observed earlier by Davies in [**7**]). The possibility of reducing the study of J-Lipschitz functions to the study of commutator J-bounded functions is very important, since it enables us to use the powerful techniques of the interpolation theory to compare the spaces $J\text{-}\operatorname{Lip}(\alpha)$ for different ideals.

Johnson and Williams [**11**] proved that commutator $\mathfrak{S}^b$-bounded functions are differentiable. Farforovskaya established in [**8**] that the continuous differentiability is not sufficient for a function to be commutator $\mathfrak{S}^b$-bounded. Williams in [**24**] asked whether commutator $\mathfrak{S}^b$-bounded functions are always continuously differentiable. The authors gave a negative answer to this question in [**14**] and constructed a commutator $\mathfrak{S}^b$-bounded function on the unit disc of $\mathbb{C}$ whose derivative is discontinuous at $z = 1$. In Theorem 3.8 below we show that on each segment $[-a, b]$, $a, b > 0$, in $\mathbb{R}$ there are commutator $\mathfrak{S}^b$-bounded functions with discontinuous derivative at $t = 0$. This, in particular, implies that the space of operator-differentiable functions on $[-a, b]$ (which are all continuously differentiable) is distinct from the space of operator-Lipschitz functions. For separable ideals J, however, the spaces of operator J-Lipschitz and of Gateaux J-differentiable functions do coincide [**15**].

In § 4 we study J-Fuglede sets. It appears that α is J-Fuglede if and only if the function $h(z) = \bar{z}$ is commutator bounded, that is, if there is $D > 0$ such that,

$$\|[A^*, X]\|_J \leqslant D\|[A, X]\|_J \quad \text{for all } A \in J_{\rm nor}(\alpha) \text{ and } X \in \mathfrak{S}^b.$$

We show that an $\mathfrak{S}^b$-Fuglede set in $\mathbb{C}$ has empty interior. Moreover, it is 'smooth': it must have a 'tangent' at each non-isolated point. As a sufficient condition we obtain that compact subsets of twice-differentiable curves are J-Fuglede for all J.

In § 5 we use Hadamard multipliers to compare the spaces J- $\mathrm{CB}(\alpha)$ for various ideals J. We prove that the spaces $\mathfrak{S}^1$- $\mathrm{CB}(\alpha)$, $\mathfrak{S}^\infty$- $\mathrm{CB}(\alpha)$ and $\mathfrak{S}^b$- $\mathrm{CB}(\alpha)$ coincide and that the space $\mathfrak{S}^2$- $\mathrm{CB}(\alpha)$ consists of all functions on α that are Lipschitz in the usual sense. We use these results in [**16**] to establish that $\mathfrak{S}^p$- $\mathrm{CB}(\alpha) \subseteq \mathfrak{S}^q$- $\mathrm{CB}(\alpha)$, for

$$\min\left(p, \frac{p}{p-1}\right) \leqslant q \leqslant \max\left(p, \frac{p}{p-1}\right),$$

and to extend inequalities (1.4) and (1.5) to all normal $A, B \in B(H)$, thereby generalizing the result of Kittaneh [**17**], who considered the case $J = \mathfrak{S}^2$.

2. Preliminaries

We will briefly discuss some properties of s.n. ideals (for full discussion see [**10**]). Let c_0 be the space of all sequences of real numbers converging to 0, let $\hat{c}$ be the subspace of c_0 of sequences with a finite number of non-zero elements, and let $\varPhi$ be the set of all symmetric norming (s.n.) functions on $\hat{c}$. Given $\xi = \{\xi_i\} \in c_0$, set $\xi^{(n)} = \{\xi_1, \dots, \xi_n, 0, \dots\}$. Then all $\xi^{(n)} \in \hat{c}$. For each $\phi \in \varPhi$, the sequence $\phi(\xi^{(n)})$ does not decrease. Set $\phi(\xi) = \lim \phi(\xi^{(n)})$ and $c^\phi = \{\xi \in c_0 : \phi(\xi) < \infty\}$.

For $A \in C(H)$, let $s(A) = \{s_i(A)\}$ be the non-increasing sequence of all eigenvalues of $(A^*A)^{1/2}$ repeated according to multiplicity. For $\phi \in \varPhi$, the set $J = J^\phi = \{A \in C(H) : s(A) \in c^\phi\}$ with norm $\|A\|_J = \phi(s(A))$ is an s.n. ideal. The closure J_0^ϕ of $\mathcal{F}$ in $\|\cdot\|_J$ is a separable s.n. ideal and $J_0^\phi \subseteq J^\phi$. An s.n. ideal is separable if and only if it coincides with some J_0^ϕ.

For many s.n. functions ϕ, the ideals J^ϕ and J_0^ϕ coincide. An important class of such functions consists of the functions

$$\phi_p(\xi) = \left(\sum |\xi_i|^p\right)^{1/p}, \quad \text{for } 1 \leqslant p < \infty, \quad \text{and} \quad \phi_\infty(\xi) = \sup |\xi_i|.$$

The corresponding ideals $\mathfrak{S}^p$ with norms $\|\cdot\|_p$ are Schatten ideals and $\mathfrak{S}^\infty = C(H)$.

For $\phi \in \varPhi$, there is the adjoint function ϕ^* such that the ideal J^{ϕ^*} is isomorphic to the dual space of J_0^ϕ: any bounded functional on J_0^ϕ has the form

$$F(X) = \mathrm{Tr}(XT) = \mathrm{Tr}(TX), \quad \text{where } T \in J^{\phi^*} \text{ and } \|F\| = \|T\|_{J^{\phi^*}}. \tag{2.1}$$

For $p \in [1, \infty]$, let p' be the conjugate exponent:

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{if } 1 < p < \infty; \qquad p' = 1 \quad \text{if } p = \infty; \qquad p' = b \quad \text{if } p = 1. \tag{2.2}$$

Then $\phi_{p'} = (\phi_p)^*$, so $\mathfrak{S}^{p'}$ is isometrically isomorphic to the dual space of $\mathfrak{S}^p$.

Remark 2.1. Our definitions of J-Lipschitz and commutator J-bounded functions depend (at least formally) on the choice of the Hilbert space. Let us clarify this dependence.

(i) Any isometry V between Hilbert spaces H and K establishes a bijection between s.n. ideals of $B(H)$ and $B(K)$. This bijection does not depend on V, so we may use the same symbol to denote s.n. ideals in different Hilbert spaces. Since the sequences $s(A)$ and $s(VAV^*)$ always coincide, the fact that J is J^ϕ or J_0^ϕ does not depend on the underlying Hilbert space. We write $J(H)$ if we need to underline that J is an s.n. ideal of $B(H)$.

Moreover, for an operator $T \in B(H,K)$ we write $T \in J$ when $(T^*T)^{1/2} \in J$.

(ii) For a non-separable Hilbert space H, the map $A \to g(A)$ is Lipschitzian if and only if its restriction to operators acting on any (some) separable subspace of H is Lipschitzian. Thus we may restrict our study to separable Hilbert spaces.

(iii) For finite-dimensional H, every ideal J coincides with $B(H)$ but has a different norm, so $J(H)$ denotes the algebra $B(H)$ endowed with $\|\cdot\|_J$. There is $\phi \in \Phi$ such that the norms $\|\cdot\|_J$ and $\|\cdot\|_{J^\phi}$ coincide.

(iv) When we say that a certain statement holds for an ideal J we mean that it holds for all ideals $J(K)$ with the same norm $\|\cdot\|_J$ in all infinite-dimensional and finite-dimensional spaces K (see (i) and (iii)).

If $0 \notin \alpha$ and $J \neq B(H)$, then the statement '$g \in J\text{-}\mathrm{Lip}(\alpha)$' means that (1.4) holds for normal operators acting on finite-dimensional spaces with spectra in α.

(v) Let P be the projection on a subspace L of H. Then $J_L = \{A \in J : A = PAP\}$ is a Banach $*$-subalgebra of J. The map $J_L \ni A \to A|_L$ is an isometric isomorphism from J_L onto $J(L)$.

(vi) Let K be the sum of n copies of H. Any $A \in J(K)$ can be represented as a block matrix $A = (A_{ij})$ with all entries from J. If $n < \infty$, $J(K)$ consists of all such matrices. If all $A_{ij} = 0$, apart from some A_{km}, then $\|A\|_{J(K)} = \|A_{km}\|_J$.

3. Operator Lipschitz functions

Recall that a function g on $\alpha \subset \mathbb{C}$ is Lipschitzian at $s \in \alpha$ if there is $D > 0$ such that $|g(t) - g(s)| \leqslant D|t - s|$ for $t \in \alpha$; it is Lipschitzian on α if this inequality holds for all $t, s \in \alpha$.

An operator A is diagonalizable if there is an orthonormal basis $\{e_n\}$ in H such that $Ae_n = \lambda_n e_n$ for all n. If the basis is understood, we say that A is diagonal and write $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n, \dots)$.

