6

Classical Register Mechanics

6.1 Introduction

Classical mechanics (CM) is conventionally formulated according to partially contextually complete Block Universe principles: an exophysical observer sitting in some frame of reference looks in on a system under observation (SUO) and assigns truth values to propositions about states of that SUO, but the details of how these truth values are obtained are not given.

There is a curious aspect to this scenario: the contextual incompleteness of CM is itself contextual. By this we mean that there are two scenarios, with different general propositional classifications (GPCs).

The Experimentalist's Perspective
From a classical experimentalist's perspective, generalized propositions (GPs) in CM take the form

$$\mathcal{P}(\boldsymbol{P}, \emptyset | \emptyset, F). \tag{6.1}$$

Here $\boldsymbol{P}$ is some proposition of scientific interest, such as "**The orbit of Jupiter is an ellipse**," and F is relative external context describing the frame of reference used to describe states of the SUO relative to the wider Universe external to that SUO. This form of GP has a GPC of 2 according to the algorithm discussed in Chapter 2.

> **Remark 6.1** Suppose an astronomer used a telescope T to test proposition $\boldsymbol{P}$. That would not upgrade the GP to one of the form $\mathcal{P}(\boldsymbol{P}, T | \emptyset, F)$ with a GPC of 3. The reason is that the use of a telescope is not essential in this context. Historically, Kepler did not use a telescope to state his laws of planetary motion: he used the data obtained by Tycho using naked eye observations. On the other hand, the observation of the Hubble red shift using telescopes fitted with spectrum analyzers would qualify for such an upgrade.

The Theorist's Perspective

Newtonian mechanics applied to the description of classical dynamically evolving
SUOs has the strength of mathematics: the classical laws of motion can be considered as a set of mathematical principles, axioms, and theorems, relative to which
propositions about classical states of SUOs can be tested. From this perspective,
the contextual information F about frames of reference becomes part of the
mathematical framework, and therefore should be included in *relative internal
context*. In this scenario, GPs take the form $\mathcal{P}(\boldsymbol{P}, \text{Laws of motion}, F | \emptyset, \emptyset)$, with
a GPC of 1, as befitting mathematical propositions.[1]

6.2 Classical Registers

We shall focus our attention on the second scenario discussed above, that of the
theorist's perspective. Therefore, we shall not discuss apparatus or the observer.
Instead, we focus our attention on the mathematical structures that we employ
to model particles moving about in continuous space and time.

The first thing is to construct a model for classical space, objectifying it as a
vast collection of containers into which, and out of which, particles can move.

Now conventionally and until relatively recently, space has always been considered to be continuous. But that is an empirically vacuous assertion. Therefore,
we take the liberty of modeling space in discrete and finite terms. This leads us
to define a classical register $\mathcal{R}^{[r]}$ of rank r, the Cartesian product of r classical
bits. We write

$$\mathcal{R}^{[r]} \equiv B^1 B^2 \dots B^r, \tag{6.2}$$

where the B^i, $i = 1, 2, \dots, r$, are the individual labeled bits and we use bit labeling
to bypass the need for the Cartesian product symbol $\times$. We will consider classical
registers of sufficiently large rank so that they can model regions of classical
physical space over which particles can move. In this approach, particle motion
is discussed in terms of the tracking of signals from a vast collection of detectors
over time. A particularly useful feature of this approach is that signality need
not be conserved, which means that classical particle creation and annihilation
is readily incorporated into the formalism.

6.3 Architecture

In any such discussion it is important to establish the relevant spatiotemporal architecture. There are two aspects of architecture relevant to the present
discussion. One has to do with the unspecified exophysical observer and their

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[1] It seems invalid to argue that in Newtonian mechanics, we have the laws of motion for
relative internal context and the observer's frame for relative external context, so that
would appear to give a GPC of 3. That would be mixing theory and experiment to an
unacceptable degree. We have made the point before that mathematics is not physics.

apparatus, and the other is to do with the dynamics of the system under observation. We shall refer to these aspects as the *external* and *internal* architectures, respectively.

External architecture involves time and persistence.

