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The Haar System in the Preduals of Hyperfinite Factors

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Abstract. We shall present examples of Schauder bases in the preduals to the hyperfinite factors of types II_1, II_∞, III_λ, $0 < \lambda \le 1$. In the semifinite (respectively, purely infinite) setting, these systems form Schauder bases in any associated separable symmetric space of measurable operators (respectively, in any non-commutative L^p-space).

1 Introduction

A sequence $\mathbf{x} = \{x_n\}_{n\ge1}$ in a Banach space X is called a (Schauder) basis of X if, for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}_{n\ge1}$ so that

$$x = \sum_{n\ge1} \alpha_n x_n.$$

A sequence $\mathbf{x}$ such that $x_n \ne 0$ for all n and the closed linear span of $\{x_n\}_{n\ge1}$ coincides with X, *i.e.*, such that $[x_n]_{n\ge1} = X$, forms a basis of X if and only if there is a constant c so that for every choice of scalars $\{\alpha_n\}_{j=1}^k$ and integers $m < k$ we have

$$\Big\| \sum_{1\le j\le m} \alpha_j x_j \Big\|_X \le c \Big\| \sum_{1\le j\le k} \alpha_j x_j \Big\|_X \quad \text{(cf. [11]).}$$

The smallest such constant c is called the basis constant of $\mathbf{x}$. In this note we shall be concerned with the construction of Schauder bases in spaces of operators associated with the hyperfinite factors of type II and III_λ, $0 < \lambda \le 1$. In the setting of symmetric spaces of measurable operators affiliated with the hyperfinite factors of type II and with some hyperfinite von Neumann algebras of type I_∞, the problem was recently considered in [4,17,18] where "non-commutative Walsh system", "non-commutative trigonometric system", and "non-commutative Vilenkin systems" were constructed. However, as with their classical counterparts, these systems fail to form a Schauder basis in the preduals. In order to construct a Schauder basis in the preduals to the hyperfinite factors, we use an analogy with another classical function system (which forms a Schauder basis in every separable symmetric function space on $(0,1)$ [9,11]), namely with the Haar system.

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2 Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra with a fixed faithful normal state ρ. Let $\mathcal{M}_*$ stand for the predual of $\mathcal{M}$, we consider this predual as a subspace of $\mathcal{M}^*$ consisting of all normal linear functionals equipped with $\|\cdot\|_* := \|\cdot\|_{\mathcal{M}^*}$, cf. [16, Theorem 1.10].

We consider several different norms on $\mathcal{M}$. $\|\cdot\|$ is the operator norm. The norm $\|\cdot\|_*$ of the predual $\mathcal{M}_*$ induces the norms $\|\cdot\|_\sharp$ and $\|\cdot\|_\flat$ on $\mathcal{M}$ by means of the left and right embeddings $x \in \mathcal{M} \to x\rho := \rho(\,\cdot\, x) \in \mathcal{M}_*$ and $x \in \mathcal{M} \to \rho x := \rho(x\,\cdot\,) \in \mathcal{M}_*$, respectively. These embeddings are injective and the ranges of $\mathcal{M}$ under these embeddings are dense in $\mathcal{M}_*$, [8,15]. Thus, if $\mathcal{M}_\sharp$ and $\mathcal{M}_\flat$ are completions of $\mathcal{M}$ with respect to the norms $\|\cdot\|_\sharp$ and $\|\cdot\|_\flat$, then these spaces are isometric to $\mathcal{M}_*$. Obviously, $\|\cdot\| \geq \|\cdot\|_\sharp$ and $\|\cdot\| \geq \|\cdot\|_\flat$, and the embeddings $\mathcal{M} \subseteq \mathcal{M}_\sharp$ and $\mathcal{M} \subseteq \mathcal{M}_\flat$ are continuous. Let us also note that the space $\mathcal{M}_\sharp$ (resp. $\mathcal{M}_\flat$) is a left (resp. right) module with respect to $\mathcal{M}$, *i.e.,* $xa \in \mathcal{M}_\sharp$ (resp. $ax \in \mathcal{M}_\flat$) provided $a \in \mathcal{M}$ and $x \in \mathcal{M}_\sharp$ (resp. $x \in \mathcal{M}_\flat$); moreover, we have

$$\|xa\|_{\mathcal{M}_\sharp} \leq \|x\|_{\mathcal{M}_\sharp}\,\|a\| \;\; (\text{resp. } \|ax\|_{\mathcal{M}_\flat} \leq \|a\|\,\|x\|_{\mathcal{M}_\flat}).$$

We introduce a left (resp. right) L^p-space associated with the algebra $\mathcal{M}$ as

$$L^p_{\sharp(\text{resp. }\flat)}(\mathcal{M}) := [\mathcal{M}, \mathcal{M}_{\sharp(\text{resp. }\flat)}]_{\frac{1}{p}}, \;\; 1 \leq p \leq \infty.$$

Here, $[\,\cdot\,,\,\cdot\,]_\theta$ is the method of complex interpolation, [2]. The space $L^p_{\sharp(\text{resp. }\flat)}(\mathcal{M})$ is isomorphic to Haagerup's L^p-spaces $L^p(\mathcal{M})$, [20]. Clearly, $L^1_{\sharp(\text{resp. },\flat)}(\mathcal{M}) = \mathcal{M}_{\sharp(\text{resp. }\flat)}$ and $L^\infty_\sharp = L^\infty_\flat = \mathcal{M}$. Moreover, the Hilbert space $L^2_{\sharp(\text{resp. }\flat)}(\mathcal{M})$ coincides with the completion of $\mathcal{M}$ with respect to the inner product $\langle x, y\rangle_\sharp := \rho(y^*x)$ (resp. $\langle x, y\rangle_\flat := \rho(xy^*)$), $x, y \in \mathcal{M}$. We refer the reader to [8, 15] for further details on this construction and also to [20] for the construction of Haagerup's L^p-spaces.

In this note, we shall prove the results for the left norm $\|\cdot\|_\sharp$. The argument for the right norm $\|\cdot\|_\flat$ is generally the same. We shall make appropriate remarks when it is necessary.

We denote by σ^ρ the modular automorphism group for the state ρ, *i.e.,* the unique automorphism group of $\mathcal{M}$ such that

(i) $\rho(x) = \rho(\sigma^\rho_t(x))$, $t \in \mathbb{R}$, $x \in \mathcal{M}$ and
(ii) for every $x, y \in \mathcal{M}$, there is a complex function $f_{x,y}(z)$ bounded in the strip $\bar{S}$ and holomorphic in S, where $S = \{z \in \mathbb{C} : \;\; 0 < \operatorname{Im} z < 1\}$ such that $\rho(\sigma^\rho_t(x)\, y) = f_{x,y}(t)$ and $\rho(y\, \sigma^\rho_t(x)) = f_{x,y}(t + i)$, cf. [7, Section 9.2].

Let $\mathcal{N} \subseteq \mathcal{M}$ be a von Neumann subalgebra such that the modular group σ^ρ (with respect to $\mathcal{M}$) leaves $\mathcal{N}$ globally invariant. In this case the restriction of σ^ρ onto $\mathcal{N}$ gives the modular group of $\rho|_{\mathcal{N}}$ in the algebra $\mathcal{N}$, and we can speak about the modular action σ^ρ without referring to the particular algebra $\mathcal{N}$ or $\mathcal{M}$. In this (and only this) setting, according to the main result of [19], there is a normal conditional expectation $\mathcal{E} \colon \mathcal{M} \to \mathcal{N}$ such that

(a) $\rho(x) = \rho(\mathcal{E}x)$, $x \in \mathcal{M}$;
(b) $\mathcal{E}(axb) = a\mathcal{E}(x)b$, $a, b \in \mathcal{N}$, $x \in \mathcal{M}$;

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(c) $0 \le \mathcal{E}(x)^*\mathcal{E}(x) \le \mathcal{E}(x^*x)$.

We now show that the existence of the conditional expectation implies that

(i) $\mathcal{N}_\sharp$ continuously embeds into $\mathcal{M}_\sharp$ and

(ii) the space $\mathcal{N}_\sharp$ is a 1-complemented subspace in $\mathcal{M}_\sharp$.