Lemma 3.1. *Let g be a continuous function on $\alpha \subset \mathbb{C}$. If $g(0) = 0$ and g is Lipschitzian at $t = 0$, then g acts on every s.n. ideal J (see (1.3)).*

Proof. If $A \in J_{\mathrm{nor}}(\alpha)$, then $A = \mathrm{diag}(t_1, \dots, t_n, \dots)$, where $t_n \in \alpha$, $t_n \to 0$. Since g is Lipschitzian at 0, $g(t) = th(t)$ and h is a bounded function. Hence $g(A) = Ah(A)$ and $h(A) = \mathrm{diag}(h(t_1), \dots, h(t_n), \dots) \in B(H)$, so $g(A) \in J$. □

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If g is Lipschitzian at $t=0$ and $g(0)\neq 0$, the function $f(t)=g(t)-g(0)$ acts on J:

$$f(A)=g(A)-g(0)\mathbf{1}\in J \quad \text{for } A\in J_{\mathrm{nor}}(\alpha). \tag{3.1}$$

Every J-Lipschitz function g, where J is an s.n. ideal or $\mathfrak{S}^b$, is Lipschitzian in the usual sense. Indeed, if Q is a rank-one projection, then $\|Q\|_J=1$ and, for $t,s\in\alpha$,

$$|g(t)-g(s)|=\|(g(t)-g(s))Q\|_J=\|g(tQ)-g(sQ)\|_J\leqslant D\|tQ-sQ\|_J=D|t-s|.$$

Example 3.2.

(i) Let g be a continuous function on $\mathbb{C}$ with Fourier transform $\hat{g}(u)$. If

$$\int_{\mathbb{C}}|\hat{g}(u)u|\,\mathrm{d}u<\infty, \tag{3.2}$$

where $u=t+\mathrm{i}s$ and $\mathrm{d}u=\mathrm{d}t\,\mathrm{d}s$, then g is J-Lipschitzian on any compact in $\mathbb{C}$ for every s.n. ideal J and for $J=\mathfrak{S}^b$. In particular, if g has continuous third-order partial derivatives, then g is J-Lipschitzian.

(ii) Let g be a continuous function on $\mathbb{R}$ with Fourier transform $\hat{g}$. If

$$\int_{\mathbb{R}}|\hat{g}(t)t|\,\mathrm{d}t<\infty, \tag{3.3}$$

then g is J-Lipschitzian on any compact in $\mathbb{R}$ for every s.n. ideal J and for $J=\mathfrak{S}^b$. In particular, if g has a continuous second derivative, it is J-Lipschitzian.

Proof. Let A be a normal operator with $\mathrm{Sp}(A)\subseteq\alpha\subseteq\mathrm{supp}(g)$. Then $A=A_1+\mathrm{i}A_2$, where A_1, A_2 are self-adjoint, commuting operators and

$$g(A)=\frac{1}{(2\pi)^2}\int_{\mathbb{C}}\hat{g}(u)\mathrm{e}^{-\mathrm{i}(tA_1+sA_2)}\,\mathrm{d}u.$$

For self-adjoint $R,T\in B(H)$ and $s\in\mathbb{R}$, we have (see [**7**]) that

$$\begin{aligned}\|\mathrm{e}^{\mathrm{i}sR}-\mathrm{e}^{\mathrm{i}sT}\|_J&=\bigg\|\int_0^s \mathrm{e}^{\mathrm{i}\tau R}(R-T)\mathrm{e}^{\mathrm{i}(s-\tau)T}\,\mathrm{d}\tau\bigg\|_J\\&\leqslant\int_0^s\|\mathrm{e}^{\mathrm{i}\tau R}\|\,\|R-T\|_J\|\mathrm{e}^{\mathrm{i}(s-\tau)T}\|\,\mathrm{d}\tau\\&=\|R-T\|_J|s|.\end{aligned}$$

Therefore, by (1.1), for normal $B=B_1+\mathrm{i}B_2$ with $\mathrm{Sp}(B)\subseteq\alpha$ and $A-B\in J$,

$$\begin{aligned}&\|\mathrm{e}^{-\mathrm{i}(tA_1+sA_2)}-\mathrm{e}^{-\mathrm{i}(tB_1+sB_2)}\|_J\\&\qquad\leqslant\|\mathrm{e}^{-\mathrm{i}tA_1}-\mathrm{e}^{-\mathrm{i}tB_1}\|_J\|\mathrm{e}^{-\mathrm{i}sA_2}\|+\|\mathrm{e}^{-\mathrm{i}tB_1}\|\,\|\mathrm{e}^{-\mathrm{i}sA_2}-\mathrm{e}^{-\mathrm{i}sB_2}\|_J\\&\qquad\leqslant\|A_1-B_1\|_J|t|+\|A_2-B_2\|_J|s|\\&\qquad\leqslant 2|u|\|A-B\|_J.\end{aligned}$$

Hence

$$\|g(A)-g(B)\|_J \leqslant \frac{1}{(2\pi)^2}\int_{\mathbb{C}} |\hat{g}(u)|\,\|\mathrm{e}^{-\mathrm{i}(tA_1+sA_2)}-\mathrm{e}^{-\mathrm{i}(tB_1+sB_2)}\|_J \,\mathrm{d}u$$
$$\leqslant \frac{2}{(2\pi)^2}\|A-B\|_J \int_{\mathbb{C}} |\hat{g}(u)u|\,\mathrm{d}u.$$

Part (i) is proved. Similarly, one can prove part (ii). □

In fact, we proved a stronger result: if g satisfies (3.2), then condition (1.4) holds for all normal A, B with spectra in α (not necessarily from $J_{\mathrm{nor}}(\alpha)$) such that $A-B \in J$. We will show in [**16**] that, for a wide class of ideals J, all J-Lipschitz functions have this property.

We write $X_n \xrightarrow{\text{sot}} X$ if operators X_n in $B(H)$ converge to X in the strong operator topology. To proceed further we need two auxiliary results. The first follows from Theorem III.5.1 of [**10**].

Lemma 3.3. *Let J be J^ϕ or $\mathfrak{S}^b$. If $B(H) \ni X_n \xrightarrow{\text{sot}} \mathbf{1}$ and $\|X_n\| \leqslant 1$, then, for each $A \in J$,*

$$\|A\|_J = \overline{\lim}\,\|X_nAX_n\|_J = \overline{\lim}\,\|AX_n\|_J = \overline{\lim}\,\|X_nA\|_J.$$

Recall that $\mathcal{F}$ denotes the set of all finite-rank operators. Let

$$\mathcal{F}_{\mathrm{nor}}(\alpha) = \{A \in \mathcal{F} : A \text{ is normal and } \mathrm{Sp}(A) \subseteq \alpha\}.$$

We write $[A,B]$ for the commutator $AB-BA$.

Proposition 3.4. *Let J be an s.n. ideal or $\mathfrak{S}^b$ and let g be a continuous function on α. Suppose that there is $D>0$ such that, for $A \in \mathcal{F}_{\mathrm{nor}}(\alpha)$ and $X \in \mathcal{F}$,*

$$\|[g(A),X]\|_J \leqslant D\|[A,X]\|_J. \tag{3.4}$$

(i) *If J is J^ϕ or $\mathfrak{S}^b$, then $[g(A),X] \in J$ and (3.4) holds for all $X \in B(H)$ and all diagonalizable $A \in B(H)$ with $\mathrm{Sp}(A) \subseteq \alpha$ such that $[A,X] \in J$.*

(ii) *Let $J = J_0^\phi$. Then $[g(A),X] \in J$ and (3.4) holds for all $X \in B(H)$ and $A \in J_{\mathrm{nor}}(\alpha)$.*

(iii) *If J is J_0^ϕ, or J^ϕ or $\mathfrak{S}^b$, then (3.4) holds for all $X \in J$ and all normal A with $\mathrm{Sp}(A) \subseteq \alpha$.*

Proof. Let A be a diagonalizable operator with $\mathrm{Sp}(A) \subseteq \alpha$. Let finite-dimensional projections $\{Q_n\}$ commute with A and $Q_n \xrightarrow{\text{sot}} \mathbf{1}$. Then $Q_nA \in \mathcal{F}_{\mathrm{nor}}(\alpha)$. For $X \in B(H)$, $Q_nXQ_n \in \mathcal{F}$ and, by (3.4),

$$\|g(Q_nA)Q_nXQ_n - Q_nXQ_ng(Q_nA)\|_J \leqslant D\|(Q_nA)Q_nXQ_n - Q_nXQ_n(Q_nA)\|_J.$$

The projections Q_n commute with $g(A)$ and $g(Q_nA) = Q_ng(A) + g(0)(\mathbf{1}-Q_n)$. Hence

$$\|Q_n[g(A),X]Q_n\|_J \leqslant D\|Q_n[A,X]Q_n\|_J.$$

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Let $[A, X] \in J$. By Lemma 3.3, $\|[A, X]\|_J = \overline{\lim}\, \|Q_n[A, X]Q_n\|_J$. Therefore,

$$\overline{\lim}\, \|Q_n[g(A), X]Q_n\|_J \leqslant D\|[A, X]\|_J. \tag{3.5}$$

Let $J = J^\phi \neq \mathfrak{S}^\infty$. The operators $Q_n[g(A), X]Q_n$ weakly converge to $[g(A), X]$. Hence it follows from Theorem III.5.1 of [**10**] and (3.5) that $[g(A), X] \in J$ and

$$\|[g(A), X]\|_J \leqslant \overline{\lim}\, \|Q_n[g(A), X]Q_n\|_J \leqslant D\|[A, X]\|_J.$$

If J is $\mathfrak{S}^\infty$ or $\mathfrak{S}^b$, we obtain from (3.5) and Lemma 3.3 that

$$\|[g(A), X]\| = \overline{\lim}\, \|Q_n[g(A), X]Q_n\| \leqslant D\|[A, X]\|.$$

Moreover, if $[A, X] \in \mathfrak{S}^\infty$, then $[P(A), X] \in \mathfrak{S}^\infty$ for all polynomials P. Therefore, $[g(A), X] \in \mathfrak{S}^\infty$. Part (i) is proved.