Time

In classical register mechanics, time is modeled as a succession of discrete stages, $\Sigma_M, \Sigma_{M+1}, \ldots, \Sigma_N$, representing the actions or lack of actions by an observer in a physical laboratory. At stage Σ_n, the state of the SUO will be denoted $\boldsymbol{\Psi}_n$, an element of a classical register $\mathcal{R}_n$ of rank r_n.

Persistence

This refers to the question of identity of a given detector over a succession of stages. In quantized detector networks (QDN), each detector is identified with a single stage only. Therefore, there is no concept of "existential persistence" in QDN. This applies as much to the observer and their apparatus as it does to the SUOs that they observe. The *first law of time* (the dictum of Heraclitus), that *everything changes*, applies here.

That does not rule out *effective persistence*, meaning that as the observer's time runs from stage to stage, their description of the apparatus seems to be constant, in terms of the rank of the registers concerned and in terms of the labeling of the detectors in those registers.

Effective persistence is the rule in ordinary experience, to the extent that humans are strongly conditioned to believe that they are moving around in time over a fixed spatial background. This belief carries over into conventional descriptions of experiments, in which apparatus is assumed to persist while states of SUOs evolve. What is really going on is that the first law of time always applies, but the time scales for significant change in apparatus are generally so great relative to those associated with the SUOs that the former time scales may be taken to be infinite compared with the latter. This is a common assumption made in high-energy particle physics scattering experiments, for instance.

Internal architecture depends on the assumed laws of mechanics, there being three distinct types.

Time-Dependent Laws of Motion

The laws of mechanics for a given SUO could be contextual, changing in some way from stage to stage, or they could be constant. An example of the former scenario is demonstrated by *phase diagrams* in chemistry. These show under which conditions of pressure and temperature a collection of molecules behaves as a gas, a liquid, or a solid. Such behavior is a manifestation of emergent processes driven by reductionist laws of physics, and the challenge is to explain the emergent behavior in each phase using those reductionist laws.

Autonomy or Not
An SUO could be *autonomous*, meaning that it has been effectively isolated from its environment. Alternatively, the exophysical observer could arrange for external agencies, such as electric and magnetic forces, to influence the dynamics of an SUO.

Signality Conservation
Signality is taken in this chapter as a marker of particle number: a signality-p state of a classical register at a given stage will be interpreted as a state with p particles. Newtonian mechanics does not readily countenance changes of particle number. If that has to be considered, then that can be readily modeled by signality nonconserving register dynamics. The reason for this is that register mechanics is more like a field theory than a particle theory.

For the rest of this chapter we shall restrict our attention to autonomous SUOs moving over a succession of classical registers, each of which has the same rank r, with stage-independent laws of dynamics, as these are generally of most interest. In principle, there would be no problem in dealing with other forms of dynamics, including those where the rank of the physical register changes with time. Indeed, that is a common scenario in the quantum register dynamics we shall discuss in the next chapter.

We could also deal with classical stochastic mechanics, which would incorporate Bayesian principles in a natural way, but for the rest of the chapter we shall deal with deterministic laws of mechanics. In the following, the set of integers $\{0, 1, 2, 4, \ldots, 2^r - 1\}$ is denoted $Z^{[r]}$.

We shall use the computational basis representation (CBR) $B_n \equiv \{\boldsymbol{k}_n : k \in Z^{[r]}\}$ to represent the 2^r labstates in a rank r classical register $\mathcal{R}_n$ at stage Σ_n. Consider the temporal evolution of a system from labstate $\boldsymbol{\Psi}_n$ in register $\mathcal{R}_n$ at stage Σ_n to labstate $\boldsymbol{\Psi}_{n+1}$ in register $\mathcal{R}_{n+1}$ at stage Σ_{n+1}. The dynamical rules will be encoded into the expression

$$\boldsymbol{\Psi}_n \rightarrow \boldsymbol{\Psi}_{n+1} \equiv \mathbb{C}_{n+1,n} \boldsymbol{\Psi}_n, \tag{6.3}$$

where $\mathbb{C}_{n+1,n}$ is a classical register operator. Such an operator maps any one of the 2^r labstates in B_n into precisely *one* of the 2^r labstates in B_{n+1}.