Indeed, (i) follows from the inequality

$$\begin{aligned} \|x\|_{\mathcal{M}_\sharp} := \|x\rho\|_{\mathcal{M}_*} &= \sup_{y\in\mathcal{M}} \|y\|^{-1}\,|\rho(yx)| \\ (\text{since } x\in\mathcal{N}) \quad &= \sup_{y\in\mathcal{M}} \|y\|^{-1}\,|\rho(\mathcal{E}(y)x)| \\ (\text{since } \|\mathcal{E}(y)\|\le\|y\|) \quad &\le \sup_{y\in\mathcal{N}} \|y\|^{-1}\,|\rho(yx)| \\ &= \|x\rho\|_{\mathcal{N}_*} = \|x\|_{\mathcal{N}_\sharp}, \ \ x\in\mathcal{N}. \end{aligned} \tag{2.1}$$

For (ii), it is sufficient to show that

$$\|\mathcal{E}(x)\|_{\mathcal{N}_\sharp} \le \|x\|_{\mathcal{M}_\sharp}, \ \ x\in\mathcal{M}. \tag{2.2}$$

Let us recall that the predual $\mathcal{M}_*$ is a subspace of $\mathcal{M}^*$ consisting of all normal linear functionals. Let us consider the mapping $\mathcal{E}'\colon \mathcal{M}_* \to \mathcal{N}_*$ given by $\mathcal{E}'(\phi) = \phi|_{\mathcal{N}}$. It follows from properties (a)–(c) of $\mathcal{E}$ above that

$$\mathcal{E}'(x\rho) = \mathcal{E}(x)\rho \ \text{ and } \ \mathcal{E}'(\rho x) = \rho\mathcal{E}(x), \ \ x\in\mathcal{M}. \tag{2.3}$$

Now (2.2) follows from (2.3), since $\mathcal{E}'$ is a norm one linear operator. It follows from (2.1) and (2.2) that the embedding $\mathcal{N}_\sharp \subseteq \mathcal{M}_\sharp$ is isometric, and therefore the space $\mathcal{N}_\sharp$ is a 1-complemented subspace of $\mathcal{M}_\sharp$.

It this paper, we only consider von Neumann subalgebras $\mathcal{N} \subseteq \mathcal{M}$, which are globally invariant under σ^ρ. Therefore, we shall refer to the norms in the spaces $\mathcal{N}_\sharp$ and $\mathcal{M}_\sharp$ simply as $\|\cdot\|_\sharp$ without specifying the particular algebra.

The assertions above may be similarly carried to the right predual space and to L^p spaces (left and right) by interpolation. That is, the space $L^p_{\sharp(\text{resp. }\flat)}(\mathcal{N})$ is 1-complemented in $L^p_{\sharp(\text{resp. }\flat)}(\mathcal{M})$, $1 \le p \le \infty$, and $L^p_{\sharp(\text{resp. }\flat)}(\mathcal{N})$ embeds isometrically into $L^p_{\sharp(\text{resp. }\flat)}(\mathcal{M})$, $1 \le p \le \infty$.

3 Matrix Spaces

Let $\nu \in \mathbb{N}$ and $0 < \alpha \le \frac{1}{2}$ be fixed throughout the text. Let $\mathcal{N}_\nu$ be the class of all complex $2^\nu \times 2^\nu$-matrices with the unit matrix $\mathbf{1}_\nu$. *Tr* is the standard trace on matrices. The state ρ_ν on $\mathcal{N}_\nu$ is given by

$$\rho_\nu(x) = Tr(x\mathrm{A}_\nu), \qquad x\in\mathcal{N}_\nu, \quad \mathrm{A}_\nu = \bigotimes_{k=1}^{\nu} \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}.$$

The definition of the state ρ_ν immediately implies that

$$\rho_{\nu+\mu}(x\otimes y) = \rho_\nu(x)\,\rho_\mu(y), \ \ x\in\mathcal{N}_\nu, y\in\mathcal{N}_\mu. \tag{3.1}$$

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We consider the ultra-weak, continuous, $*$-isomorphic embedding $i_\nu\colon \mathcal{N}_\nu \to \mathcal{N}_{\nu+1}$ given by

$$i_\nu(x) = x \otimes \mathbf{1}_1, \quad x \in \mathcal{N}_\nu. \tag{3.2}$$

Due to (3.1), we have that $\rho_{\nu+1}(i_\nu(x)) = \rho_\nu(x)$, $x \in \mathcal{N}_\nu$, *i.e.*, the restriction of the state $\rho_{\nu+1}$ onto the subalgebra $i_\nu(\mathcal{N}_\nu)$ is equal to the state ρ_ν.

The modular automorphism group of the state ρ_ν is given by $\sigma_t^{\rho_\nu}(x) = \mathrm{A}_\nu^{it} x \mathrm{A}_\nu^{-it}$, $t \in \mathbb{R}$, $x \in \mathcal{N}_\nu$. Indeed,

$$\rho_\nu(\sigma_t^{\rho_\nu}(x)\, y) = Tr(\mathrm{A}_\nu^{1+it} x \mathrm{A}_\nu^{-it} y) = f_{x,y}(t),$$
$$\rho_\nu(y\, \sigma_t^{\rho_\nu}(x)) = Tr(\mathrm{A}_\nu^{it} x \mathrm{A}_k^{1-it} y) = f_{x,y}(t+i),$$

where the holomorphic function $f_{x,y}$ is given by $f_{x,y}(z) = Tr(\mathrm{A}_\nu^{1+iz} x \mathrm{A}_\nu^{-iz} y)$, $x, y \in \mathcal{N}_\nu$. Therefore, it is readily seen that the group $\sigma^{\rho_{\nu+1}}$ leaves the subalgebra $i_\nu(\mathcal{N}_\nu)$ globally invariant. According to the preceding section, the space $i_\nu(\mathcal{N}_{\nu,\sharp})$ is 1-complemented in $\mathcal{N}_{\nu+1,\sharp}$ and the mapping i_ν embeds $\mathcal{N}_{\nu,\sharp}$ isometrically into $\mathcal{N}_{\nu+1,\sharp}$. We denote $\mathcal{E}_\nu$ the norm one projection $\mathcal{N}_{\nu+1,\sharp} \to i_\nu(\mathcal{N}_{\nu,\sharp})$. From now on, we shall refer to the norms in the spaces $\mathcal{N}_{\nu,\sharp}$ simply as $\|\cdot\|_\sharp$, omitting the index ν.

Similarly, we introduce the L^p-space $\mathcal{L}^p_{\nu,\sharp} := L^p_\sharp(\mathcal{N}_\nu)$, $\nu \geq 1$ and we refer to the norm in this space as $\|\cdot\|_{\sharp,p}$.

We also introduce the p-th Schatten–von Neumann norm $\|\cdot\|_{\mathcal{C}_p}$, $1 \leq p < \infty$ on $\mathcal{N}_\nu$ as

$$\|x\|_{\mathcal{C}_p} := (Tr((x^*x)^{\frac{p}{2}}))^{\frac{1}{p}}, \ \ x \in \mathcal{N}_\nu.$$

Let $\|\cdot\|_{\mathcal{C}_\infty}$ stand for the operator norm. We denote by $\mathcal{C}_p^{(\nu)}$ the matrix space $\mathcal{N}_\nu$ equipped with the norm $\|\cdot\|_{\mathcal{C}_p}$, $1 \leq p \leq \infty$. We now may express the norms $\|\cdot\|_\sharp$ and $\|\cdot\|_\flat$ as

$$\|x\|_\sharp = \|x\mathrm{A}_\nu\|_{\mathcal{C}_1}, \quad \|x\|_\flat = \|\mathrm{A}_\nu x\|_{\mathcal{C}_1}, \quad x \in \mathcal{N}_\nu. \tag{3.3}$$

The last identities may be carried to the L^p-spaces associated with $\mathcal{N}_\nu$ as follows.