Let $\{e_n\}$ be a basis in H and let Y be an operator on H such that $Ye_2 = e_1$ and $Ye_n = 0$ for $n \neq 2$. For $t, s \in \alpha$, let $A(t, s) = \operatorname{diag}(t, s, 0, \dots)$. By (3.4),

$$|g(t) - g(s)| = \|[g(A(t, s)), Y]\|_J \leqslant D\|[A(t, s), Y]\|_J = D|t - s|.$$

Hence g is Lipschitzian on α. Let $J = J_0^\phi$ and $A \in J_{\mathrm{nor}}(\alpha)$. By (3.1), $[g(A), X] \in J$ for $X \in B(H)$. Part (ii) follows from (i), since

$$\|[g(A), X]\|_J = \|[g(A), X]\|_{J^\phi} \leqslant \|[A, X]\|_{J^\phi} = \|[A, X]\|_J.$$

Let A be normal. There are diagonal operators A_n commuting with A such that $\operatorname{Sp}(A_n) \subseteq \alpha$ and $\|A - A_n\| \to 0$, as $n \to \infty$. Let J be J_0^ϕ, or J^ϕ or $\mathfrak{S}^b$, and let $X \in J$. By (i), for all n,

$$\|[g(A_n), X]\|_J = \|[g(A_n), X]\|_{J^\phi} \leqslant D\|[A_n, X]\|_{J^\phi} = D\|[A_n, X]\|_J.$$

Since $\|[A, X] - [A_n, X]\|_J \leqslant 2\|A - A_n\|\,\|X\|_J$, we have $\|[A_n, X]\|_J \to \|[A, X]\|_J$. Since g is continuous, $\|g(A) - g(A_n)\| \to 0$. Hence $\|[g(A_n), X]\|_J \to \|[g(A), X]\|_J$. Thus

$$\|[g(A), X]\|_J = \lim \|[g(A_n), X]\|_J \leqslant D \lim \|[A_n, X]\|_J = D\|[A, X]\|_J,$$

which completes the proof. □

The theorem below establishes that condition (3.4) can be considered as a linearization of (1.4) and that a function is J^ϕ-Lipschitzian if it satisfies (1.4) for finite-rank operators.

Theorem 3.5. *Let J be an s.n. ideal or $\mathfrak{S}^b$ and let g be a continuous function on $\alpha \subset \mathbb{C}$.*

(I) *The following conditions are equivalent:*

- (i) *g is J-Lipschitzian on α;*
- (ii) *there is $D > 0$ such that (3.4) holds for all $A \in J_{\mathrm{nor}}(\alpha)$ and all $X = X^* \in B(H)$.*

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(II) *The following conditions are equivalent:*

(i) *there is $D > 0$ such that (1.4) holds for all $A, B \in \mathcal{F}_{\mathrm{nor}}(\alpha)$;*

(ii) *there is $D > 0$ such that (3.4) holds for all $A \in \mathcal{F}_{\mathrm{nor}}(\alpha)$ and all $X = X^* \in \mathcal{F}$.*

(III) *If J is J_0^{ϕ} or J^{ϕ} or $\mathfrak{S}^b$, then* (I) *and* (II) *are equivalent.*

Proof. (I) (i) $\Rightarrow$ (I) (ii). Let $A \in J_{\mathrm{nor}}(\alpha)$. For unitary U, $\mathrm{Sp}(UAU^*) = \mathrm{Sp}(A) \subseteq \alpha$. Since $g(UAU^*) = Ug(A)U^*$, we obtain from (1.2) and (1.4) that

$$\begin{aligned}\|[g(A), U]\|_J &= \|[g(A), U]U^*\|_J = \|g(A) - Ug(A)U^*\|_J \\ &= \|g(A) - g(UAU^*)\|_J \leqslant D\|A - UAU^*\|_J = D\|[A, U]U^*\|_J = D\|[A, U]\|_J.\end{aligned}$$

Let X be self-adjoint. Then $\mathrm{e}^{\mathrm{i}tX}$ is unitary and therefore

$$\begin{aligned}\|g(A)\mathrm{e}^{\mathrm{i}tX} - \mathrm{e}^{\mathrm{i}tX}g(A)\|_J &= \|\mathrm{i}t[g(A), X] + O(t^2)\|_J \\ &\leqslant D\|A\mathrm{e}^{\mathrm{i}tX} - \mathrm{e}^{\mathrm{i}tX}A\|_J = D\|\mathrm{i}t[A, X] + O(t^2)\|_J,\end{aligned}$$

as $t \to 0$. Dividing by t, we obtain that $\|[g(A), X]\|_J \leqslant D\|[A, X]\|_J$.

(I) (ii) $\Rightarrow$ (I) (i). Let $A, B \in J_{\mathrm{nor}}(\alpha)$. The operator

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

on $H \oplus H$ also belongs to $J_{\mathrm{nor}}(\alpha)$ (see Remark 2.1). The operator

$$X = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

on $H \oplus H$ is self-adjoint and unitary. Hence $\|[g(L), X]\|_J \leqslant D\|[L, X]\|_J$. By (1.2),

$$\|[g(L), X]X\|_J = \|[g(L), X]\|_J \leqslant D\|[L, X]\|_J = D\|[L, X]X\|_J.$$

Since

$$g(L) = \begin{pmatrix} g(A) & 0 \\ 0 & g(B) \end{pmatrix},$$

we have

$$[g(L), X]X = \begin{pmatrix} g(A) - g(B) & 0 \\ 0 & g(B) - g(A) \end{pmatrix} \quad \text{and} \quad [L, X]X = \begin{pmatrix} A - B & 0 \\ 0 & B - A \end{pmatrix}.$$

Therefore, it follows from Remark 2.1 that

$$\|g(A) - g(B)\|_J \leqslant \|[g(L), X]X\|_J \leqslant D\|[L, X]X\|_J \leqslant 2D\|A - B\|_J.$$

The equivalence of (II) (i) and (II) (ii) can be proved similarly. Clearly, (I) implies (II). If J is J_0^{ϕ}, or J^{ϕ} or $\mathfrak{S}^b$, then, repeating the proof of parts (i) and (ii) of Proposition 3.6 for self-adjoint operators, we obtain that (II) implies (I). $\square$

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Corollary 3.6. *Let J be an s.n. ideal or $\mathfrak{S}^b$. Then*

(i) $J\text{-}\mathrm{CB}(\alpha) \subseteq J\text{-}\mathrm{Lip}(\alpha)$ *for each* $\alpha \subset \mathbb{C}$*;*

(ii) $J\text{-}\mathrm{CB}(\alpha) = J\text{-}\mathrm{Lip}(\alpha)$ *for each* $\alpha \subset \mathbb{R}$.

Proof. Part (i) follows from Theorem 3.5 (I).

Let $g \in J\text{-}\mathrm{Lip}(\alpha)$. By Theorem 3.5, there exists $D > 0$ such that $\|[g(A), X]\|_J \leqslant D\|[A, X]\|_J$ for $A \in J_{\mathrm{nor}}(\alpha)$ and $X = X^* \in B(H)$.

Let $X = Y + \mathrm{i}Z \in B(H)$, where $Y = \frac{1}{2}(X + X^*)$, $Z = -\frac{1}{2}\mathrm{i}(X - X^*)$ are self-adjoint. Then

$$\begin{aligned} \|[g(A), X]\|_J &\leqslant \|[g(A), Y]\|_J + \|[g(A), Z]\|_J \\ &\leqslant D(\|[A, Y]\|_J + \|[A, Z]\|_J) \\ &\leqslant D(\|[A, X]\|_J + \|[A, X^*]\|_J). \end{aligned} \tag{3.6}$$

Since $\alpha \subset \mathbb{R}$, A is self-adjoint, so $\|[A, X^*]\|_J = \|-[A, X]^*\|_J = \|[A, X]\|_J$. Hence

$$\|[g(A), X]\|_J \leqslant 2D\|[A, X]\|_J.$$

Hence $g \in J\text{-}\mathrm{CB}(\alpha)$, so $J\text{-}\mathrm{CB}(\alpha) = J\text{-}\mathrm{Lip}(\alpha)$. □

For $J = \mathfrak{S}^p$, $1 < p < \infty$, the above result was noticed by Davies in [**7**]. He also proved that the spaces of $\mathfrak{S}^p$-Lipschitz functions on $\mathbb{R}$ contain non-differentiable functions, for example, $g(t) = |t|$.

Johnson and Williams [**11**] showed that the functions on $\alpha \subseteq \mathbb{C}$, satisfying condition (3.4) for $J = \mathfrak{S}^b$, all $A \in \mathfrak{S}^b_{\mathrm{nor}}(\alpha)$ and $X \in B(H)$, are differentiable on α. Combining this with Corollary 3.6 yields the following corollary.

Corollary 3.7. *Any $\mathfrak{S}^b$-Lipschitz function on $\alpha \subset \mathbb{R}$ is differentiable on α.*

We will now show that there are $\mathfrak{S}^b$-Lipschitz functions on compacts in $\mathbb{R}$ with discontinuous derivative.

Theorem 3.8. *On each infinite compact subset of $\mathbb{R}$ there are $\mathfrak{S}^b$-Lipschitz functions which are not continuously differentiable.*

Proof. To prove the theorem it suffices to show that the function

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(t) = t^2 \sin\left(\frac{1}{t}\right), \quad \text{for } t \neq 0,$$

is $\mathfrak{S}^b$-Lipschitzian on all segments $[-n, n]$ and, hence, on all infinite compacts in $\mathbb{R}$.