The reason for this constraint on any classical register operator comes from the basic principles of CM: states in CM are well-defined, single-valued elements of phase space at any given time. This translates in the present context to the requirement that at any stage Σ_n, the classical state of an SUO is precisely one of the 2^r possible states of the register $\mathcal{R}_n$. In consequence, any admissible classical register operator acts on single elements of $\mathcal{R}_n$ and maps them into single elements of $\mathcal{R}_{n+1}$. Note that this does not rule out nonsurjective or noninjective operators; that is, there may be elements in B_{n+1} that are not mapped into (nonsurjective), and it could be the case that more than one element in B_n may be mapped into the same element in B_{n+1} (noninjective).

Taking any initial state, there are 2^r possible final states. Since there is a total of 2^r possible initial states, we immediately deduce that there is a total of $(2^r)^{2^r}$ different possible classical register operators mapping from $\mathcal{R}_n$ into $\mathcal{R}_{n+1}$.

Even for low-rank registers, the number of possible operators can be impressive. For a rank-one system, we deduce that there should be $2^2 = 4$ different possible such operators. Indeed, that is exactly what we saw in Chapter 3, where we identified the four classical bit operators $\boldsymbol{I}, \boldsymbol{F}, \boldsymbol{U}$, and $\boldsymbol{D}$. The number of possibilities grows rapidly with r. For example, there are 256 different operators that can map a rank-two register into another rank-two register, and a total of $8^8 = 16,777,216$ different operators mapping a rank-three register into another rank-three register.

Given a potentially vast number of possible evolution operators, some criteria need to be imposed in order to reduce the discussion to manageable and realistic proportions. What comes to our assistance here is that most of the possible evolution operators over a classical register will not be useful. Many of them will correspond to irreversible and/or unphysical dynamical evolution and only a small subset will be of interest. We need to find some principles to guide us in our choice of evolution operator.

We turn to standard classical mechanics as discussed in Hamiltonian mechanics. The first thing to note is that classical phase spaces are generally constant in time. This corresponds to taking the rank of successive classical registers to be constant, i.e., $r_n = r_{n+1} \equiv r$ for some integer r and for all n in the temporal interval under discussion. We shall make this assumption from this point on.

Next, we recall that in standard CM of the Hamiltonian variety, Hamilton's equations of motion lead to Liouville's theorem. This tells us that as we track a small volume element along a classical trajectory,[2] this volume remains constant in magnitude though not necessarily constant in shape or orientation. This leads to the idea that a system of many noninteracting particles moving along classical trajectories in phase space behaves as an incompressible fluid, such a phenomenon being referred to as a Hamiltonian flow.

An important characteristic of Hamiltonian flows is that flow lines never cross. We shall encode this idea into our development of classical register mechanics. There are two versions of this mechanics, one of which does not necessarily conserve signality while the other does. We consider the first one now.

6.4 Permutation Flows

A classical rank-r register $\mathcal{R}_n$ contains 2^r labstates denoted by $\boldsymbol{k}_n$, $k \in Z^{[r]}$, in the CBR. Consider a permutation P_n of the integers k, such that under P_n, $k_n \to k_{n+1} \equiv P_n(k_n) \in Z^{[r]}$. Define the evolution of the labstate $\boldsymbol{k}_n$ over one

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[2] A classical trajectory is a phase space trajectory that satisfies Hamilton's equations of motion.

stage by $\boldsymbol{k}_n \rightarrow \boldsymbol{k}_{n+1} \equiv P_n(\boldsymbol{k}_n)$. Such a process is reversible and will be referred to as a *permutation flow*.

Example 6.2   Mathematicians often represent a permutation of, say, five objects in the form $(135)(24)$, which means the permutation $1 \rightarrow 3 \rightarrow 5 \rightarrow 1, 2 \rightarrow 4 \rightarrow 2$, and so on, where the notation $a \rightarrow b \rightarrow c$ means that original element a is replaced by, or goes to, element b, original element b is replaced by element c, and so on. A group of elements within a given pair of brackets is called a *cycle*.