Remark 3.1 We fix $\nu \in \mathbb{N}$ and consider the function $f^\sharp\colon \mathbb{C} \times (\mathcal{N}_\nu + \mathcal{N}_{\nu\sharp}) \to \mathcal{C}_\infty^{(\nu)} + \mathcal{C}_1^{(\nu)}$ given by

$$f^\sharp(z,x) = x\mathrm{A}_\nu^z, \quad z \in \mathbb{C}, \quad x \in \mathcal{N}_\nu + \mathcal{N}_{\nu,\sharp}.$$

For every fixed $z \in \mathbb{C}$, $f_z^\sharp(\,\cdot\,) := f^\sharp(z,\,\cdot\,)$ is a linear operator $\mathcal{N}_\nu + \mathcal{N}_{\nu,\sharp} \to \mathcal{C}_\infty^{(\nu)} + \mathcal{C}_1^{(\nu)}$. Thus, we may consider $f^\sharp_{(\,\cdot\,)}$ as a function on the complex plane with values in $B(\mathcal{N}_\nu + \mathcal{N}_{\nu,\sharp}, \mathcal{C}_\infty^{(\nu)} + \mathcal{C}_1^{(\nu)})$. Here $B(X,Y)$ is the Banach space of all bounded linear operators $X \to Y$. The function $f^\sharp_{(\,\cdot\,)}$ is holomorphic on $0 < \operatorname{Re} z < 1$. It follows from (3.3) that the mapping $f^\sharp_{1+it}$ is an isometry between $\mathcal{C}_1^{(\nu)}$ and $\mathcal{N}_{\nu,\sharp}$, for every $t \in \mathbb{R}$. On the other hand, the mapping $f^\sharp_{it}$ is clearly an isometry between $\mathcal{C}_\infty^{(\nu)}$ and $\mathcal{N}_\nu$, for every $t \in \mathbb{R}$. Thus, interpolating, we obtain that the mapping $f^\sharp_{1/p}$ is an isometry between $\mathcal{C}_p^{(\nu)}$ and $\mathcal{L}^p_{\nu,\sharp}$, *i.e.*,

$$\|x\|_{p,\sharp} = \|x\mathrm{A}_\nu^{\frac{1}{p}}\|_{\mathcal{C}_p}, \quad x \in \mathcal{L}^p_{\nu,\sharp}. \tag{3.4}$$

A similar argument for the right spaces gives

$$(3.5)\qquad \|x\|_{p,\flat} = \|\mathrm{A}_\nu^{\frac{1}{p}} x\|_{\mathcal{C}_p}, \quad x \in \mathcal{L}^p_{\nu,\flat}.$$

Let $\mathbf{e}_\nu = \{e_j^{(\nu)}\}_{0\le j<4^\nu}$ be the matrix units in $\mathcal{N}_\nu$ given in the shell enumeration.[1] The system of matrix units $\mathbf{e}_\nu$ forms a basis of $\mathcal{C}_p$ with the basis constant 2, cf. [10, Theorem 2.1]. The next lemma shows that this system remains a basis with the same basis constant 2 with respect to the norms $\|\cdot\|_{p,\sharp}$ and $\|\cdot\|_{p,\flat}$, $1 \le p \le \infty$.

Lemma 3.2 *For every $0 \le m < 4^\nu$, every $1 \le p \le \infty$, and any complex numbers $\alpha_j \in \mathbb{C}$, $0 \le j < 4^\nu$, we have*

$$\Big\| \sum_{0\le j\le m} \alpha_j\, e_j^{(\nu)} \Big\|_{p,\sharp(resp.\ \flat)} \le 2 \Big\| \sum_{0\le j<4^\nu} \alpha_j e_j^{(\nu)} \Big\|_{p,\sharp(resp.\ \flat)}.$$

Proof Let $P_{\nu,m}$ be the basis projection corresponding to the number $0 \le m < 4^\nu$. The projection $P_{\nu,m}$ is a Schur multiplier, *i.e.*,

$$P_{\nu,m}(x) = p_{\nu,m} \circ x,$$

where $\circ$ is the Schur (entrywise) product of matrices and

$$p_{\nu,m} = \sum_{0\le j<m} e_j^{(\nu)}.$$

Let us note that the Schur product is commutative and multiplication by a diagonal matrix is a special case of Schur multiplier. Thus, the claim of the lemma follows from the result [10, Theorem 2.1], the identities (3.3), (3.4), (3.5), and the fact that the operator $P_{\nu,m}$ commutes with left and right multiplication by a diagonal matrix, *i.e.*,

$$P_{\nu,m}(x\mathrm{A}_\nu^{1/p}) = P_{\nu,m}(x)\,\mathrm{A}_\nu^{1/p} \ \text{ and } \ P_{\nu,m}(\mathrm{A}_\nu^{1/p}x) = \mathrm{A}_\nu^{1/p}\,P_{\nu,m}(x). \qquad \blacksquare$$

At the end of the section we establish the explicit formula of the projection $\mathcal{E}_\nu$ on elementary tensors, *i.e.*,

$$(3.6)\qquad \mathcal{E}_\nu(x\otimes y) = \rho_1(y)\, i_\nu(x), \ \ x \in \mathcal{N}_\nu, y \in \mathcal{N}_1.$$

To this end, consider the Hilbert spaces $\mathcal{H}^\sharp_\nu := \mathcal{L}^2_{\nu,\sharp}$, which is the matrix space $\mathcal{N}_\nu$, equipped with the inner product $\langle x, y\rangle_\nu = \rho_\nu(y^*x)$, $x, y \in \mathcal{N}_\nu$ and observe that the projection $\mathcal{E}_\nu$ is an orthogonal projection in the Hilbert space $\mathcal{H}^\sharp_{\nu+1}$ onto the subspace $i_\nu(\mathcal{H}^\sharp_\nu)$. If $\{f_j\}_{0\le j<4^\nu}$ is an orthonormal basis in $\mathcal{H}^\sharp_\nu$, then, using (3.1), we obtain (3.6) as follows

$$\begin{aligned}\mathcal{E}_\nu(x\otimes y) &= \sum_{0\le j<4^\nu} \langle i_\nu(f_j), x\otimes y\rangle_{\nu+1}\, i_\nu(f_j)\\ &= \sum_{0\le j<4^\nu} \rho_1(y)\,\langle f_j, x\rangle_\nu\, i_\nu(f_j)\\ &= \rho_1(y)\, i_\nu(x), \ \ x \in \mathcal{N}_\nu, y \in \mathcal{N}_1.\end{aligned}$$

[1] In the shell enumeration, the couple (k, m) with $k, m \ge 1$ is assigned the index $j(k, m) = (m-1)^2+k$, if $k \le m$ and $k^2 - m + 1$ if $k > m$, see [1, 10].

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4 The Haar System

We shall construct the Haar system on $\mathcal{N}_\nu$ with respect to ρ_ν inductively. Let us first note that we construct two Haar systems: the left and the right Haar system, which coincide when ρ_ν is a tracial state. We shall show the construction of the left Haar system. At the outset, we fix an orthonormal basis of $\mathcal{L}^2_{1,\sharp}$, that is, elements $r_j \in \mathcal{N}_1$, $0 \le j \le 3$ such that

$$\rho_1(r_j^* r_k) = \delta_{jk}, \quad 0 \le j, k \le 3. \tag{4.1}$$

We define the Haar system inductively. The Haar system $\mathbf{h}_1$ in $\mathcal{N}_1$ is the system $\mathbf{h}_1 = \{r_0, r_1, r_2, r_3\}$. If $\mathbf{h}_\nu = \{h_j^{(\nu)}\}_{0\le j<4^\nu}$ is the Haar system in $\mathcal{N}_\nu$, then the system $\mathbf{h}_{\nu+1} = \{h_j^{(\nu+1)}\}_{0\le j<4^{\nu+1}}$ given by

$$h_j^{(\nu+1)} = \begin{cases} i_\nu(h_k^{(\nu)}) \cdot (\mathbf{1}_\nu \otimes r_0), & \text{if } q = 0; \\ i_\nu(e_k^{(\nu)}) \cdot (\mathbf{1}_\nu \otimes r_q), & \text{if } q \ne 0; \end{cases}, \quad 0 \le j < 4^{\nu+1}, \tag{4.2}$$
$$j = 4^\nu q + k, \quad 0 \le q \le 3, \quad 0 \le k < 4^\nu$$

is the Haar system in $\mathcal{N}_{\nu+1}$.

We shall now present an inductive estimate of the basis constant of the system $\mathbf{h}_\nu$ in the space $\mathcal{N}_{\nu,\sharp}$.

