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LIMIT POINT CRITERIA FOR DIFFERENTIAL EQUATIONS

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Introduction. For certain ordinary differential operators L of order $2n$, this paper considers the problem of determining the number of linearly independent solutions of class $L_2[a, \infty)$ of the equation $L(y) = \lambda y$. Of central importance is the operator

$$(0.1) \qquad L(y) = (-1)^n(p_0 y^{(n)})^{(n)} + (-1)^{n-1}(p_1 y^{(n-1)})^{(n-1)} + \ldots + p_n y$$

where the coefficients p_i are real. For this L, classical results give that the number m of linearly independent $L_2[a, \infty)$ solutions of $L(y) = \lambda y$ is the same for all non-real λ, and is at least n [**10**, Chapter V]. When $m = n$, the operator L is said to be in the *limit-point* condition at infinity. We consider here conditions on the coefficients p_i of L which imply $m = n$. These conditions are in the form of limitations on the growth of the coefficients.

For $n = 1$ in (0.1), numerous limit point criteria for (0.1) have been given. Notable among these are the criteria of N. Levinson [**8**] and E. C. Titchmarsh [**11**]. The fourth-order equation has been less investigated. However, effective limit point criteria have been given by W. N. Everitt [**5**; **6**] and W. N. Everitt and J. Chanduri [**7**].

Limit point criteria for (0.1) for $n \geqq 2$ obtained by the use of asymptotic methods have long been known (cf. [**10**, Chapter VIII]). Such criteria usually require considerable differentiability on the coefficients, and in addition to calculating the deficiency indices, give information on the asymptotic behavior of all solutions. A comprehensive such treatment for $n = 2$ has recently been given by P. Walker [**12**; **13**].

A. Devinatz [**1**] has given a very general theorem for the calculation of deficiency indices of a class of formally self-adjoint operators by the use of asymptotic methods. This work contains, as special cases, many of the known results on deficiency indices. The paper of Devinatz also shows how to construct a differential operator with given deficiency indices (m, m), $n \leqq m \leqq 2n$. While we require here much weaker conditions on the coefficients p_i than in the papers using asymptotic methods, it appears that our techniques are limited to the case where the deficiency indices are (n, n).

Our proofs depend on inequalities for a system of differential equations. These are given in § 1. The techniques are similar to those used by W. N. Everitt in [**6**]

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together with certain linear transformations. In § 2 the equation (0.1) is studied. The limit point criteria obtained is similar to known results for $n = 1$ or $n = 2$. In particular, we obtain the theorem that

$$(-1)^n y^{(2n)} + q(t)y = \lambda y$$

is in the limit point condition at infinity if for some $K > 0$, $-q(t) \leqq Kt^{2n/(2n-1)}$. This extends the known criteria when $n = 1$, and when $n = 2$ [**6**].

In § 3 we treat the general fourth-order self-adjoint operator with complex coefficients. In the last section certain nonself-adjoint equations are considered.

1. Inequalities for a system of equations. In this section we establish inequalities which will be used in the remainder of the paper. Consider the system of differential equations

$$X' = wBX, \tag{1.1}$$

where $X = (x_1, \ldots, x_m)^T$ is a column vector, w is a positive continuous function on a ray $[a, \infty)$, and $B = \{b_{ij}\}$ is an $m \times m$ matrix of measurable, locally integrable, complex valued functions on $[a, \infty)$ satisfying

$$b_{ij} = \begin{cases} 0 & \text{if} \quad j > i+1 \\ \pm 1 & \text{if} \quad j = i+1. \end{cases}$$

THEOREM 1.1. *Suppose X is a solution of* (1.1) *and that for some $k \leqq m$, b_{ij} is bounded on $[a, \infty)$ for $i \leqq k$. Let*

$$I_i = I_i(t) \equiv \max\left\{1, \int_a^t w|x_i|^2 ds\right\} \quad (i = 1, \ldots, m)$$

and suppose $I_1(\infty) < \infty$.