Consider a rank-three classical register $\mathcal{R}^{[3]}$. This has a total of eight states $\{\boldsymbol{k} : k = 0, 1, 2, \ldots, 7\}$. Under the permutation $(0346)(1)(25)(7)$, the evolution is given by

$$0 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 5 \rightarrow 2, 7 \rightarrow 7. \qquad (6.4)$$

This permutation flow does not conserve signality. The simplest proof of this is to note that the signality zero state $\boldsymbol{0}$ evolves into a state with nonzero signality.

There is a total of $n!$ distinct permutations of n objects, so there are $(2^r)!$ possible distinct permutation flow processes. For large r, the number of permutation flows is a rapidly decreasing fraction of the number $(2^r)^{2^r}$ of all possible forms of rank-preserving classical register processes.

Permutation flows are restricted to constant rank registers, and even then, are of debatable value in the following sense. Given two successive classical registers $\mathcal{R}_n$ and $\mathcal{R}_{n+1}$ of the same rank, the relationship between the labeling of states in the two registers is contextual. By this we mean that the identification of element $\boldsymbol{1}_n$ with element $\boldsymbol{1}_{n+1}$, $\boldsymbol{2}_n$ with $\boldsymbol{2}_{n+1}$, and so on, is up to the observer. This is analogous to the parallel transport problem in general relativity, where contextual information relating initial and final tangent spaces along a path is required before a notion of parallelism can be established.

Example 6.3   Consider a permutation written as a transformation from $\mathcal{R}_0$ to $\mathcal{R}_1$ in the form

$$\boldsymbol{i}_0 \rightarrow \boldsymbol{p}(i)_1 : i = 0, 1, 2, \ldots, 2^r - 1. \qquad (6.5)$$

Now suppose that we passively relabel the 2^r states in $\mathcal{R}_1$ by the rule $\boldsymbol{p}(i)_1 \rightarrow \boldsymbol{i}_1$. Then the permutation reduces to the "identity" transformation

$$\boldsymbol{i}_0 \rightarrow \boldsymbol{i}_1, \quad i = 0, 1, 2, \ldots, 2^r - 1. \qquad (6.6)$$

This relabeling seems to have eliminated the significance of permutation dynamics.

This apparent paradox is resolved by *context*: if we had no knowledge of the original permutation (6.5) or of the physical meaning of the original states and

the final states, then (6.6) would indeed be trivial. If we had such contextual information, however, then the relabeling $\boldsymbol{p}(i)_1 \to \boldsymbol{i}_1$ would have no physical significance, and the real dynamics would be understood by the observer.

Permutation flows have the following features with analogues in CM.

Reversibility
Permutations form a group, which means that given a permutation P, then its inverse P^{-1} always exists. Hence permutation flows are reversible.

Orbits
Any SUO evolving under stage-independent permutation dynamics will demonstrate patterns known as *orbits* or *cycles*. Given a permutation of 2^r objects, there will exist subsets known as *cycles* of the objects being permuted such that only elements within a given cycle replace each other under the permutation. This is relevant here because we have chosen to discuss time-independent autonomous systems, the evolution of which is given by repeated applications of the same permutation. Therefore, the structure of the cycles does not change and so each cycle is stable, consisting of the same p elements with a dynamical period p. For example, the identity permutation gives a trivial form of mechanics where nothing changes. It has 2^r cycles each of period 1. At the other end of the spectrum, the permutation denoted by $(0 \to 1 \to 2 \to \cdots \to 2^r - 1 \to 0)$ has no cycles except itself and has period 2^r.

Any physical register evolving under time-independent, autonomous permutation mechanics must return to its initial labstate no later than after 2^r time steps. This is the analogue of the Poincaré recurrence theorem (Poincaré, 1890).

6.5 Signality-Conserving Flows

Most permutation flows will not conserve signality, as Example 6.2 shows. Suppose now we have decided to identify signality with *particle number*. Then experience with Hamiltonian mechanics, where particle conservation is the general rule, leads us to investigate signality-conserving flows.