Let $A = A^*$. First assume that $\mathrm{Sp}(A) \subseteq [-n, -\varepsilon] \cup [\varepsilon, n]$ for some $\varepsilon > 0$. Then $\|A\| \leqslant n$, $\|\sin(A^{-1})\| \leqslant 1$ and $\varphi(A) = A\sin(A^{-1})A$. For $X \in B(H)$,

$$\begin{aligned} \|[\varphi(A), X]\| &= \|[A, X]\sin(A^{-1})A + A[\sin(A^{-1}), X]A + A\sin(A^{-1})[A, X]\| \\ &\leqslant \|[A, X]\|\,\|\sin(A^{-1})\|\,\|A\| + \|A[\sin(A^{-1}), X]A\| + \|A\|\,\|\sin(A^{-1})\|\,\|[A, X]\| \\ &\leqslant 2n\|[A, X]\| + \|A[\sin(A^{-1}), X]A\|. \end{aligned}$$

Since $\sin(A^{-1}) = (1/2\mathrm{i})(\exp(\mathrm{i}A^{-1}) - \exp(-\mathrm{i}A^{-1}))$,

$$\|A[\sin(A^{-1}), X]A\| \leqslant \tfrac{1}{2}\|A[\exp(\mathrm{i}A^{-1}), X]A\| + \tfrac{1}{2}\|A[\exp(-\mathrm{i}A^{-1}), X]A\|.$$

It follows from Lemma 2 of [**21**] that, for each $B \in B(H)$,

$$[\exp(B), X] = \int_0^1 \exp(tB)[B, X]\exp((1-t)B)\,\mathrm{d}t.$$

Therefore, since $\exp(t\mathrm{i}A)$ and $\exp((1-t)\mathrm{i}A)$ are unitary operators, we have

$$\begin{aligned}\|A[\exp(\mathrm{i}A^{-1}), X]A\| &= \left\|\int_0^1 \exp(\mathrm{i}tA^{-1})A[\mathrm{i}A^{-1}, X]A\exp(\mathrm{i}(1-t)A^{-1})\,\mathrm{d}t\right\| \\ &\leqslant \int_0^1 \|\exp(\mathrm{i}tA^{-1})\|\,\|A[\mathrm{i}A^{-1}, X]A\|\,\|\exp(\mathrm{i}(1-t)A^{-1})\|\,\mathrm{d}t \\ &= \|A[A^{-1}, X]A\| = \|[A, X]\|.\end{aligned}$$

Similarly, $\|A[\exp(-\mathrm{i}A^{-1}), X]A\| = \|[A, X]\|$, so that

$$\|[\varphi(A), X]\| \leqslant (2n+1)\|[A, X]\|. \tag{3.7}$$

To consider self-adjoint A with $0 \in \mathrm{Sp}(A)$ we need the following lemma.

Lemma 3.9. *Let g be a continuous function on α in $\mathbb{R}$ and let λ be a non-isolated point in α. Assume that for all self-adjoint A with $\mathrm{Sp}(A) \subseteq \alpha$ and $\mathrm{Ker}(A - \lambda\mathbf{1}) = \{0\}$,*

$$\|[g(A), X]\| \leqslant D\|[A, X]\| \quad \text{for } X \in B(H). \tag{3.8}$$

Then (3.8) also holds for all self-adjoint A with $\mathrm{Sp}(A) \subseteq \alpha$.

Proof. Let P be the projection on $M = \mathrm{Ker}(A - \lambda\mathbf{1}) \neq 0$. Set $B = A|_{H\ominus M}$ and $T_n = B \oplus \lambda_n P$, where $\lambda_n \neq \lambda$ belong to α and converge to λ. Then $\mathrm{Ker}(T_n - \lambda\mathbf{1}) = \{0\}$ and $\mathrm{Sp}(T_n) \subseteq \alpha$. Therefore, $\|[g(T_n), X]\| \leqslant D\|[T_n, X]\|$. Since $T_n \to A$ and $g(T_n) = g(B) \oplus g(\lambda_n)P \to g(A)$, we have $\|[g(A), X]\| \leqslant D\|[A, X]\|$. □

Now continue the proof of Theorem 3.8. By Lemma 3.9, in order to show that the function φ is $\mathfrak{S}^b$-Lipschitzian, it suffices to prove (3.8) for the case when $\mathrm{Sp}(A) \subseteq [-n, n]$ and $\mathrm{Ker}(A) = \{0\}$. Let $E(\lambda)$ be the spectral function of A. Set $P(\varepsilon) = \mathbf{1} - (E(\varepsilon) \ominus E(-\varepsilon))$ for $\epsilon > 0$. Since $\mathrm{Ker}(A) = \{0\}$, we have $E(\varepsilon) \xrightarrow{\text{sot}} E(0)$, as $\varepsilon \to 0$. Therefore, $P(\varepsilon) \xrightarrow{\text{sot}} \mathbf{1}$. Hence, by Lemma 3.3,

$$\|[\varphi(A), X]\| = \overline{\lim_{\varepsilon\to 0}}\, \|P(\varepsilon)[\varphi(A), X]P(\varepsilon)\|. \tag{3.9}$$

Since the projections $P(\varepsilon)$ commute with A, they commute with $\varphi(A)$ and

$$\varphi(P(\varepsilon)A) = P(\varepsilon)\varphi(A) + \varphi(0)(\mathbf{1} - P(\varepsilon)) = P(\varepsilon)\varphi(A).$$

Set $A_\varepsilon = P(\varepsilon)A$. Then $\mathrm{Sp}(A_\varepsilon) \subseteq [-n, -\varepsilon] \cup [\varepsilon, n]$ and we have, from (3.7),

$$\begin{aligned}\|P(\varepsilon)[\varphi(A), X]P(\varepsilon)\| &= \|[\varphi(A_\varepsilon), P(\varepsilon)XP(\varepsilon)]\| \leqslant (2n+1)\|[A_\varepsilon, P(\varepsilon)XP(\varepsilon)]\| \\ &= (2n+1)\|P(\varepsilon)[A, X]P(\varepsilon)\| \leqslant (2n+1)\|[A, X]\|.\end{aligned}$$

By (3.9), $\|[\varphi(A), X]\| \leqslant (2n+1)\|[A, X]\|$. Applying Corollary 3.6, we complete the proof. □

4. Commutator-bounded functions: Fuglede sets

It follows from Proposition 3.4 that, if J is J_0^ϕ, J^ϕ or $\mathfrak{S}^b$, then commutator J-bounded functions could have been equivalently defined as those for which inequality (1.5) holds for all $A \in J_{\mathrm{nor}}(\alpha)$ and $X \in J$, or even for all $A \in \mathcal{F}_{\mathrm{nor}}(\alpha)$ and $X \in \mathcal{F}$.

By Theorem 3.5, a function is J-Lipschitzian if and only if (1.5) holds for all self-adjoint X. Therefore, each commutator J-bounded function is J-Lipschitzian. If $\alpha \subset \mathbb{R}$, then, by Corollary 3.6, the converse is true. In particular, all functions satisfying (3.3) are commutator J-bounded for all s.n. ideals J on each compact α in $\mathbb{R}$.

For $\alpha \not\subseteq \mathbb{R}$, the situation changes. To see this, note that, by Corollary 4.3 of [**11**], there are operators $X_n \in B(H)$ and a diagonal operator A such that

$$\|AX_n - X_nA\| \to 0 \quad \text{and} \quad \|A^*X_n - X_nA^*\| \geqslant 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Thus, although the function $h(z) = \bar{z}$ is J-Lipschitzian for all s.n. ideals J and for $J = \mathfrak{S}^b$, it is not commutator $\mathfrak{S}^b$-bounded.

The following result clarifies the relation between J-Lipschitz and commutator J-bounded functions.

Proposition 4.1. *Let J be an s.n. ideal or $\mathfrak{S}^b$ and let g be a continuous function on $\alpha \subset \mathbb{C}$. The following conditions are equivalent.*

(i) *g is commutator J-bounded on α.*

(ii) *There exists $D > 0$ such that for all $A, B \in J_{\mathrm{nor}}(\alpha)$ and $X \in B(H)$,*

$$\|g(A)X - Xg(B)\|_J \leqslant D\|AX - XB\|_J. \tag{4.2}$$

If J is J_0^ϕ or J^ϕ or $\mathfrak{S}^b$, then (i) and (ii) are also equivalent to condition (iii) that follows.

(iii) *g satisfies (4.2) for all $A, B \in \mathcal{F}_{\mathrm{nor}}(\alpha)$ and all X in $\mathcal{F}$.*

Proof. (i) $\Rightarrow$ (ii). Let $A, B \in J_{\mathrm{nor}}(\alpha)$ and $X \in B(H)$. Set

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

The operator L on $H \oplus H$ also belongs to $J_{\mathrm{nor}}(\alpha)$,

$$g(L) = \begin{pmatrix} g(A) & 0 \\ 0 & g(B) \end{pmatrix}$$

and

$$[g(L), \tilde{X}]U = \begin{pmatrix} g(A)X - Xg(B) & 0 \\ 0 & 0 \end{pmatrix}, \qquad [L, \tilde{X}]U = \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix}.$$

If g is commutator J-bounded, it follows from (1.2) and Remark 2.1 that

$$\begin{aligned} \|g(A)X - Xg(B)\|_J &= \|[g(L), \tilde{X}]U\|_J = \|[g(L), \tilde{X}]\|_J \\ &\leqslant D\|[L, \tilde{X}]\|_J = D\|[L, \tilde{X}]U\|_J = D\|AX - XB\|_J. \end{aligned}$$

(ii) $\Rightarrow$ (i). Set $B = A$.