(i) *If $k < m$, then for $i = 1, \ldots, k$, the following order relations hold as $t \to \infty$:*

$$I_i = O(I_{i+1}{}^{(i-1)/i}) \quad \text{and} \quad |x_i|^2 = O(I_{i+1}{}^{(2i-1)/2i}). \tag{1.2}$$

(ii) *If $k = m$, then for $i = 1, \ldots, m$ and as $t \to \infty$, $I_i = O(1)$ and $|x_i|^2 = O(1)$.*

Proof. Since $I_1 = O(1)$, we have $I_1 = O(I_2{}^{0/1})$. From

$$|x_i(t)|^2 = |x_i(a)|^2 + \int_a^t [x_i\bar{x}_i' + x_i'\bar{x}_i]ds \tag{1.3}$$

and $x_1' = w[b_{11}x_1 + b_{12}x_2]$, we obtain by the Cauchy-Schwarz inequality,

$$|x_1(t)|^2 = O(I_1 + [I_1 I_2]^{1/2}) = O(I_2{}^{1/2}).$$

If $k = 1$, the proof of (i) is complete. Assume now $1 < k < m$ and that (1.2) holds for $i = 1, \ldots, p$ for some $p \leqq k - 1$. Then for $i \leqq p$, $I_i = O(I_p) = O(I_{p+1}{}^{(p-1)/p})$, and (for $I_{p+1} > 1$),

$$
\begin{aligned}
I_{p+1} &= \int_a^t w\bar{x}_{p+1}x_{p+1}ds \\
&= \int_a^t \bar{x}_{p+1}(b_{p,p+1})\left[x_p' - w\sum_{j=1}^{p} b_{p,j}x_j\right]ds \\
&= (b_{p,p+1})\bar{x}_{p+1}x_p\Big|_a^t - \int_a^t (b_{p,p+1})\bar{x}_{p+1}'x_pds + O([I_pI_{p+1}]^{1/2}) \\
&= (b_{p,p+1})\bar{x}_{p+1}x_p\Big|_a^t + O([I_pI_{p+1}]^{1/2}) - \int_a^t (b_{p,p+1})x_pw\sum_{j=1}^{p+2} \bar{b}_{p+1,j}\bar{x}_jds \\
&= (b_{p,p+1})\bar{x}_{p+1}x_p\Big|_a^t + O([I_pI_{p+1}]^{1/2}) + O(I_p^{1/2}[I_{p+1}^{1/2} + I_{p+2}^{1/2}]).
\end{aligned}
$$

Using (1.3) for $i = p + 1$, $I_p = O(I_{p+1})$, and $x_{p+1}' = w\sum_{j=1}^{p+2} b_{p+1,j}x_j$, we have

$$|x_{p+1}|^2 = O(I_{p+1}^{1/2}[I_{p+1}^{1/2} + I_{p+2}^{1/2}]). \tag{1.4}$$

Applying (1.4), $|x_p|^2 = O(I_{p+1}^{(2p-1)/2p})$, and $I_p = O(I_{p+1}^{(p-1)/p})$ to the above equation for I_{p+1} yields

$$
\begin{aligned}
I_{p+1} = O\{&I_{p+1}^{1/4}[I_{p+1}^{1/2} + I_{p+2}^{1/2}]^{1/2}I_{p+1}^{(2p-1)/4p} + I_{p+1}^{(2p-1)/2p} \\
&+ I_{p+1}^{(p-1)/2p}[I_{p+1}^{1/2} + I_{p+2}^{1/2}]\}.
\end{aligned} \tag{1.5}
$$