We can readily identify the subset of the permutation flows that conserve signality by using the signality classes discussed in the previous chapter. Suppose $\mathbb{U}_{n+1,n}$ is an evolution operator with the following characteristics:

1. $\mathbb{U}_{n+1,n}\mathbf{0}_n = \mathbf{0}_{n+1}$: such an evolution is called *calibrated*.
2. $\mathbb{U}_{n+1,n}$ permutes signality-1 states, that is, $\mathbb{U}_{n+1,n}\widehat{\mathbb{A}}_n^i \mathbf{0}_n = \widehat{\mathbb{A}}_{n+1}^{i_P}\mathbf{0}_{n+1}$, for $i = 1, 2, \ldots, r$ and with i_P the number into which i is transformed under the permutation.
3. Likewise for each signality class, until finally, we have for the fully saturated signal state $\mathbb{U}_{n+1,n}\underline{\mathbf{2^r - 1}}_n = \underline{\mathbf{2^r - 1}}_{n+1}$.

Then clearly, signality is conserved under such a dynamical scheme.

The total number of distinct permutations of r objects is $r!$, so there are that many distinct forms of signal permutation dynamics for a rank-r classical register. Since there are $(2^r)!$ distinct forms of permutation dynamics, the set of signal permutation dynamics forms a rapidly decreasing fraction of the set of all possible permutation dynamics.

6.6 Evolution and Measurement

Any experiment consists of several distinct phases. An important phase is the process of measurement itself, which ends with the extraction of classical information from an SUO. Typically this information will be in the form of real numbers, and these can always be expressed in binary form, justifying our approach.

Context plays a vital role here. When, for example, an observer reports that a particle has been observed at position $x = 1.5$, what they mean is that positive signals have been detected at some normal detector or detectors associated with the number $x = 1.5$. This assignment is based on the context of the experiment: the observer will know on the basis of prior theoretical and empirical knowledge, gained during the process of calibration (preparation of apparatus) what those detectors mean in terms of the physics of the SUO concerned, and therefore, what values of some measurable quantity those signals represent.

There is room here for error, in that the observer could associate the wrong context to the signals being observed. Such a process occurred in the Mars Climate Orbiter disaster in 1999, when there was a "failure to use metric units in the coding of a ground software file" (Mars Climate Orbiter Mishap Investigation Board, 1999). In the following, we assume all contexts have been interpreted correctly by the observer.

So far we have discussed the evolution of labstates. For each run or repetition of the experiment, this is modeled by the action of an evolution operator $\mathbb{U}_{N,0}$ mapping initial labstates at stage Σ_0 into final labstates at stage Σ_N. We need now to discuss how numbers are extracted at the end of an experiment consisting of a number of runs.

With reference to the position measurement discussed above, we model the measurement process in terms of *weighted relevant questions*. What this means is this. Suppose the final physical register $\mathcal{R}_N$ has rank r_N. Assuming the experimentalist has established that each detector is working normally, then there will be a total of 2^{r_N} possible normal labstates in this register. Therefore, the observer could ask a total of 2^{r_N} maximal questions. These questions are represented by the dual labstates $\left\{ \overline{\boldsymbol{k}} : k = 0, 1, \ldots, 2^{r_N} - 1 \right\}$. Given a final labstate $\boldsymbol{\Psi}_N$, the answer *yes* or *no* to each question $\overline{\boldsymbol{k}_N} \equiv \textbf{is it true that } \boldsymbol{\Psi}_N = \boldsymbol{k}?$ is represented by the number one or zero, respectively, and given by the answer $\overline{\boldsymbol{k}_N} \boldsymbol{\Psi}_N$.

Now the observer will generally have some theory as to what each answer $\overline{\boldsymbol{k}_N} \boldsymbol{\Psi}_N$ means physically. In many experiments, this will be some real number X^k. Therefore, the actual number $\langle X \rangle_{\Psi_N}$ obtained at time t_N at end of a single run of the experiment can be written in the form

$$\langle X \rangle_{\Psi_N} = \overline{\boldsymbol{\Psi}_N} \mathbb{X}_N \boldsymbol{\Psi}_N, \tag{6.7}$$

where $\mathbb{X}_N \equiv \sum_{k=0}^{2^{r_N}-1} \boldsymbol{k}_N X^k \overline{\boldsymbol{k}_N}$ is an *classical observable*, a sum of dyadics representing a weighted relevant question.