(ii) $\Rightarrow$ (iii). This is evident.

(iii) $\Rightarrow$ (i). Set $B = A$ in (4.2). Using Proposition 3.4, we obtain that g is commutator J-bounded. $\square$

We will now consider some sufficient conditions for a function to be commutator J-bounded.

Example 4.2. Let a function g be analytic in a neighbourhood Ω of α. Then there is $D > 0$ such that, for any s.n. ideal J, for any normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and any X in $B(H)$, the condition $[A, X] \in J$ implies $[g(A), X] \in J$ and (1.5) holds. In particular, g is commutator J-bounded on α.

Indeed, let γ be a contour in Ω surrounding α and let $R(A, \lambda) = (\lambda\mathbf{1} - A)^{-1}$. Then $\rho = \inf\{|s - \lambda| : \lambda \in \gamma,\ s \in \alpha\} > 0$ and, for each normal A with $\mathrm{Sp}(A) \subseteq \alpha$,

$$g(A) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma R(A, \lambda) g(\lambda)\,\mathrm{d}\lambda.$$

We have

$$\|R(A, \lambda)\| \leqslant \Big(\inf_{s\in\alpha} |s - \lambda|\Big)^{-1} \leqslant \rho^{-1} < \infty$$

if $\lambda \in \gamma$. If $[A, X] \in J$, then

$$[R(A, \lambda), X] = R(A, \lambda)[X, A]R(A, \lambda) \in J \quad \text{and} \quad \|[R(A, \lambda), X]\|_J \leqslant \rho^{-2}\|[A, X]\|_J.$$

Hence

$$[g(A), X] = \frac{1}{2\pi\mathrm{i}} \oint_\gamma [R(A, \lambda), X] g(\lambda)\,\mathrm{d}\lambda \in J$$

and

$$\|[g(A), X]\|_J \leqslant \frac{\rho^{-2}}{2\pi} \|[A, X]\|_J \oint_\gamma |g(\lambda)|\,\mathrm{d}\lambda = D\|[A, X]\|_J.$$

$\square$

We will now consider a much wider class of commutator J-bounded functions. For compact subsets α, β of $\mathbb{C}$, the Varopoulos algebra $V(\alpha, \beta) = C(\alpha) \hat{\otimes} C(\beta)$ (the projective tensor product of the algebras of continuous functions) consists of all functions

$$\varphi(z, u) = \sum_{n=1}^{\infty} a_n(z) b_n(u) \in C(\alpha \times \beta),$$

where $a_n \in C(\alpha)$, $b_n \in C(\beta)$, such that,

$$\|\varphi\| = \inf\bigg\{\sum_{n=1}^{\infty} \|a_n\|\,\|b_n\| : \varphi(z, u) = \sum_{n=1}^{\infty} a_n(z) b_n(u)\bigg\} < \infty.$$

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Set $V(\alpha) = V(\alpha, \alpha)$ and let A be a normal operator with $\mathrm{Sp}(A) \subseteq \alpha$. If J is an s.n. ideal or $\mathfrak{S}^b$, the operators L_A, R_A of left and right multiplication by A on J ($L_A(X) = AX$, $R_A(X) = XA$) commute and $\|L_A\| = \|R_A\| = \|A\|$. For $\varphi \in V(\alpha)$, set

$$\varphi(L_A, R_A) = \sum_{n=1}^{\infty} L_{a_n(A)} R_{b_n(A)}.$$

Using the properties of the projective tensor product, we have that the map $\theta : \varphi \to \varphi(L_A, R_A)$ is a contractive homomorphism from $V(\alpha)$ to the algebra of all bounded operators on J.

Proposition 4.3. *Suppose that a function g on α satisfies the condition*

$$g(z) - g(u) = \varphi(z, u)(z - u) \quad \textit{for } z, u \in \alpha, \tag{4.3}$$

with $\varphi \in V(\alpha)$. Then, for each s.n. ideal J and for $J = \mathfrak{S}^b$, g is commutator J-bounded on α.

Proof. The function $f(z) = g(z) - g(0)$ is Lipschitzian at 0 and $f(0) = 0$. By Lemma 3.1, it acts on any ideal J. If $A \in J_{\mathrm{nor}}(\alpha)$, then $[g(A), X] = [f(A), X] \in J$ for $X \in B(H)$. Applying the homomorphism θ to both parts of (4.3), we have

$$L_{g(A)} - R_{g(A)} = \varphi(L_A, R_A)(L_A - R_A).$$

Hence

$$\begin{aligned} \|[g(A), X]\|_J = \|(L_{g(A)} - R_{g(A)})X\|_J &= \|\varphi(L_A, R_A)(L_A - R_A)X\|_J \\ &\leqslant \|\varphi(L_A, R_A)\| \, \|[A, X]\|_J \leqslant \|\varphi\|_{V(\alpha)} \|[A, X]\|_J. \end{aligned}$$

Thus g is J-Lipschitzian. □

Condition (4.3) and its integral analogues have been extensively studied and used by Arazy *et al.* [**1**], Birman and Solomyak [**3**], Peller [**20**] and others. In particular, Peller's proof of the inclusion of the Besov class of functions $B^1_{\infty 1}(a, b)$ in the class of all $\mathfrak{S}^b$-Lipschitz functions on $[a, b]$ in $\mathbb{R}$ is based on the proof of (4.3) for all $g \in B^1_{\infty 1}(a, b)$. We denote the space of functions on α satisfying (4.3) with $\varphi \in V(\alpha)$ by $\mathrm{BSP}(\alpha)$ (i.e. the Birman–Solomyak–Peller space).

Remark 4.4.

(i) If $g \in \mathrm{BSP}(\alpha)$ and $\beta \subseteq \alpha$, then $g|_\beta$ belongs to $\mathrm{BSP}(\beta)$.

(ii) Let $\alpha = [a, b]$, $\beta = [c, d]$ and let $\varphi(t, s)$ be a Lipschitz function on $\alpha \times \beta$. Varopoulos proved (see [**23**, Theorem 7.1.1]) that $\varphi \in V(\alpha, \beta)$.

(iii) Peller proved in [**20**] that $B^1_{\infty 1}(a, b) \subseteq \mathrm{BSP}(a, b)$. Hence any function with continuous second derivative belongs to $\mathrm{BSP}(a, b)$.

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In the rest of this section we investigate the following question: for which sets α are properties (1.4) and (1.5) equivalent. In the discussion before Proposition 4.1 it was shown that the function $h(z) = \bar{z}$ is not commutator $\mathfrak{S}^b$-bounded for some $\alpha \not\subseteq \mathbb{R}$. The following result shows that this function plays a crucial role in the theory of commutator J-bounded and J-Lipschitz functions on complex domains.

Proposition 4.5. *Let $h(z) = \bar{z}$ be commutator J-bounded on $\alpha \subset \mathbb{C}$. Then $J\text{-}\operatorname{Lip}(\alpha) = J\text{-}\operatorname{CB}(\alpha)$.*

Proof. In the proof of Corollary 3.6 (ii) we used the condition $\alpha \subset \mathbb{R}$ only once: to show that $\|[A, X^*]\|_J = \|[A, X]\|_J$ and to substitute this in (3.6). In our case, h is commutator J-bounded on α and therefore there is $C > 0$ such that

$$\|[A, X^*]\|_J = \|[A^*, X]\|_J \leqslant C\|[A, X]\|_J \quad \text{for } A \in J_{\text{nor}}(\alpha) \text{ and } X \in B(H).$$

Substituting this in (3.6), we have $\|[g(A), X]\|_J \leqslant (1 + C)\|[A, X]\|_J$. □

Taking into account Proposition 4.5, we introduce a special class of compact sets.

Definition 4.6. A compact set α in $\mathbb{C}$ is called J-Fuglede if the function $h(z) = \bar{z}$ is commutator J-bounded on α, that is, there is $C > 0$ such that

$$\|[A^*, X]\|_J \leqslant C\|[A, X]\|_J \quad \text{for all } A \in J_{\text{nor}}(\alpha) \text{ and } X \in B(H). \tag{4.4}$$

It follows from Proposition 4.5 that the spaces of $J\text{-}\operatorname{Lip}(\alpha)$ and $J\text{-}\operatorname{CB}(\alpha)$ coincide on J-Fuglede compacts α (and only on them). Clearly, every compact set in $\mathbb{R}$ is J-Fuglede for any ideal J and for $J = \mathfrak{S}^b$. To study further properties of J-Fuglede sets, we need the following extension of Rosenblum's theorem.

Lemma 4.7. *Let α, β be compacts in $\mathbb{C}$ and let $\alpha \cap \beta = \varnothing$. There is $C = C(\alpha, \beta) > 0$ such that, for all unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, for all Banach left $\mathcal{A}$- and right $\mathcal{B}$-modules $\mathfrak{X}$, and for all normal $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $\operatorname{Sp}(A) \subseteq \alpha$, $\operatorname{Sp}(B) \subseteq \beta$,*

$$\|AX - XB\|_{\mathfrak{X}} \geqslant C\|X\|_{\mathfrak{X}} \quad \textit{for all } X \in \mathfrak{X}.$$

Proof. Repeating the proof of Rosenblum's theorem (see [**22**]), we obtain that, for any normal $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $\operatorname{Sp}(A) \subseteq \alpha$, $\operatorname{Sp}(B) \subseteq \beta$, the operator $X \to AX - XB$ on $\mathfrak{X}$ is invertible. Hence there is $C(A, B) > 0$ such that $\|AX - XB\|_{\mathfrak{X}} \geqslant C(A, B)\|X\|_{\mathfrak{X}}$ for all $X \in X$.