Theorem 4.1 *If $c_{\nu,\sharp}$ is the basis constant for $\mathbf{h}_\nu$, then, we have*

$$c_{1,\sharp} \le \sum_{0\le j\le 3} \|r_j\| \, \|r_j\|_\sharp$$

and

$$c_{\nu+1,\sharp} \le \max\Big\{ c_{\nu,\sharp} \|r_0\|^2, \|r_0\|^2 + 2 \sum_{q=1}^{3} \|r_q\|^2 \Big\}.$$

Proof For the first inequality, it is sufficient to note that, if $x = \sum_{j=0}^3 \alpha_j r_j$, then $\alpha_j = \rho_1(r_j^* x)$, $0 \le j \le 3$, see (4.1), therefore, $|\alpha_j| \le \|r_j\| \, \|x\|_\sharp$ and

$$\Big\| \sum_{0\le j\le m} \alpha_j r_j \Big\|_\sharp \le \sum_{0\le j\le 3} |\alpha_j| \, \|r_j\|_\sharp \le \|x\|_\sharp \sum_{0\le j\le 3} \|r_j\| \, \|r_j\|_\sharp, \quad 0 \le m < 4.$$

Let $r_{j,\nu} := \mathbf{1}_\nu \otimes r_j$, $0 \le j \le 3$. We shall estimate the constant $c_{\nu+1,\sharp}$ in the inequality

$$\Big\| \sum_{0\le j\le m} \alpha_j \, h_j^{(\nu+1)} \Big\|_\sharp \le c_{\nu+1,\sharp} \Big\| \sum_{0\le j<4^{\nu+1}} \alpha_j \, h_j^{(\nu+1)} \Big\|_\sharp, \tag{4.3}$$

where $0 \le m < 4^{\nu+1}$. We first establish the estimate

$$\Big\| \sum_{0\le j<4^\nu} \alpha_j \, h_j^{(\nu)} \Big\|_\sharp \le \|r_0\| \Big\| \sum_{0\le j<4^{\nu+1}} \alpha_j \, h_j^{(\nu+1)} \Big\|_\sharp. \tag{4.4}$$

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This inequality follows from the observations that (cf. (3.6), (4.1) and (4.2))

$$i_\nu^{-1}(\mathcal{E}_\nu(r_{0,\nu}^*\, h_j^{(\nu+1)})) = \begin{cases} h_j^{(\nu)}, & \text{if } 0 \le j < 4^\nu; \\ 0, & \text{if } j \ge 4^\nu; \end{cases}$$

and that the left multiplication is a bounded operation in $\|\cdot\|_\sharp$. It follows from (4.4) that, for $0 \le m < 4^\nu$, we have

$$\begin{aligned}\Big\| \sum_{0\le j\le m} \alpha_j h_j^{(\nu+1)} \Big\|_\sharp &\le \|r_0\| \, \Big\| \sum_{0\le j\le m} \alpha_j \, i_\nu(h_j^{(\nu)}) \Big\|_\sharp \\ &= \|r_0\| \, \Big\| \sum_{0\le j\le m} \alpha_j h_j^{(\nu)} \Big\|_\sharp \le c_{\nu,\sharp} \, \|r_0\| \, \Big\| \sum_{0\le j<4^\nu} \alpha_j h_j^{(\nu)} \Big\|_\sharp \\ &\le c_{\nu,\sharp} \, \|r_0\|^2 \, \Big\| \sum_{0\le j<4^{\nu+1}} \alpha_j h_j^{(\nu+1)} \Big\|_\sharp .\end{aligned}$$

Therefore, if $0 \le m < 4^\nu$, then the constant in (4.3) admits the estimate

$$c_{\nu+1,\sharp} \le c_{\nu,\sharp} \, \|r_0\|^2.$$

Let us next establish the estimate

$$\Big\| \sum_{q4^\nu \le j \le q4^\nu + m} \alpha_j \, h_j^{(\nu+1)} \Big\|_\sharp \le 2 \, \|r_q\|^2 \, \Big\| \sum_{0\le j<4^{\nu+1}} \alpha_j \, h_j^{(\nu+1)} \Big\|_\sharp, \tag{4.5}$$

$$1 \le q \le 3, \;\; 0 \le m < 4^\nu.$$

To this end, we observe that, according to (3.6) and (4.1), for $1 \le q \le 3$

$$i_\nu^{-1}(\mathcal{E}_\nu(r_{q,\nu}^*\, h_j^{(\nu+1)})) = \begin{cases} e_{j-q4^\nu}^{(\nu)}, & \text{if } q4^\nu \le j < (q+1)4^\nu; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if $P_{\nu,m}$ is the projection from Lemma 3.2, then the left side can be obtained from the right side in (4.5) via the mapping $x \to r_{q,\nu}\, i_\nu(P_{\nu,m} i_\nu^{-1}(\mathcal{E}_\nu(r_{q,\nu}^*\, x)))$ and, therefore, (4.5) follows.

Finally, if $4^\nu \le m < 4^{\nu+1}$, then, combining (4.4) and (4.5), we obtain the estimate

$$c_{\nu+1,\sharp} \le \|r_0\|^2 + 2 \sum_{q=1}^{3} \|r_q\|^2$$

for the constant in (4.3). The theorem is proved. ∎

Remark 4.2 For the right Haar system the construction is the same, except we start with the system $\{r_0, r_1, r_2, r_3\}$ such that

$$\rho_1(r_j r_k^*) = \delta_{jk}, \ \ 0 \le j, k \le 3. \tag{4.6}$$

Clearly, in the proof of Theorem 4.1, all references to the left multiplication should be replaced with those to the right multiplication. Thus, we obtain that the right Haar system basis constant $c_{\nu,\flat}$ admits the similar estimates:

$$c_{1,\flat} \le \sum_{0 \le j \le 3} \|r_j\| \, \|r_j\|_\flat$$

and

$$c_{\nu+1,\flat} \le \max\Big\{ c_{\nu,\flat} \, \|r_0\|^2, \|r_0\|^2 + 2 \sum_{q=1}^{3} \|r_q\|^2 \Big\}.$$

Remark 4.3 Inspection of the proof of Theorem 4.1 shows that the main ingredients of the proof are

(i) Lemma 3.2 and
(ii) the fact that the left multiplication by a bounded operator is continuous in the norm $\|\cdot\|_\sharp$ uniformly $\nu \ge 1$.

Clearly, both these ingredients hold in the space $\mathcal{L}^p_{\nu,\sharp}$, $1 \le p \le \infty$. Thus, the Haar system (4.2) is a basis in $\mathcal{L}^p_{\nu,\sharp}$. More precisely, the constant $c^{(p)}_{\nu,\sharp}$ that guarantees the inequality

$$\Big\| \sum_{0 \le j \le m} \alpha_j h_j^{(\nu)} \Big\|_{p,\sharp} \le c^{(p)}_{\nu,\sharp} \Big\| \sum_{0 \le j < 4^\nu} \alpha_j h_j^{(\nu)} \Big\|_{p,\sharp},$$

for every $0 \le m < 4^\nu$ and every complex scalars α_j admits the estimate

$$c^{(p)}_{1,\sharp} \le \sum_{0 \le j \le 3} \|r_j\| \, \|r_j\|_{p,\sharp}$$

and

$$c^{(p)}_{\nu+1,\sharp} \le \max\Big\{ c^{(p)}_{\nu,\sharp} \, \|r_0\|^2; \|r_0\|^2 + 2 \sum_{q=1}^{3} \|r_q\|^2 \Big\}.$$

Similar estimates hold true for the right spaces $\mathcal{L}^p_{\nu,\flat}$ and the right Haar system.

Remark 4.4 The construction of the Haar system may be generalized as follows: in the inductive definition (4.2) for each inductive step from $\mathbf{h}_\nu$ to $\mathbf{h}_{\nu+1}$, we can use its own set $\{r_0^{(\nu)}, r_1^{(\nu)}, r_2^{(\nu)}, r_3^{(\nu)}\}$, which possesses the property (4.1) (or (4.6), if we build a right Haar system). Theorem 4.1 remains valid in this case with obvious changes to the estimates of the constants $c_{\nu,\sharp(\text{resp. } \flat)}$.

We shall refer to the system $\mathbf{h}_\nu$ constructed above as $\mathbf{h}_\alpha^{(\nu)}(r_0, r_1, r_2, r_3)$ in the sequel, to stress the fact that the Haar system $\mathbf{h}_\alpha^{(\nu)}$ depends on $0 < \alpha \le \frac{1}{2}$ and $\{r_j\}_{0 \le j \le 3}$.

Corollary 4.5 *The system* $\mathbf{h}_\alpha^{(\nu)}(r_0, r_1, r_2, r_3)$, *where* $\{r_j\}_{0\le j\le 3}$ *satisfies* (4.1) *(resp.,* (4.6)*) and* $\|r_0\| \le 1$, *is a basis in* $\mathcal{N}_{\nu,\sharp}$ *(resp.* $\mathcal{N}_{\nu,\flat}$*) with the basis constant uniformly bounded with respect to* $\nu \in \mathbb{N}$.