Two comments are relevant. First, despite appearances, this is still a classical theory at this point. The final labstate $\boldsymbol{\Psi}_N$ is a single element in the final physical register, $\mathcal{R}_N$, not a superposition of elements. Second, there is nothing in classical mechanics that rules out weighted sums of dyadics. For any element in $\mathcal{R}_N$, all the possible answers $\overline{\boldsymbol{k}_N}\boldsymbol{\Psi}_N$ are zero except for one of them, so (6.7) returns a physically sensible value for $\langle X \rangle_{\boldsymbol{\Psi}_N}$.

A further refinement, anticipating the possibility of random variations in the initial state and the extension of these ideas to quantum mechanics, is to write

$$\langle X \rangle_{\boldsymbol{\Psi}_N} = Tr\{\mathbb{X}_N \boldsymbol{\varrho}_N\}, \tag{6.8}$$

where Tr represents the familiar trace process and $\boldsymbol{\varrho}_N$ is the dyadic $\boldsymbol{\Psi}_N \overline{\boldsymbol{\Psi}_N}$ analogous to a pure state density operator in quantum mechanics.

We note that $\boldsymbol{\Psi}_N = \mathbb{U}_{N,0}\boldsymbol{\Psi}_0$ and $\overline{\boldsymbol{\Psi}_N} = \overline{\boldsymbol{\Psi}_N}\,\overline{\mathbb{U}}_{N,0}$, where the evolution operator $\mathbb{U}_{N,0}$ maps elements of $\mathcal{R}_0$ into elements of $\mathcal{R}_N$ and similarly for the dual evolution operator $\overline{\mathbb{U}}_{N,0}$. In general, it will be true that

$$\overline{\mathbb{U}}_{N,0}\mathbb{U}_{N,0} = \mathbb{I}_0, \tag{6.9}$$

the identity operator for $\mathcal{R}_0$. However, because there is no requirement formally in this approach for the rank r_N of the final physical register $\mathcal{R}_N$ to equal the rank r_0 of the initial physical register $\mathcal{R}_0$, it is possible that $\mathbb{U}_{N,0}\overline{\mathbb{U}}_{N,0}$ does not equal $\mathbb{I}_N$. This corresponds to irreversible dynamics. In the analogous quantum formalism that we will discuss in the next chapter, such evolution operators are referred to as *semi-unitary*, and $\overline{\mathbb{U}}_{N,0}$ is the *retraction* of $\mathbb{U}_{N,0}$.

Using (6.9) in (6.8), we may write $\langle X \rangle_{\boldsymbol{\Psi}_N} = Tr\{\mathbb{X}_N \mathbb{U}_{N,0}\boldsymbol{\varrho}_0 \overline{\mathbb{U}}_{N,0}\}$, where $\boldsymbol{\varrho}_0$ is the initial dyadic $\boldsymbol{\Psi}_0 \overline{\boldsymbol{\Psi}_0}$.

6.7 Random Initial States

Real experiments normally consist of a large number of repetitions or runs of a basic process. However, it cannot always be guaranteed that the initial labstate is always the same. In principle, we could start with any one of 2^{r_0} initial labstates. In such a case, a statistical approach can be taken.

Consider an extremely large number R of runs, such that there is a total of R^k runs starting with initial labstate $\boldsymbol{k}_0$, for $k = 0, 1, \ldots, 2^{r_0} - 1$. Clearly, $\sum_{k=0}^{2^{r_0}-1} R^k = R$. Then in the limit of R tending to infinity, we would assign a probability $\omega^k \equiv \lim_{R \to \infty} R^k/R$ for the initial labstate to be in state $\boldsymbol{k}_0$.

In such a scenario we define the initial density matrix

$$\boldsymbol{\varrho}_0 \equiv \sum_{k=0}^{2^{r_0}-1} \omega^k \boldsymbol{k}_0 \overline{\boldsymbol{k}_0}, \tag{6.10}$$

where $\boldsymbol{k}_0$ is any one of the 2^{r_0} elements of the initial physical register $\mathcal{R}_0$ and the ω^k are probabilities summing to unity. The formalism outlined above then gives the expectation values of operators.