If $I_{p+1} = O(1)$, then $I_{p+1} = O(I_{p+2}^{p/(p+1)})$; otherwise, a division of (1.5) by I_{p+1} gives

$$1 = O\{[I_{p+1}^{-1/2p} + I]^{1/2} + I_{p+1}^{-1/2p} + I\}$$

where $I = I_{p+2}^{1/2}/I_{p+1}^{(p+1)/2p}$; hence limit inf $I(t) > 0$ as $t \to \infty$. The relation $I_{p+1} = O(I_{p+2}^{p/(p+1)})$ is immediate, and from (1.4),

$$|x_{p+1}|^2 = O([I_{p+1}I_{p+2}]^{1/2}) = O(I_{p+2}^{(2p+1)/(2p+2)});$$

thus the proof of (i) is complete. For $k = m$, we have from (i) that (1.2) holds for $i \leqq m - 1$. Using $I_i = O(I_m^{(m-2)/(m-1)}) = O(I_m)$ for $i \leqq m - 1$ and (1.3) for $i = m$ gives $|x_m|^2 = O(I_m)$. This relation and $|x_{m-1}|^2 = O(I_m^{(2m-3)/(2m-2)})$ gives (for $I_m > 1$),

$$
\begin{aligned}
I_m &= \int_a^t w\bar{x}_mx_mds = \int_a^t \bar{x}_m(b_{m-1,m})\left[x'_{m-1} - w\sum_{j=1}^{m=1} b_{m-1,j}x_j\right]ds \\
&= b_{m-1,m}\bar{x}_mx_{m-1}\Big|_a^t - \int_a^t \bar{x}_m'(b_{m-1,m})x_{m-1} + O(I_m^{1/2}I_m^{(m-2)/(2m-2)}) \\
&= O(I_m^{1/2}I_m^{(2m-3)/(4m-4)}) + O(I_m^{1/2}I_m^{(m-2)/(2m-2)}) \\
&= O(I_m^{(4m-5)/(4m-4)}).
\end{aligned}
$$

Thus $I_m = O(1)$ and the proof of (ii) is complete.

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2. Equations of order $2n$. We consider now the differential operator

$$L(y) = (-1)^n(p_0y^{(n)})^{(n)} + (-1)^{n-1}(p_1y^{(n-1)})^{(n-1)} + \ldots + p_ny. \tag{2.1}$$

The coefficients of L are assumed to be real measurable functions on a ray $[a, \infty)$ with $p_0 > 0$ having n continuous derivatives and $p_1, \ldots, p_n$ Lebesgue integrable on compact intervals. The operator L is formally self-adjoint. The theory of deficiency indices tells us that the equation $L(y) = \lambda y$, $\operatorname{Im} \lambda \neq 0$, has at least n linearly independent solutions in the Hilbert space $L_2[a, \infty)$ of quadratically summable complex functions on $[a, \infty)$.

It is assumed throughout this section that ρ is a positive function on $[a, \infty)$ with n continuous derivatives and λ denotes a complex number with $\operatorname{Re} \lambda = 0$. We consider the conditions:

$$|p_i|\rho^{4i}/p_0 = O(1) \text{ as } t \to \infty \qquad (i = 1, \ldots, n-1). \tag{2.2}$$

$$\text{For some } K > 0, \qquad -p_n\rho^{4n}/p_0 \leqq K. \tag{2.3}$$

$$\rho'\rho \text{ and } \rho^2p_0'/p_0 \text{ are } O(1) \text{ as } t \to \infty. \tag{2.4}$$

$$\int_a^\infty [\rho^{4n-2}/p_0]dt = \infty. \tag{2.5}$$

$$\text{For } j = 1, \ldots, n,\ [\rho^{4n-2}/p_0]^{(j)} = O(\rho^{4n-2-2j}/p_0) \text{ and } \tag{2.6}$$

$$[\rho^{4n}/p_0]^{(j)} = O(\rho^{4n-2j}/p_0) \text{ as } t \to \infty.$$

For (2.1), the quasi-derivatives $y^{[i]}$ are defined by $y^{[i]} = y^{(i)}$, $i = 0, \ldots, n-1$, $y^{[n]} = p_0y^{(n)}$, and $y^{[n+i]} = p_iy^{(n-i)} - (y^{[n+i-1]})'$, $i = 1, \ldots, n$. Then $L(y) = y^{[2n]}$ and the equation $L(y) = \lambda y$ has the vector formulation