5 The Hyperfinite Factors II_1 and III_λ, $0 < \lambda < 1$

The collection of the algebras $\{(\mathcal{N}_\nu, \rho_\nu)\}_{\nu\in\mathbb{N}}$ together with the embedding (3.2) forms *a directed system of* C^**-algebras*, [7, Section 11.4]. *The inductive limit* of this system possesses a state ρ_α, induced by ρ_ν, $\nu \ge 1$. We denote the GNS representation of this inductive limit with respect to the state ρ_α as $\mathcal{R}_\alpha$. $\mathcal{R}_\alpha$ is a factor of type III_λ if $0 < \alpha < \frac{1}{2}$, with $\lambda = \frac{\alpha}{1-\alpha}$ and a factor of type II_1 if $\alpha = \frac{1}{2}$. The properties of the factor $\mathcal{R}_\alpha$ are collected in the following lemma. We also refer the reader to [7, Section 12.3], where the representation of the factor $\mathcal{R}_\alpha$ as a discrete crossed product is given.

Lemma 5.1 *The factor* $\mathcal{R}_\alpha$ *possesses a distinguished faithful normal state* ρ_α. *With* $\mathcal{N}_\nu$, ρ_ν, $\nu \in \mathbb{N}$ *defined in the previous section, there are ultra-weakly continuous* $*$*-isomorphic embeddings* $\pi_\nu \colon \mathcal{N}_\nu \to \mathcal{R}_\alpha$, $\nu \in \mathbb{N}$ *such that*

(i) *the embedding* $i_\nu \colon \mathcal{N}_\nu \to \mathcal{N}_{\nu+1}$, *given in* (3.2), *carries into* $\pi_\nu(\mathcal{N}_\nu) \subseteq \pi_{\nu+1}(\mathcal{N}_{\nu+1})$*;*

(ii) *the state* ρ_ν *is induced by* ρ_α *and* π_ν, i.e., $\rho_\nu(x) = \rho_\alpha(\pi_\nu(x))$, $x \in \mathcal{N}_\nu$, *moreover, the automorphism group* σ^{ρ_α} *leaves every subalgebra* $\pi_\nu(\mathcal{N}_\nu)$ *globally invariant;*

(iii) *the set* $\cup_{\nu\ge 1}\pi_\nu(\mathcal{N}_\nu)$ *is dense in* $\mathcal{R}_\alpha$ *with respect to the weak operator topology.*

From now on we shall identify the algebras $\mathcal{N}_\nu$ with $\pi_\nu(\mathcal{N}_\nu)$, $\nu \in \mathbb{N}$. Since the group σ^{ρ_α} leave the subalgebras $\mathcal{N}_\nu$, $\nu \in \mathbb{N}$ globally invariant, it follows from the preliminaries that, for every $1 \le p \le \infty$, we have

$$\mathcal{L}^p_{1,\sharp} \subseteq \cdots \subseteq \mathcal{L}^p_{\nu,\sharp} \subseteq \mathcal{L}^p_{\nu+1,\sharp} \subseteq \cdots \subseteq L^p_\sharp(\mathcal{R}_\alpha), \tag{5.1}$$

and all embeddings here are isometric. Thus, we may refer to the norms in all these spaces as $\|\cdot\|_{p,\sharp}$, omitting the index $\nu \in \mathbb{N}$. Moreover, these embeddings are 1-complemented, *i.e.,* there is norm one projection

$$\mathcal{E}_\nu \colon L^p_\sharp(\mathcal{R}_\alpha) \to \mathcal{L}^p_{\nu,\sharp}, \ \ \nu \in \mathbb{N}.$$

We also note that $\cup_{\nu\in\mathbb{N}}\mathcal{L}^p_{\nu,\sharp}$ is norm dense in $L^p_\sharp(\mathcal{R}_\alpha)$, $1 \le p < \infty$, since

$$\lim_{\nu\to\infty} \|\mathcal{E}_\nu(x) - x\|_{p,\sharp} = 0, \ \ x \in L^p_\sharp(\mathcal{R}_\lambda), \ \ 1 \le p < \infty. \tag{5.2}$$

The last statement (and its right counterpart) is established in [5, Theorem 8].

Since the space $\mathcal{R}_\alpha$ is not separable, the convergence (5.2) cannot be extended to the norm $\|\cdot\|$. Nonetheless, we have the ultra-weak convergence in this case, namely

Lemma 5.2 *For every* $x \in \mathcal{R}_\alpha$ *and* $\phi \in \mathcal{R}_{\alpha,*}$, *we have* $\lim_{\nu\to\infty} \phi(\mathcal{E}_\nu(x) - x) = 0$.

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Proof The proof is straightforward. First, from (5.2), we have

$$|\rho_\alpha(y\,(\mathcal{E}_\nu(x)-x))| \leq \|y\|\,\|\mathcal{E}_\nu(x)-x\|_\sharp \to 0, \text{ as } \nu\to\infty, \ \ x,y\in\mathcal{R}_\alpha.$$

The latter means that we proved the lemma for the special case $\phi = \rho y$, $y \in \mathcal{R}_\alpha$. Since the linear subspace $\{\rho y\}_{y\in\mathcal{R}_\alpha}$ is norm dense in $\mathcal{R}_{\alpha,*}$ and the projections $\mathcal{E}_\nu\colon \mathcal{R}_\alpha \to \mathcal{N}_\nu$ are uniformly bounded, $\nu\in\mathbb{N}$, the general case now follows from [16, Lemma 1.2]. ∎

Alternatively, we may look at the tower (5.1) from an *inductive limit* point of view as follows (see [13, p. 135] for the definition of inductive limits of Banach spaces).

Theorem 5.3 *If* $1 \leq p < \infty$, *the collection of Banach spaces* $\{\mathcal{L}^p_{\nu,\sharp}\}_{\nu\in\mathbb{N}}$ *together with the embedding* (3.2) *form* a directed system of Banach spaces. *The inductive limit of this system is isomorphic to* $\mathcal{L}^p_\sharp(\mathcal{R}_\alpha)$.

Let us further note that the identities (3.3) and (3.4) mean that the Banach spaces $\mathcal{L}^p_{\nu,\sharp}$ and $\mathcal{C}^{(\nu)}_p$ are isometric with the isometry given by $x \in \mathcal{L}^p_{\nu,\sharp} \to x\mathrm{A}^{1/p}_\nu \in \mathcal{C}^{(\nu)}_p$. Applying this isometry to the directed system $\{\mathcal{L}^p_{\nu,\sharp}\}_{\nu\in\mathbb{N}}$ together with the embedding i_ν, we obtain the following.

Corollary 5.4 *The collection of matrix spaces* $\{\mathcal{C}^{(\nu)}_p\}_{\nu\in\mathbb{N}}$, $1 \leq p < \infty$, *equipped with the* p*-th Schatten-von Neumann norm, together with the embedding*

$$x \in \mathcal{C}^{(\nu)}_p \to x\otimes A_1^{1/p} \in \mathcal{C}^{(\nu+1)}_p$$

is a directed system of Banach spaces with the inductive limit isomorphic to Haagerup's space $L^p(\mathcal{R}_\alpha)$.

6 Haar System (cont.)

Consider the left (resp. right) Haar system $\mathbf{h}^{(\nu)}_\alpha = \mathbf{h}^{(\nu)}_\alpha(r_0,r_1,r_2,r_3)$, where the system $\{r_j\}_{0\leq j\leq 3}$ satisfies (4.1) (resp. (4.6)) such that $r_0 = \mathbf{1}_1$. It then follows from (3.2) and (4.2) that $h^{(\nu+1)}_j = i_\nu(h^{(\nu)}_j)$, $0 \leq j < 4^\nu$, $\nu\in\mathbb{N}$. Thus, we can construct a unified left (resp. right) Haar system $\mathbf{h}_\alpha = \mathbf{h}_\alpha(r_0,r_1,r_2,r_3) = \{h_j\}_{j\geq 0}$ in $\mathcal{R}_\alpha$ as $h_j = \pi_\nu(h^{(\nu)}_j)$, provided $0 \leq j < 4^\nu$. As a corollary of Theorem 4.1, we now have the following.

Theorem 6.1 *The left (resp. right) Haar system* $\mathbf{h}_\alpha$ *forms a basis in the space* $\mathcal{R}_{\alpha,\sharp(\mathit{resp.}\ \flat)}$.

Proof To prove that $\mathbf{h}_\alpha$ is a basis, we need to prove

(i) $\mathbf{h}_\alpha$ is a basic sequence and
(ii) the linear span of $\mathbf{h}_\alpha$ is dense in $\mathcal{R}_{\alpha,\sharp}$ (resp. $\mathcal{R}_{\alpha,\flat}$).