Assume that there are C^*-algebras $\mathcal{A}_i$ and $\mathcal{B}_i$, Banach modules $\mathfrak{X}_i$ and normal $A_i \in \mathcal{A}_i$, $B_i \in \mathcal{B}_i$ such that $C(A_i, B_i) \to 0$. Let $\mathfrak{X}$ be the Banach space of all bounded sequences $X = (X_1, \dots, X_n, \dots)$, $X_i \in \mathfrak{X}_i$, with norm $\|X\|_{\mathfrak{X}} = \sup \|X_i\|_{\mathfrak{X}_i}$ and let $\mathcal{A}$ be the C^*-algebra of all bounded sequences $R = (R_1, \dots, R_n, \dots)$, $R_i \in \mathcal{A}_i$, with norm $\|R\|_{\mathcal{A}} = \sup \|R_i\|_{\mathcal{A}_i}$. Similarly, we define the C^*-algebra $\mathcal{B}$. Then $\mathfrak{X}$ is a Banach left $\mathcal{A}$- and right $\mathcal{B}$-module.

Set $A = (A_1, \dots, A_n, \dots)$, $B = (B_1, \dots, B_n, \dots)$. Since all A_i, B_i are normal and $\operatorname{Sp}(A_i) \subseteq \alpha$, $\operatorname{Sp}(B_i) \subseteq \beta$, there is $K > 0$ such that $\|A_i\|_{\mathcal{A}_i} \leqslant K$, $\|B_i\|_{\mathcal{B}_i} \leqslant K$. Thus

$A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\mathrm{Sp}(A) \subseteq \alpha$, $\mathrm{Sp}(B) \subseteq \beta$. Hence, as above, there is $C > 0$ such that $\|AX - XB\|_{\mathfrak{X}} \geqslant C\|X\|_{\mathfrak{X}}$ for all $X \in \mathfrak{X}$. If $\hat{X}_i = (0, \dots, 0, X_i, 0, \dots)$, for $X_i \in \mathfrak{X}_i$, then

$$C\|X_i\|_{\mathfrak{X}_i} = C\|\hat{X}_i\|_{\mathfrak{X}} \leqslant \|A\hat{X}_i - \hat{X}_i B\|_{\mathfrak{X}} = \|A_i X_i - X_i B_i\|_{\mathfrak{X}_i},$$

which contradicts our assumption. □

Proposition 4.8.

(i) *Any compact subset of a J-Fuglede set is J-Fuglede.*

(ii) *If α_1, α_2 are disjoint J-Fuglede sets, the set $\alpha_1 \cup \alpha_2$ is also J-Fuglede.*

Proof. (i) is evident.

There is $K > 0$ such that $\|A\| \leqslant K$ for any normal A with $\mathrm{Sp}(A) \subseteq \alpha_1 \cup \alpha_2$. If $A \in J$, then $H = H_1 \oplus H_2$ and $A = A_1 \oplus A_2$, where $A_i = A|_{H_i}$ and $\mathrm{Sp}(A_i) \subseteq \alpha_i$. Let P_i be the projections on H_i. Then

$$\|[A^*, X]\|_J \leqslant \sum_{i,j=1}^{2} \|P_i[A^*, X]P_j\|_J = \sum_{i,j=1}^{2} \|[(P_iA)^*, P_iXP_j]\|_J. \tag{4.5}$$

The operators $X_{ij} = P_iXP_j$, $i, j = 1, 2$, act from H_j into H_i. It follows from Remark 2.1 that $\|[P_iA, P_iXP_i]\|_J = \|[A_i, X_{ii}]\|_J$. Since α_i are J-Fuglede, there are $c_i > 0$ such that $\|[A_i^*, X_{ii}]\|_J \leqslant c_i\|[A_i, X_{ii}]\|_J$. Hence,

$$\|[(P_iA)^*, P_iXP_i]\|_J \leqslant c_i\|[A_i, X_{ii}]\|_J = c_i\|P_i[A, X]P_i\|_J \leqslant c_i\|[A, X]\|_J. \tag{4.6}$$

Let $J = \mathfrak{S}^b$. Then

$$\|[(P_iA)^*, P_iXP_j]\| \leqslant 2\|A\|\,\|P_iXP_j\| \leqslant 2K\|X_{ij}\| \quad \text{for } i \neq j.$$

Since the α_i are disjoint and $\mathrm{Sp}(A_i) \subseteq \alpha_i$, it follows from Lemma 4.7 that

$$\|X_{ij}\| \leqslant c\|A_iX_{ij} - X_{ij}A_i\| = c\|P_i[A, X]P_j\| \leqslant c\|[A, X]\|,$$

for some $c > 0$. Combining this with (4.5) and (4.6), we obtain that there is $D > 0$ such that $\|[A^*, X]\| \leqslant D\|[A, X]\|$, so $\alpha_1 \cup \alpha_2$ is $\mathfrak{S}^b$-Fuglede.

Let $J \neq \mathfrak{S}^b$. Since α_1, α_2 are disjoint, only one of them may contain 0. Let $0 \notin \alpha_2$. Since $\mathrm{Sp}(A_2) \subseteq \alpha_2$ and A_2 is compact, H_2 is finite dimensional, so $\mathfrak{X} = \{P_1XP_2 : X \in B(H)\}$ consists of finite-rank operators. Hence $(\mathfrak{X}, \|\cdot\|_J)$ is a Banach left $B(H_1)$-module and right $B(H_2)$-module and

$$\|[(P_1A)^*, P_1XP_2]\|_J \leqslant 2\|A\|\,\|P_1XP_2\|_J \leqslant 2K\|P_1XP_2\|_J.$$

Since the α_i are disjoint and $\mathrm{Sp}(A_i) \subseteq \alpha_i$, it follows from Lemma 4.7 that

$$\|P_1XP_2\|_J \leqslant c\|A_1P_1XP_2 - P_1XP_2A_2\|_J = c\|P_1[A, X]P_2\|_J \leqslant c\|[A, X]\|_J,$$

for some $c > 0$. Hence $\|[(P_1A)^*, P_1XP_2]\|_J \leqslant 2cK\|[A, X]\|_J$.

Similarly, there is $d > 0$ such that $\|[(P_2A)^*, P_2XP_1]\|_J \leqslant 2dK\|[A, X]\|_J$. Combining this with (4.5) and (4.6), we obtain that there is $D > 0$ such that $\|[A^*, X]\|_J \leqslant D\|[A, X]\|_J$. Thus $\alpha_1 \cup \alpha_2$ is J-Fuglede. □

We say that a simple Jordan line in $\mathbb{C}$ is 2-*smooth* if it has a parametrization $z = k(t)$, $t \in \mathbb{R}$, where k is twice continuously differentiable and the derivative k' does not vanish.

Theorem 4.9. *A compact subset α of a 2-smooth line $z = k(t) = x(t) + \mathrm{i}y(t)$ is J-Fuglede for any s.n. ideal J and for $J = \mathfrak{S}^b$.*

Proof. It follows from Proposition 4.8 (ii) that we only need to prove the theorem in the case when α is connected. Define a function φ on $\alpha \times \alpha$ by

$$\varphi(z,u) = \frac{\overline{z-u}}{z-u}, \quad \text{for } z \neq u, \quad \text{and} \quad \varphi(z,z) = \frac{\overline{k'(k^{-1}(z))}}{k'(k^{-1}(z))}.$$

Since $h(z)-h(u) = \varphi(z,u)(z-u)$, we have from Proposition 4.3 that to prove the theorem it is sufficient to show that $\varphi \in V(\alpha)$. To do this, it suffices (see [**23**]) to establish that, for any $(z,u) \in \alpha \times \alpha$, there are compact neighbourhoods β of z and γ of u in α such that $\varphi|_{\beta\times\gamma} \in V(\beta,\gamma)$.

Since k' does not vanish, there are $\delta = [a,b]$, $\sigma = [c,d]$ in $\mathbb{R}$ and $t_0 \in (a,b)$, $s_0 \in (c,d)$ such that

$$k(\delta), k(\sigma) \subset \alpha, \quad z = k(t_0), \quad u = k(s_0) \quad \text{and} \quad k \text{ is injective on } \delta \text{ and } \sigma. \tag{4.7}$$

The function $\theta(t,s) = \varphi(k(t),k(s))$ belongs to $V(\delta,\sigma)$ if and only $\varphi \in V(k(\delta),k(\sigma))$.

Let $z \neq u$. Choose δ and σ such that, in addition to (4.7), $k(\delta) \cap k(\sigma) = \varnothing$. Then θ is twice continuously differentiable on $\delta \times \sigma$, so it is Lipschitzian. By Remark 4.4 (ii), θ belongs to $V(\delta,\sigma)$. Hence $\varphi \in V(k(\delta),k(\sigma))$.