$$Y' = AY, \tag{2.7}$$

where $Y = (y^{[0]}, \ldots, y^{[2n-1]})^T$ and

$$A = \begin{bmatrix} 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & 1 & & & & & \\ & & & 0 & 1/p_0 & & & & \\ & & & p_1 & 0 & -1 & & & \\ & & \iddots & & & \ddots & \ddots & & \\ & \iddots & & & & & \ddots & \ddots & \\ & p_{n-1} & & & & & & \ddots & -1 \\ p_n - \lambda & & & & & & & & 0 \end{bmatrix}.$$

We transform the equation (2.7) by the transformation $X = MY$ where M is the diagonal matrix

$$M = \text{diagonal } \{\rho, \rho^3, \ldots, \rho^{2n-1}, \rho^{2n+1}/p_0, \ldots, \rho^{4n-1}/p_0\}.$$

The vector X satisfies

$$X' = (1/\rho^2)BX, \qquad B = \rho^2[MAM^{-1} + M'M^{-1}]. \tag{2.8}$$

Condition (2.4) implies b_{ij} is bounded for $i \leqq n$ with $b_{i,i+1} = 1$. The transformation $X = MY$ gives the integral relations

$$\begin{aligned} \int_a^t (1/\rho^2)|x_i|^2 ds &= \int_a^t \rho^{4i-4}|y^{(i-1)}|^2 dt \qquad (i = 1, \ldots, n+1), \\ \int_a^t (1/\rho^2)|x_i|^2 ds &= \int_a^t (\rho^{4i-4}/p_0^2)|y^{[i-1]}|^2 dt \quad (i = n+2, \ldots, 2n). \end{aligned} \tag{2.9}$$

The Lagrange identity for (2.1) is

$$L(y)\bar{z} - y\overline{L(z)} = [y, z]', \tag{2.10}$$

where

$$[y, z] = \sum_{i=0}^{n-1} \{y^{[i]}\bar{z}^{[2n-i-1]} - y^{[2n-i-1]}\bar{z}^{[i]}\}.$$

We note that $L(y) = \lambda y$ and $L(z) = \bar{\lambda} z$ implies $[y, z]' \equiv 0$. For $L(y) = \lambda y$, the quadratic expression

$$-\lambda|y|^2 + \sum_{i=0}^{n} p_{n-i}|y^{(i)}|^2 = \left\{\sum_{i=0}^{n-1} y^{[2n-i-1]}\bar{y}^{[i]}\right\}', \tag{2.11}$$

holds. We also make use of the vector spaces

$$V = \{y | L(y) = \lambda y\},$$
$$V_1 = \{y | L(y) = \lambda y, y \in L_2[a, \infty)\},$$

and

$$V_2 = \{z | L(z) = \bar{\lambda} z, z \in L_2[a, \infty)\}.$$

LEMMA 2.1. *If* $\dim V_1 + \dim V_2 > 2n$, *then there is a* $y \in V_1$ *and* $z \in V_2$ *such that* $[y, z] \equiv 1$.

Proof. Define the linear transformation T on V by $T(y)$ is the unique $w \in V$ with initial values

$$(w^{[0]}, \ldots, w^{[2n-1]})(a) = (y^{[2n-1]}, \ldots, y^{[n]}, -y^{[n-1]}, \ldots, -y^{[0]})(a).$$