The first part is contained in Theorem 4.1 and the second one is guaranteed by (5.2). ∎

As an example of the system $\mathbf{h}_\alpha^{(1)}$, we now take the system

$$r_0 = \mathbf{1}_1,\ r_1 = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix},\ r_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ r_3 = \begin{bmatrix} 0 & -\sqrt{\lambda} \\ \frac{1}{\sqrt{\lambda}} & 0 \end{bmatrix}, \tag{6.1}$$

where $\lambda = \frac{\alpha}{1-\alpha}$. The system $\mathbf{h}_\alpha^{(1)}$ satisfies (4.1) (resp. (4.6), if r_3 replaced with r_3^*). Thus, from Theorem 6.1, we have the following.

Corollary 6.2 *The left (resp. right) Haar system* $\mathbf{h}_\alpha = \mathbf{h}_\alpha(r_0, r_1, r_2, r_3)$, *where* $\{r_j\}_{0\le j\le 3}$ *are given in* (6.1), *is a basis in the predual of the hyperfinite factor* III_λ.

For the special case $\alpha = \frac{1}{2}$, the system (6.1) turns into

$$\hat{r}_0 = \mathbf{1}_1,\ \hat{r}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \hat{r}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \hat{r}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and we also have the following.

Corollary 6.3 *The Haar system* $\mathbf{h}_{\frac{1}{2}} = \mathbf{h}_{\frac{1}{2}}(\hat{r}_0, \hat{r}_1, \hat{r}_2, \hat{r}_3)$ *is a basis in the predual of the hyperfinite factor* II_1.

Let us next consider the diagonal subalgebras $\mathcal{A}_\nu \subseteq \mathcal{N}_\nu$, $\nu \in \mathbb{N}$. The weak-operator closure of $\cup_{\nu\in\mathbb{N}}\pi_\nu(\mathcal{A}_\nu)$ forms an Abelian subalgebra $\mathcal{A}_\alpha$ in $\mathcal{R}_\alpha$, which is isomorphic to $L_\infty([0,1), m_\alpha)$, cf. [7, Section 12.3], the algebra of all essentially bounded m_α-measurable functions on $[0,1)$, where the measure m_α is given by

$$m_\alpha\Big(\Big[\frac{k}{2^\nu}, \frac{k+1}{2^\nu}\Big]\Big) = \prod_{s=0}^{\nu-1}\big[(1-\epsilon_s)\alpha + \epsilon_s(1-\alpha)\big],$$

where $0 \le k < 2^\nu$ and ϵ_s are binary digits of k, *i.e.*, $\epsilon_s = 0, 1$ such that

$$k = \epsilon_0 2^0 + \epsilon_1 2^1 + \cdots + \epsilon_{\nu-1}2^{\nu-1}.$$

Since $\mathcal{A}_\alpha$ is commutative, we have that $\mathcal{A}_{\alpha,\sharp} = \mathcal{A}_{\alpha,\flat} = \mathcal{A}_{\alpha,*}$. The modular automorphism group σ^{ρ_α} (resp. σ^{ρ_ν}) leaves the subalgebra $\mathcal{A}_\alpha$ (resp. $\mathcal{A}_\nu$) globally invariant. Thus, the embedding $\mathcal{A}_{\alpha,*} \subseteq \mathcal{R}_{\alpha,\sharp(\text{resp. }\flat)}$ (resp. $\mathcal{A}_{\nu,*} \subseteq \mathcal{N}_{\nu,\sharp(\text{resp. }\flat)}$) is isometric and complemented. Let $\mathcal{E}$ be the norm one projection $\mathcal{E}\colon \mathcal{R}_\alpha \to \mathcal{A}_\alpha$. We denote by the same letter $\mathcal{E}$ the norm one projection $\mathcal{E}\colon \mathcal{R}_{\alpha,\sharp(\text{resp. }\flat)} \to \mathcal{A}_{\alpha,*}$. The projection $\mathcal{E}\colon \mathcal{N}_\nu \to \mathcal{A}_\nu$ vanishes on all non-diagonal matrix entries. Hence, we obtain that, if $\mathbf{h}_\alpha = \mathbf{h}_\alpha(r_0, r_1, r_2, r_3) = \{h_j\}_{j\ge 0}$ is the left (or right) Haar system, with respect to (6.1), then

(6.2) $\mathcal{E}(h_j) = h_j$, if $4^\nu \le j < 2 \cdot 4^\nu$ and $h_j = i_\nu(e_k^{(\nu)}) \cdot (\mathbf{1}_\nu \otimes r_1)$ with $k = j - 4^\nu$ and $e_k^{(\nu)}$ being a diagonal matrix unit, see (4.2);

$\mathcal{E}(h_j) = 0$, otherwise.

Clearly, this implies that the non-zero subsystem $\chi_\alpha = \{\chi_j\}_{j\ge 0}$ of $\mathcal{E}(\mathbf{h}_\alpha)$ forms a basis of $\mathcal{A}_{\alpha,*}$. From (4.2) and (6.2), we obtain that $\chi_0 = r_0$, $\chi_1 = r_1$ and if $2^\nu \le j < 2^{\nu+1}$,

$$\chi_j = \varepsilon_k^{(\nu)} \cdot (\mathbf{1}_\nu \otimes r_1), \quad j = 2^\nu + k, \ \ 0 \le k < 2^\nu, \tag{6.3}$$

where $\varepsilon_k^{(\nu)}$ is the k-th diagonal matrix unit in $\mathcal{N}_\nu$. When $\alpha = \frac{1}{2}$, the system $\chi_{\frac{1}{2}}$ is the classical Haar system, cf. [12, Section 2.c]. Thus, we have

Corollary 6.4 *The system χ_α, given in* (6.3) *is a basis of* $L_1([0,1), m_\alpha)$. *In particular, for $\alpha = \frac{1}{2}$ this system coincides with the classical Haar system on* $L_1(0,1)$, [12, Section 2.c].

Remark 6.5 Every result in this section extends to the L^p-spaces associated with the factors III_λ and II_1. If $p = \infty$, then the results still hold true with norm convergence replaced by ultra-weak convergence, cf. Lemma 5.2.

Remark 6.6 In analogy to the classical Haar system, we shall call the system (6.1) and its derivatives $r_{j,\nu} = \mathbf{1}_\nu \otimes r_j$, $0 \le j \le 3$, $\nu \in \mathbb{N}$ *the (non-commutative) Rademacher system.* Due to unconditionality of martingale differences in the spaces $L^p(\mathcal{R}_{\frac{1}{2}})$, $1 < p < \infty$, [14, 17, 18], the Rademacher system is an unconditional basis sequence in $L^p(\mathcal{R}_{\frac{1}{2}})$, $1 < p < \infty$.

7 Factors of Type III_1 and II_∞

Here we shall consider the construction of bases in the preduals of the factors of type III_1 and II_∞. Since these two factors may be reduced to the factors of type III_λ, II_1, and I_∞ by means of tensor products, we shall first consider the extension of the Haar system construction over preduals of tensor products. To this end, it is useful to recall the notion of *Schauder decomposition*, [11].

Let $\mathbf{D} = \{D_j\}_{j\ge1}$ be a system of projections in a Banach space X such that $D_jD_k = 0$, $j \ne k$. The system $\mathbf{D}$ is a Schauder decomposition of the Banach space X if and only if the series $\sum_{j=1}^\infty D_jx$ converges to x in the norm of X, for every $x \in X$. As for bases, we have the equivalent criteria for the system $\mathbf{D}$ to be a Schauder decomposition of X, [11]. Namely, a system $\mathbf{D} = \{D_j\}_{j\ge1}$, $D_jD_k = 0$, $j \ne k$ is a Schauder decomposition of X if and only if (i) $[D_j(X)]_{j\ge1} = X$; (ii) there is a constant c such that

$$\Big\| \sum_{1\le j\le m} D_jx \Big\|_X \le c \Big\| \sum_{1\le j\le n} D_jx \Big\|_X, \quad x \in X, \ 1 \le m \le n.$$

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We fix two von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$ equipped with faithful normal states ρ and ϕ, respectively. The tensor product algebra $\mathcal{N}\bar{\otimes}\mathcal{M}$ with respect to the product state $\rho \otimes \phi$ is the weak operator completion of the GNS representation of the tensor product C^*-algebra $\mathcal{N} \otimes \mathcal{M}$ with respect to the state $\rho \otimes \phi$, cf. [7, Chapter 11].