Let $z = u = k(t_0)$. Since $k'(t_0) \neq 0$, $|x'(t_0)| + |y'(t_0)| \neq 0$. Assume that $x'(t_0) \neq 0$. Choose $\delta = \sigma$ so that, in addition to (4.7), $x'(t) \neq 0$ for $t \in \delta$. Then

$$\theta(t,s) = \frac{\overline{k(t)-k(s)}}{k(t)-k(s)} = \frac{\tilde{x}(t,s) - \mathrm{i}\tilde{y}(t,s)}{\tilde{x}(t,s) + \mathrm{i}\tilde{y}(t,s)},$$

where

$$\tilde{y}(t,s) = \frac{y(t)-y(s)}{t-s}, \quad \tilde{x}(t,s) = \frac{x(t)-x(s)}{t-s} \quad \text{for } t \neq s,$$

and

$$\tilde{y}(t,t) = y'(t), \qquad \tilde{x}(t,t) = x'(t).$$

Since x, y are twice continuously differentiable functions, by Remark 4.4 (iii), they belong to $\mathrm{BSP}(\delta)$. Hence the functions $\tilde{x}$ and $\tilde{y}$ belong to $V(\delta)$. Since $\tilde{x} \neq 0$ on $\delta \times \delta$, the function $\tilde{x} + \mathrm{i}\tilde{y}$ does not vanish on $\delta \times \delta$. Since $\delta \times \delta$ is the space of maximal ideals of the algebra $V(\delta)$, $\tilde{x} + \mathrm{i}\tilde{y}$ is invertible in $V(\delta)$. Hence $\theta \in V(\delta)$, so $\varphi \in V(k(\delta))$. $\square$

We will see now that some smoothness conditions are necessary for a compact α to be $\mathfrak{S}^b$-Fuglede. We say that α is *smooth* at a non-isolated point $w \in \alpha$ if it has a 'tangent' at w, that is, there is a straight line L such that $w \in L$ and $\operatorname{dist}(z,L) = o(|z-w|)$ when $\alpha \ni z \to w$.

Proposition 4.10. *$\mathfrak{S}^b$-Fuglede compacts are smooth at all non-isolated points.*

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Proof. For $A \in B(H)$, the map $\delta_A : X \to [A, X]$ is a derivation on $B(H)$. If α is $\mathfrak{S}^b$-Fuglede, there is $D > 0$ such that $\|[h(A), X]\| \leqslant D\|[A, X]\|$ for all normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $X \in B(H)$. It follows from Corollary 4.7 of [**11**] that the range of the derivation $\delta_{h(A)}$ on $B(H)$ is included in the range of δ_A for any such A. By Theorem 4.1 of [**11**], this implies that h is differentiable relative to α at each non-isolated point, which is, clearly, equivalent to the smoothness of α. □

The above result shows that the set $\alpha = \{x + \mathrm{i}|x| : -1 \leqslant x \leqslant 1\}$ and any compact subset of α with 0 as a cluster point for points with $x > 0$ and for points with $0 < x$ are not $\mathfrak{S}^b$-Fuglede. It also shows that the union of two $\mathfrak{S}^b$-Fuglede sets can be non-Fuglede and that the compacts with non-empty interior are not $\mathfrak{S}^b$-Fuglede.

5. Hadamard multipliers

To study commutator J-bounded functions further we need to use the notion of the Hadamard multiplier. Every orthogonal basis $E = \{e_i\}$ in H defines a map φ_E from $B(H)$ into the set $\mathfrak{M}$ of all matrices: $\varphi_E : T \to (t_{ij})$, where $t_{ij} = (Te_j, e_i)$. Any matrix $M = (m_{ij}) \in \mathfrak{M}$ acts on $\mathfrak{M}$ by the formula $M \circ X = (m_{ij}x_{ij})$ for $X = (x_{ij}) \in \mathfrak{M}$. We identify J and $\varphi_E(J)$ and write $M \circ T$ for $T \in J$.

A matrix M is called a *Hadamard J-multiplier*, if $M \circ T \in J$ for each $T \in J$. Denote by $\mathbb{M}_J$ the set of all Hadamard J-multipliers. If $M \in \mathbb{M}_J$, then, by the closed graph theorem, the map $T \to M \circ T$ on J is bounded; its norm we denote by $\|M\|_J$. If $J = \mathfrak{S}^p$, we write $\mathbb{M}_p$ and $\|M\|_p$. The set $\varphi_E(\mathfrak{S}^2)$ consists of all matrices X with $\sum_{i,j} |x_{ij}|^2 < \infty$. Hence $M \in \mathbb{M}_2$ if and only if

$$\|M\|_2 = \sup |m_{ij}| < \infty. \tag{5.1}$$

The next result may well be known, but we could not find a reference.

Lemma 5.1. *Let $\phi \in \Phi$ and let ϕ^* be its adjoint. Then*

$$\mathbb{M}_{J_0^\phi} = \mathbb{M}_{J^\phi} = \mathbb{M}_{J_0^{\phi^*}} = \mathbb{M}_{J^{\phi^*}}$$

and the norms coincide. In particular, $\mathbb{M}_b = \mathbb{M}_1 = \mathbb{M}_\infty$ and $\mathbb{M}_p = \mathbb{M}_{p'}$ (see (2.2)).

Proof. Let $\{e_i\}$ be a basis in H and let V be the anti-linear isometry on H:

$$V\biggl(\sum_i \lambda_i e_i\biggr) = \sum_i \bar{\lambda}_i e_i.$$

Let $T = \sum_i s_i(T)(x_i \otimes y_i)$ be a Schmidt decomposition (see [**10**]) of a compact operator T, where $\{x_i\}$, $\{y_i\}$ are orthonormal sets in H, $x_i \otimes y_i$ are rank-one operators: $(x_i \otimes y_i)z = (z, x_i)y_i$, and $s_i(T)$ are the eigenvalues of $(T^*T)^{1/2}$. The transpose T' of T in $\{e_i\}$ has the form

$$T' = VT^*V = \sum_i s_i(T)(Vy_i \otimes Vx_i).$$

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Since $\{Vx_i\}$, $\{Vy_i\}$ are orthonormal sets, it is a Schmidt decomposition of T'. Thus, if J is J^ϕ, J_0^ϕ or $\mathfrak{S}^b$, and $T \in J$, then $T' \in J$ and $\|T\|_J = \|T'\|_J$.

Let $M \in \mathbb{M}_J$ and let M' be its transpose. For $T \in J$, we have $M' \circ T = (M \circ T')' \in J$ and $\|M' \circ T\|_J = \|M \circ T'\|_J \leqslant \|M\|_J \|T\|_J$. Hence $M' \in \mathbb{M}_J$ and $\|M'\|_J \leqslant \|M\|_J$. Since $M'' = M$, we have $\|M'\|_J = \|M\|_J$.

Set $J = J^\phi$ and $J_0 = J_0^\phi$. Let P_n be the projections on the subspaces spanned by $\{e_i\}_{i=1}^n$, $n < \infty$. If $M \in \mathbb{M}_J$ and $T \in J_0$, then $P_n(M \circ T)P_n = M \circ (P_n T P_n) \in J_0$ and

$$\|M \circ T - P_n(M \circ T)P_n\|_J = \|M \circ (T - P_n T P_n)\|_J \leqslant \|M\|_J \|T - P_n T P_n\|_J \to 0,$$

since $\|T - P_n T P_n\|_J \to 0$ (see [**10**, Theorem III.6.3]). Hence $M \circ T \in J_0$, so $M \in \mathbb{M}_{J_0}$. Clearly, $\|M\|_{J_0} \leqslant \|M\|_J$.

Set $I = J^{\phi^*}$. For $M \in \mathbb{M}_{J_0}$ and $A \in I$ it follows from (2.1) that the functional

$$F_{M \circ A}(T) = \mathrm{Tr}(T(M \circ A)) = \mathrm{Tr}((M' \circ T)A) = F_A(M' \circ T)$$

on J_0 is bounded. Hence $M \circ A \in I$ and $\|M \circ A\|_I \leqslant \|M'\|_{J_0} \|A\|_I$, so $M \in \mathbb{M}_I$ and $\|M\|_I \leqslant \|M\|_{J_0}$. Thus $\mathbb{M}_J \subseteq \mathbb{M}_{J_0} \subseteq \mathbb{M}_I$ and $\|M\|_I \leqslant \|M\|_{J_0} \leqslant \|M\|_J$, for $M \in \mathbb{M}_J$.

Set $I_0 = J_0^{\phi^*}$. Since $(\phi^*)^* = \phi$, we obtain similarly that $\mathbb{M}_I \subseteq \mathbb{M}_{I_0} \subseteq \mathbb{M}_J$ and $\|M\|_J \leqslant \|M\|_{I_0} \leqslant \|M\|_I$, for $M \in \mathbb{M}_I$. This proves the lemma for $J \neq \mathfrak{S}^p$, $p = 1, \infty, b$.

Repeating the above argument for $J = \mathfrak{S}^b$, $J_0 = \mathfrak{S}^\infty$, $I = \mathfrak{S}^1$, we complete the proof. □

Let $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n, \dots)$ be a diagonal operator with respect to a basis $\{e_n\}$. Let g be a continuous function on $\alpha \subset \mathbb{C}$ and let α contain all λ_n. Set

$$m_{ij} = \begin{cases} \dfrac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j}, & \text{if } \lambda_i \neq \lambda_j, \\ m_{ij} = 0, & \text{if } \lambda_i = \lambda_j. \end{cases} \tag{5.2}$$

Consider the matrix $M(A, g) = (m_{ij})$. The following result generalizes Lemma 3.3 of [**11**].

Proposition 5.2.

(i) *Let $M(A, g)$ be a Hadamard J-multiplier. Then*

$$\|[g(A), X]\|_J \leqslant D \|[A, X]\|_J \quad \textit{for all } X \in J \tag{5.3}$$

with $D = \|M(A, g)\|_J$.