Then T is nonsingular and $\dim T(V_1) = \dim V_1$. Let S be the linear transformation from V_2 into V defined by $S(z)$ is the unique $y \in V$ with initial values

$$(y^{[0]}, \ldots, y^{[2n-1]})(a) = (z^{[0]}, \ldots, z^{[2n-1]})(a).$$

Then S is one-one and $\dim S(V_2) = \dim V_2$. Since $\dim V = 2n$, we have

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$\dim[T(V_1) \cap S(V_2)] \geqq 1$. Choose $S(z) \in T(V_1) \cap S(V_2)$, $z \neq 0$. Then there is a $y \in V_1$ such that

$$(z^{[0]}, \ldots, z^{[2n-1]})(a) = (y^{[2n-1]}, \ldots, y^{[n]}, -y^{[n-1]}, \ldots, -y^{[0]})(a).$$

Hence

$$[y, z] = -\sum_{i=0}^{2n-1} |y^{[i]}(a)|^2 \neq 0. \tag{2.12}$$

Multiplication of (2.12) by an appropriate constant completes the proof.

LEMMA 2.2. *Let* $y \in V_1$, $z \in V_2$, *assume conditions* (2.2), (2.4), (2.6), *and define*

$$J_1(t) = \max\left\{1, \int_a^t \rho^{4n}|y^{(n)}|^2 ds\right\} \quad \text{and} \quad J_2(t) = \max\left\{1, \int_a^t \rho^{4n}|z^{(n)}|^2 ds\right\}.$$

Then for $i = n, \ldots, 2n-1$, *and* $(w_1, w_2) = (y, z)$ *or* (z, y),

$$\text{(i)} \quad \int_a^t w_1^{[i]}\bar{w}_2^{[j]}\{(1-s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)} ds = O([J_1 J_2]^{1/2})$$

as $t \to \infty$ *for all* j, k *such that* $i + j + k = 2n - 1$.

$$\text{(ii)} \quad \int_a^t w_1^{[i]}\bar{w}_1^{[j]}\{(1-s/t)^n \rho^{4n}/p_0\}^{(k)} ds = O(J_1^{(2n-1)/2n})$$

as $t \to \infty$ *for all* j, k *such that* $k \geqq 1$ *and* $i + j + k = 2n$.

Proof of (i). Applying Theorem 1.1 to (2.8), we have from (2.9) for $i \leqq n$,

$$\begin{aligned} \int_a^t \rho^{4i-4}|y^{(i-1)}|^2 ds &= \int_a^t (1/\rho^2)|x_i|^2 ds \\ &= O\left(\max\left\{1, \left[\int_a^t (1/\rho^2)|x_{n+1}|^2 ds\right]^{(n-1)/n}\right\}\right) \\ &= O(J_1^{(n-1)/n}) = O(J_1); \end{aligned} \tag{2.13}$$

similarly, for $i \leqq n$,

$$\int_a^t \rho^{4i-4}|z^{(i-1)}|^2 ds = O(J_2^{(n-1)/n}) = O(J_2). \tag{2.14}$$

Consider $(w_1, w_2) = (y, z)$ (the case $(w_1, w_2) = (z, y)$ has a similar proof). We note that $\rho\rho' = O(1)$ implies that $\rho^2(t) = O(t)$. Hence conditions (2.4) and (2.6) give for $k \leqq n - 1$ and $s \leqq t$,

$$\begin{aligned} \frac{d}{ds^k}\{(1-s/t)^{n-1}\rho(s)^{4n-2}/p_0(s)\} &= \sum_{u=0}^{k} \binom{k}{u}\{(1-s/t)^{n-1}\}^{(k-u)}\{\rho^{4n-2}/p_0\}^{(u)} \\ &= \sum_{u=0}^{k} O(1/t^{k-u})O(\rho^{4n-2-2u}/p_0) \\ &= O(\rho^{4n-2-2k}/p_0); \end{aligned} \tag{2.15}$$

thus with $i = n$ in (i) and $j + k = n - 1$, it follows from (2.14),

$$\int_a^t y^{[n]}\bar{z}^{[j]}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)}ds = \int_a^t y^{[n]}z^{(j)}O(\rho^{2n+2j}/p_0)ds$$