We shall consider the algebra $\mathcal{N}$ as a von Neumann subalgebra of $\mathcal{N} \otimes \mathcal{M}$ under the embedding $x \to x \otimes \mathbf{1}$, $x \in \mathcal{N}$. It clearly follows from the modular condition that

$$\sigma_t^{\rho\otimes\phi}(x \otimes y) = \sigma_t^{\rho}(x) \otimes \sigma_t^{\phi}(y), \quad x \in \mathcal{N}, y \in \mathcal{M}, t \in \mathbb{R}.$$

Thus, the modular group $\sigma^{\rho\otimes\phi}$ leaves subalgebra $\mathcal{N}$ globally invariant, and therefore, the results in the preliminaries are applicable. In particular, the left predual $\mathcal{N}_\sharp$ isometrically embeds into $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$, and the space $\mathcal{N}_\sharp$ is 1-complemented in $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$. Let us denote the corresponding projection as $\mathcal{E}_{\mathcal{N}}$. As in (3.6), we obtain the explicit formula for the projection $\mathcal{E}_{\mathcal{N}}$ on elementary tensors

$$\mathcal{E}_{\mathcal{N}}(x \otimes y) = (x \otimes \mathbf{1})\, \phi(y), \quad x \in \mathcal{N}, \ y \in \mathcal{M}. \tag{7.1}$$

Let us fix an orthonormal basis $\mathbf{y} = \{y_j\}_{j\geq 1} \subseteq \mathcal{M}$ in the predual $\mathcal{M}_\sharp$, $i = 1, 2$. We assume that

$$\phi(y_j^* y_k) = \delta_{jk}. \tag{7.2}$$

Having basis $\mathbf{y}$ and the expectation $\mathcal{E}_{\mathcal{N}}$ at our disposal, we can construct the associated system of projections $\mathbf{D} = \{D_j\}_{j\geq 1}$ of the predual $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$ by

$$D_j z = (\mathbf{1} \otimes y_j)\, \mathcal{E}_{\mathcal{N}}((\mathbf{1} \otimes y_j)^*\, z), \quad z \in \mathcal{N}\bar{\otimes}\mathcal{M}. \tag{7.3}$$

The left multiplication by an element of the tensor product $\mathcal{N}\bar{\otimes}\mathcal{M}$ is a bounded operator on $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$. Therefore, the operators D_j are indeed bounded linear operators on $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$. The fact that D_j are projections and that $D_j D_k = 0$ if $j \neq k$ follows from (7.1) and (7.2). Furthermore, the identities (7.1) and (7.2) give the explicit formula for the projection D_j on the algebraic tensor product $\mathcal{N} \otimes \mathcal{M}$. Indeed, if $a_k \in \mathcal{N}$, $b_k \in \mathcal{M}$, $1 \leq k \leq n$ and

$$b_k = \sum_{s\geq 1} \alpha_{ks}\, y_s, \quad \alpha_{ks} \in \mathbb{C}$$

is the expansion of the element b_k with respect to the system $\mathbf{y}$ in $\mathcal{M}_\sharp$, then

$$D_j\Big(\sum_{1\leq k\leq n} a_k \otimes b_k\Big) = \sum_{1\leq k\leq n} \alpha_{kj}\, a_k \otimes y_j. \tag{7.4}$$

Since the algebraic tensor product $\mathcal{N} \otimes \mathcal{M}$ is norm dense in $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$ and the norm in the space $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$ is a cross-norm, we get $[D_j(\mathcal{N}\bar{\otimes}\mathcal{M})]_{j\geq 1} = (\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$. In general, it is not the case that the system $\mathbf{D}$ constructed above is a Schauder decomposition of the Banach space $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$.

Theorem 7.1 *If $\mathbf{x} = \{x_j\}_{j\geq 1}$ and $\mathbf{y} = \{y_k\}_{k\geq 1}$ are two bases in $\mathcal{N}_\sharp$ and $\mathcal{M}_\sharp$, respectively, such that the associated systems of projections $\mathbf{E} = \{E_j\}_{j\geq 1}$ and $\mathbf{D} = \{D_k\}_{k\geq 1}$, defined in* (7.3)*, are Schauder decompositions of the Banach space $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$, then the product basis $\mathbf{z} = \mathbf{x}\otimes\mathbf{y} := \{x_j \otimes y_k\}_{j,k\geq 1}$, taken in the shell enumeration, is a basis in the predual $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$.*

Proof The proof is rather standard. The shell enumeration assigns to a pair (j,k), $j,k \geq 1$ the number $s(j,k)$ defined by

$$s(j,k) = \begin{cases} (k-1)^2 + j, & \text{if } j \leq k; \\ j^2 - k + 1, & \text{if } j > k. \end{cases}$$

Let $\mathbf{z} = \{z_s\}_{s\geq 1}$ be the system $\mathbf{z}$ in the shell enumeration. It is clear that the linear span of the system $\mathbf{z}$ is dense in $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$. Thus, to prove the theorem, we have to establish, that there is a constant c such that

$$\Big\| \sum_{1\leq s\leq m} \alpha_s z_s \Big\|_\sharp \leq c \Big\| \sum_{1\leq s\leq n} \alpha_s z_s \Big\|_\sharp, \quad \alpha_s \in \mathbb{C}, 1 \leq m \leq n.$$

Without loss of generality we may assume that $n = n_1^2$, for some $n_1 \geq 1$. There is an integer $m_1 \geq 1$ such that either of the relations is true (i) $m_1^2 + 1 \leq m \leq m_1^2 + m_1$ or (ii) $m_1^2 + m_1 + 1 \leq m \leq (m_1+1)^2$. Let us consider the first option. For the second one the argument is similar. We have

$$\begin{aligned} \Big\| \sum_{1\leq s\leq m} \alpha_s z_s \Big\|_\sharp &\leq \Big\| \sum_{1\leq s\leq m_1^2} \alpha_s z_s \Big\|_\sharp + \Big\| \sum_{m_1^2+1\leq s\leq m} \alpha_s z_s \Big\|_\sharp \\ &= \Big\| \sum_{1\leq j,k\leq m_1} \alpha_{s(j,k)}\, x_j \otimes y_k \Big\|_\sharp \\ &\quad + \Big\| \sum_{1\leq j\leq m-m_1^2-1} \alpha_{s(j,m_1)}\, x_j \otimes y_{m_1} \Big\|_\sharp. \end{aligned} \tag{7.5}$$

Letting

$$z = \sum_{1\leq s\leq n} \alpha_s z_s = \sum_{1\leq j,k\leq n_1} \alpha_{s(j,k)} x_j \otimes y_k,$$

it then follows from (7.4) that the latter two terms on the right-hand side of (7.5) are given by

$$\sum_{1\leq j,k\leq m_1} \alpha_{s(j,k)}\, x_j \otimes y_k = P_{m_1} Q_{m_1}(z)$$

and

$$\sum_{1\leq j\leq m-m_1^2-1} \alpha_{s(j,m_1)}\, x_j \otimes y_{m_1} = P_{m-m_1^2-1} D_{m_1}(z),$$

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where P_j and Q_k are partial sum projections with respect to the decompositions **E** and **D**, respectively, *i.e.*, $P_j = \sum_{1\le s\le j} E_s$ and $Q_k = \sum_{1\le s\le k} D_s$. Thus, we continue

$$\Big\| \sum_{1\le s\le m} \alpha_s z_s \Big\|_\sharp \le \|P_{m_1} Q_{m_1}(z)\|_\sharp + \|P_{m-m_1^2-1} D_{m_1}(z)\|_\sharp \le c\, \|z\|_\sharp.$$

The latter inequality is due to the fact that the partial sum projections are uniformly bounded. The claim of the theorem follows. ■

We consider two specific examples. First, we take $\mathcal{M} = B(\ell_n^2)$. In this setting the algebra $\mathcal{N} \otimes \mathcal{M}$ may be considered as the space of all bounded $n \times n$-block matrices with entries in $\mathcal{N}$. The latter observation means, in particular, that the system of projections $\mathbf{E} = \{E_j\}_{j\ge 1}$, (7.3), associated with the matrix unit basis **e** is a Schauder decomposition of the predual $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$; see the proof of Lemma 3.2. Let us also note that the Schauder constant of the system **E** is uniformly bounded with respect to the dimension of ℓ_n^2.