(ii) *Let J be J_0^ϕ or J^ϕ or $\mathfrak{S}^b$. If (5.3) holds, then $M(A, g)$ is a Hadamard J-multiplier and $\|M(A, g)\|_J \leqslant 2D$.*

Proof. Let $X = (x_{ij})$, $[A, X] = (y_{ij})$, $[g(A), X] = (z_{ij})$. Then $y_{ij} = (\lambda_i - \lambda_j)x_{ij}$ and $z_{ij} = (g(\lambda_i) - g(\lambda_j))x_{ij}$, so $z_{ij} = m_{ij} y_{ij}$. Therefore,

$$[g(A), X] = M(A, g) \circ [A, X]. \tag{5.4}$$

If $M(A, g)$ is a Hadamard J-multiplier, then (5.3) holds. Part (i) is proved.

Suppose that J is J_0^ϕ, or J^ϕ or $\mathfrak{S}^b$, and that (5.3) holds. Let $\lambda_{i(k)}$ be all distinct eigenvalues of A and let Q_k be the projections on the corresponding eigenspaces. We have from Theorem III.4.2 of [**10**] that, for any operator B in J,

$$\hat{B} = \sum_k Q_k B Q_k \quad \text{belongs to } J \text{ and } \|\hat{B}\|_J \leqslant \|B\|_J. \tag{5.5}$$

Denote by P_n the projections on the subspaces of H spanned by $\{e_i\}_{i=1}^n$. All P_n and Q_k commute. Let $Y \in \mathcal{F}$ and let $Y = P_n Y P_n$ for some n. Set $Y^\# = Y - \hat{Y} = (y_{ij})$. Then $Y^\# = P_n Y^\# P_n$ and $y_{ij} = 0$ if $\lambda_i = \lambda_j$. Consider $X = (x_{ij})$, where

$$x_{ij} = \begin{cases} \dfrac{y_{ij}}{\lambda_i - \lambda_j}, & \text{if } \lambda_i \neq \lambda_j, \\ x_{ij} = 0, & \text{if } \lambda_i = \lambda_j. \end{cases}$$

Then $X = P_n X P_n$, $Y^\# = [A, X]$ and $M(A, g) \circ Y = M(A, g) \circ Y^\# = M(A, g) \circ [A, X]$.

By (5.5), $\|[A, X]\|_J = \|Y^\#\|_J \leqslant 2\|Y\|_J$, so it follows from (5.3) and (5.4) that

$$\|M(A, g) \circ Y\|_J = \|M(A, g) \circ [A, X]\|_J = \|[g(A), X]\|_J \leqslant D\|[A, X]\|_J \leqslant 2D\|Y\|_J. \tag{5.6}$$

Hence $M(A, g)$ generates a bounded operator S on the subspace J_{fin} of J of all finite-rank operators X such that $X = P_n X P_n$ for some n.

If $J = J_0^\phi$, it follows from Theorem III.6.3 of [**10**] that $\|X - P_n X P_n\|_J \to 0$ for $X \in J$. Therefore, J_{fin} is dense in J and S extends to a bounded operator on J, which we also denote by S. Let $X = (x_{ij}) \in J$. For any i, j,

$$(SXe_i, e_j) = \lim_n (SP_n X P_n e_i, e_j) = \lim_n ((M(A, g) \circ P_n X P_n) e_i, e_j) = m_{ij} x_{ij}.$$

Hence $SX = M(A, g) \circ X$. Thus $M(A, g)$ is a Hadamard J-multiplier and, by (5.6),

$$\begin{aligned} \|M(A, g) \circ X\|_J &= \lim_n \|P_n (M(A, g) \circ X) P_n\|_J = \lim_n \|M(A, g) \circ (P_n X P_n)\|_J \\ &\leqslant 2D \lim_n \|P_n X P_n\|_J = 2D\|X\|_J, \end{aligned}$$

so that $\|M(A, g)\|_J \leqslant 2D$.

If $J = J^\phi$ is not separable (respectively, if $J = \mathfrak{S}^b$), consider the separable ideal $I = J_0^\phi$ in J (respectively, $I = C(H)$). By the above argument, $M(A, g)$ is a Hadamard I-multiplier. It follows from Lemma 5.1 that $M(A, g)$ is also a Hadamard J-multiplier. $\square$

Making use of Propositions 3.4 and 5.2, we obtain the following corollary.

Corollary 5.3. *Let J be J_0^ϕ, or J^ϕ or $\mathfrak{S}^b$, and let g be a continuous function on $\alpha \subset \mathbb{C}$. Then the following conditions are equivalent.*

(i) *g is commutator J-bounded on α.*

(ii) *There is $D > 0$ such that, for any $A \in J_{\text{nor}}(\alpha)$ (for any diagonal A in $J_{\text{nor}}(\alpha)$ if $J = \mathfrak{S}^b$), the matrix $M(A, g)$ is a Hadamard J-multiplier and $\|M(A, g)\|_J \leqslant D$.*

(iii) *There exists $D > 0$ such that $\|M(A, g)\|_J \leqslant D$, for all $A \in \mathcal{F}_{\mathrm{nor}}(\alpha)$.*

The following result echoes the relations between Hadamard multipliers.

Corollary 5.4.

(i) *For an s.n. ideal J, let $\phi \in \Phi$ be such that $J_0^\phi \subseteq J \subseteq J^\phi$ (see Proposition 2.1 of [**16**]), and let ϕ^* be the adjoint of ϕ. Then each commutator J-bounded function g on α is commutator J_0^ϕ-, J^ϕ-, $J_0^{\phi^*}$- and J^{ϕ^*}-bounded on α.*

(ii) *The following conditions are equivalent:*

(1) *g is commutator $\mathfrak{S}^b$-bounded on α;*

(2) *g is commutator $\mathfrak{S}^\infty$-bounded on α;*

(3) *g is commutator $\mathfrak{S}^1$-bounded on α.*

Proof. The function $f(t) = g(t) - g(0)$ (see (3.1)) acts on all s.n. ideals. By Proposition 2.1 of [**16**], the norms $\|\cdot\|_J$ and $\|\cdot\|_{J^\phi}$ coincide on J_0^ϕ. Hence f is commutator J_0^ϕ-bounded on α. It follows from Lemma 5.1 and Corollary 5.3 that f is J^ϕ-, $J_0^{\phi^*}$- and J^{ϕ^*}-bounded on α. Hence g is J^ϕ-, $J_0^{\phi^*}$- and J^{ϕ^*}-bounded on α. The proof of (ii) is the same. $\square$

We now consider the most important case $J = \mathfrak{S}^b$ and the simplest case $J = \mathfrak{S}^2$.

Proposition 5.5.

(i) *A function g is commutator $\mathfrak{S}^b$-bounded on $\alpha \subset \mathbb{C}$ if and only if there exists $D > 0$ such that, for any distinct numbers $\{\lambda_n\}_{n=1}^k$, $k < \infty$, in α, the $k \times k$ matrix $M(\{\lambda_n\}, g) = (m_{ij})$, $1 \leqslant i, j \leqslant k$, where*

$$m_{ii} = 0 \quad \text{and} \quad m_{ij} = \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j}, \quad \text{if } i \neq j,$$

is a Hadamard multiplier on $B(\mathbb{C}^k)$ with norm less than or equal to D.

(ii) *A function on α is commutator $\mathfrak{S}^2$-bounded if and only if it is Lipschitzian on α in the usual sense.*

Proof. By Corollary 5.3, g is commutator $\mathfrak{S}^b$-bounded if and only if the norms of all multipliers $M(A, g)$, for $A \in \mathcal{F}(\alpha)$, are bounded by a mutual constant.

Let $A \in \mathcal{F}_{\mathrm{nor}}$, $\mathrm{Sp}(A) = \{\lambda_n\}_{n=1}^k \subseteq \alpha$ and $\{H_n\}_{n=1}^k$ be the eigenspaces of A. Then $H = \sum_{n=1}^k \oplus H_n$ and any $T \in B(H)$ has a block-matrix form (T_{ij}), where T_{ij} are bounded operators from H_j into H_i. The $k \times k$ matrix $M(\{\lambda_n\}, g)$ is a Hadamard multiplier on $B(\mathbb{C}^k)$ and also defines a bounded operator $\tilde{M}$ on $B(H)$ by the formula $\tilde{M}(T) = (m_{ij}T_{ij})$, for $T \in B(H)$. It is easy to see that $\tilde{M}(T) = M(A, g) \circ T$. It was proved in Lemma 4.5 of [**13**] that $\|M(A, g)\|_{\mathfrak{S}^b} = \|M(\{\lambda_n\}, g)\|_{\mathbb{C}^k}$. This completes the proof of (i).

By Corollary 5.3, g is commutator $\mathfrak{S}^2$-bounded if and only if there is $D > 0$ such that $\|M(A, g)\|_2 \leqslant D$ for $A \in \mathcal{F}_{\mathrm{nor}}(\alpha)$. Hence (ii) follows from (5.1) and (5.2). $\square$

If H is finite dimensional, all norms on $B(H)$ are equivalent. Hence, for each ideal $J = J(H)$, a function on α is commutator J-bounded (respectively, J-Lipschitzian) if and only if it is commutator $\mathfrak{S}^2$-bounded (respectively, $\mathfrak{S}^2$-Lipschitzian).

Corollary 5.6. *Let H be a finite-dimensional space and let g be a function on α. The following conditions are equivalent.*

(i) *g is Lipschitzian on α in the usual sense.*

(ii) *g is commutator J-bounded on α for all s.n. ideals $J = J(H)$.*

(iii) *g is J-Lipschitzian on α for all s.n. ideals $J = J(H)$.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from Proposition 5.5 (ii) and from the comment before the corollary.

(ii) $\Rightarrow$ (iii). This follows from Theorem 3.5.

(iii) $\Rightarrow$ (i). This was proved at the beginning of § 3. □

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