As another example, we take $\mathcal{M} = \mathcal{R}_\alpha$, $0 < \alpha \le \frac{1}{2}$. We fix the Haar system $\mathbf{h}_\alpha = \mathbf{h}_\alpha(r_0, r_1, r_2, r_3) = \{h_j\}_{j\ge 1}$, where the Rademachers $\{r_j\}_{0\le j\le 3}$ are given in (6.1). We denote the associated system of projections by $\mathbf{H}_\alpha = \mathbf{H}_\alpha(r_0, r_1, r_2, r_3) = \{H_j\}_{j\ge 1}$. To establish that the system $\mathbf{H}_\alpha$ is a Schauder decomposition in $(\mathcal{N}\bar{\otimes}\mathcal{M})_\sharp$ we only need to verify that there is a constant c such that

$$\Big\| \sum_{1\le j\le m} H_j z \Big\|_\sharp \le c \Big\| \sum_{1\le j\le n} H_j z \Big\|_\sharp, \ \ z \in \mathcal{N} \otimes \mathcal{M}, \ \ 1 \le m \le n.$$

The proof of the latter inequality is based on the following theorem.

Theorem 7.2 *Let $(\mathcal{N}_\nu, \rho_\nu)$ be the algebras from the preceding sections and let $\mathcal{N}\bar{\otimes}\mathcal{N}_\nu$ be the tensor product von Neumann algebras equipped with the product states $\rho \otimes \rho_\nu$, $\nu \in \mathbb{N}$. Let $\mathbf{h}_\alpha^{(\nu)} = \mathbf{h}_\alpha^{(\nu)}(r_0, r_1, r_2, r_3) = \{h_j^{(\nu)}\}_{j\ge 1}$, where $\{r_j\}_{0\le j\le 3}$ satisfies* (4.1)*, be the left Haar system in $\mathcal{N}_\nu$. If $\mathbf{H}_\alpha^{(\nu)} = \mathbf{H}_\alpha^{(\nu)}(r_0, r_1, r_2, r_3)$ is the associated decomposition defined by* (7.3)*, then the minimal constant c, which guarantees the inequality*

$$\Big\| \sum_{0\le j<m} H_j^{(\nu)} z \Big\|_\sharp \le c_{\nu,\sharp} \Big\| \sum_{0\le j<4^\nu} H_j^{(\nu)} z \Big\|_\sharp, \ \ z \in \mathcal{N}\bar{\otimes}\mathcal{N}_\nu, \ 0 \le m < 4^\nu,$$

admits the same inductive estimate as that in Theorem 4.1.

The proof of Theorem 7.2 is essentially a repetition of that of Theorem 4.1; we leave details to the reader. Thus, we obtain the following.

Theorem 7.3 *The system $\mathbf{H}_\alpha = \mathbf{H}_\alpha(r_0, r_1, r_2, r_3) = \{H_j\}_{j\ge 1}$, associated with the Haar basis $\mathbf{h}_\alpha(r_0, r_1, r_2, r_3)$, where $\{r_j\}_{0\le j\le 3}$ is given by* (6.1)*, is a Schauder decomposition of the Banach space $(\mathcal{N}\bar{\otimes}\mathcal{R}_\alpha)_\sharp$.*

Now we may apply the results above to the hyperfinite factors III_1 and II_∞. It is known that $\mathrm{III}_1 = \mathrm{III}_{\lambda_1}\bar{\otimes}\mathrm{III}_{\lambda_2}$, where $\log\lambda_1/\log\lambda_2 \notin \mathbb{Q}$ and $\mathrm{II}_\infty = \mathrm{II}_1\bar{\otimes}\mathrm{I}_\infty$, cf. [3,6]. See the definitions of $\mathbf{h}_\alpha$ and $\mathbf{h}_{\frac{1}{2}}$ in Corollaries 6.2 and 6.3. We obtain the following.

Corollary 7.4 *The system* $\mathbf{z} = \mathbf{h}_{\alpha_1} \otimes \mathbf{h}_{\alpha_2}$, $\alpha_i = \frac{\lambda_i}{\lambda_i + 1}$, $i = 1, 2$, $\frac{\log \lambda_1}{\log \lambda_2} \notin \mathbb{Q}$, *is a basis in the predual of the hyperfinite factor of type* III_1.

Corollary 7.5 *The system* $\mathbf{x} = \mathbf{h}_{\frac{1}{2}} \otimes \mathbf{e}$, *is a basis in the predual of the hyperfinite factor of type* II_∞.

Remark 7.6 Similarly to the preceding sections, all the results above hold true in the setting of left and right L^p-spaces associated with the factors of type III_1 and II_∞. Moreover, the system $\mathbf{h}_{\frac{1}{2}}$ (resp. $\mathbf{h}_{\frac{1}{2}} \otimes \mathbf{e}$) from Corollary 6.3 (resp. 7.5) forms a basis in any symmetric operator space $E(\mathcal{R}_{\frac{1}{2}})$ (resp. $E(\mathcal{M})$), where $\mathcal{R}_{\frac{1}{2}}$ (resp. $\mathcal{M}$) is a hyperfinite factor II_1 (resp. II_∞) and E is separable rearrangement invariant function space (see definitions and further references in [4]).

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References

[1] J. Arazy, *Some remarks on interpolation theorems and the boundness of the triangular projection in unitary matrix spaces.* Integral Equations Operator Theory **1**(1978), no. 4, 453–495. doi:10.1007/BF01682937

[2] J. Bergh and J. Löfström, *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, 223, Springer-Verlag, Berlin-New York, 1976.

[3] A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. of Math. (2) **104**(1976), no. 1, 73–115. doi:10.2307/1971057

[4] P. G. Dodds, S. V. Ferleger, B. de Pagter, and F. A. Sukochev, *Vilenkin systems and generalized triangular truncation operator.* Integral Equations Operator Theory **40**(2001), no. 4, 403–435. doi:10.1007/BF01198137

[5] S. Goldstein, *Conditional expectations in* L^p*-spaces over von Neumann algebras.* In: Quantum probability and applications, II (Heidelberg, 1984), Lecture Notes in Math., 1136, Springer, Berlin, 1985, pp. 233–239.

[6] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1. Acta Math. **158**(1987), no. 1–2, 95–148. doi:10.1007/BF02392257

[7] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory.* Graduate Studies in Mathematics, 16, American Mathematical Society, Providence, RI, 1997.

[8] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: noncommutative* L^p*-spaces.* J. Funct. Anal. **56**(1984), no. 1, 29–78. doi:10.1016/0022-1236(84)90025-9

[9] S. G. Kreĭn, Y. Ī. Petunīn, and E. M. Semënov, *Interpolation of linear operators*, Translations of Mathematical Monographs, 54, American Mathematical Society, Providence, RI, 1982.

[10] S. Kwapień and A. Pełczyński, *The main triangle projection in matrix spaces and its applications.* Studia Math. **34**(1970), 43–68.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I. Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 92, Springer-Verlag, Berlin-New York, 1977.

[12] ———, *Classical Banach spaces. II. Function spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 97, Springer-Verlag, Berlin-New York, 1979.

[13] G. Pisier, *Factorization of linear operators and geometry of Banach spaces.* CBMS Regional Conference Series in Mathematics, 60, American Mathematical Society, Providence, RI, 1986.

[14] G. Pisier and Q. Xu, *Non-commutative martingale inequalities.* Comm. Math. Phys. **189**(1997), no. 3, 667–698. doi:10.1007/s002200050224

[15] Y. Raynaud, L_p*-spaces associated with a von Neumann algebra without trace: a gentle introduction via complex interpolation.* In: Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math., 321, American Mathematical Society, Providence, RI, 2003, pp. 245–273.

[16] Ş. Strătilă and L. Zsidó, *Lectures on von Neumann algebras.* Editura Academiei, Bucharest, 1979.

[17] F. A. Sukochev and S. V. Ferleger, *Harmonic analysis in symmetric spaces of measurable operators. (Russian)* Dokl. Akad. Nauk **339**(1994), no. 3, 307–310.

[18] ———, *Harmonic analysis in UMD-spaces: applications to basis theory. (Russian)* Mat. Zametki **58**(1995), no. 6, 890–905, 960.

[19] M. Takesaki, *Conditional expectations in von Neumann algebras.* J. Funct. Anal. **9**(1972), 306–321. doi:10.1016/0022-1236(72)90004-3

[20] M. Terp, *L^p-spaces associated with von Neumann algebras.* Copenhagen University, 1981.

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