$$= O\left(\left[\int_a^t \rho^{4n}|y^{(n)}|^2 ds \int_a^t \rho^{4j}|z^{(j)}|^2 ds\right]^{1/2}\right)$$

$$= O([J_1 J_2^{(n-1)/n}]^{1/2}) = O([J_1 J_2]^{1/2}).$$

Assume now (i) holds for some i, $n \leqq i < 2n - 1$. Then for

$$(i + 1) + j + k = 2n - 1,$$

$$(2.16) \quad \int_a^t y^{[i+1]}\bar{z}^{(j)}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)}ds$$

$$= \int_a^t \{p_{i+1-n}y^{(2n-i-1)} - y^{[i]\prime}\}\bar{z}^{(j)}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)}ds$$

$$= \int_a^t p_{i+1-n}y^{(2n-i-1)}\bar{z}^{(j)}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)}ds$$

$$+ O(1) + \int_a^t y^{[i]}[\bar{z}^{(j)}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)}]'ds.$$

By the induction hypothesis the second integral on the right hand side of (2.16) is $O([J_1 J_2]^{1/2})$. By (2.2), (2.6), and (2.15),

$$p_{i+1-n}\{(1 - s/t)^{n-1}\rho^{4n-2}/p_0\}^{(k)} = p_{i+1-n}O(\rho^{4n-2-2k}/p_0) = O(\rho^{8n-4i-6-2k}).$$

Now $\rho^{8n-4i-6-2k} = \rho^{4n-2i-2+2j}$ since $(i + 1) + j + k = 2n - 1$; thus the first integral on the right hand side of (2.16) is

$$O\left(\left[\int_a^t \rho^{4(2n-i-1)}|y^{(2n-i-1)}|^2 ds \int_a^t \rho^{4j}|z^{(j)}|^2 ds\right]^{1/2}\right)$$

which is $O([J_1 J_2]^{1/2})$ by application of (2.13) and (2.14). This inductive step completes the proof of (i). The proof of (ii) is similar.

LEMMA 2.3. *Let F be a nonnegative, continuous function on $[a, \infty)$ and define*

$$H(t) = \int_a^t (t - s)^n F(s)ds.$$

If as $t \to \infty$, $H(t) = O(t^n[H^{(n)}]^\alpha)$ where $\alpha = (2n - 1)/2n$, then

$$\int_a^t F(s)ds = O(1)$$

as $t \to \infty$.

Proof. Suppose to the contrary that

$$\int_a^\infty F(s)ds = \infty.$$

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Then by L'Hôpital's rule $H^{(i)}/t^{n-i} \to \infty$ as $t \to \infty$ for $i = 0, \ldots n$. We now prove by induction that for $i = 0, \ldots, n-1$, there is a constant $K_i > 0$ such that for all large t,

$$H^{\beta_i} \leqq K_i t^{n/\alpha} H^{(n-i)}[H']^i, \qquad \beta_i = i + 1/\alpha. \tag{2.17}$$

For $i = 0$, (2.17) is a consequence of the hypothesis. If (2.17) holds for some i, $0 \leqq i < n-1$, then

$$\begin{aligned} H'H^{\beta_i} &\leqq K_i t^{n/\alpha} H^{(n-i)}[H']^{i+1} \\ &\leqq K_i [t^{n/\alpha} H^{(n-i-1)}(H')^{i+1}]'. \end{aligned} \tag{2.18}$$

An integration of (2.18) yields (2.17) for $i+1$ and the induction is complete. From (2.17) for $i = n-1$, i.e.,

$$H^{n-1+1/\alpha} \leqq K_{n-1} t^{n/\alpha} (H')^n,$$

we obtain

$$1/t^{1/\alpha} \leqq K_{n-1}^{1/n} H'/H^\beta, \qquad \beta = 1 + 1/n(2n-1).$$

An integration of this inequality over $[t, \infty)$ gives

$$1/t^{1/(2n-1)} \leqq K_{n-1}^{1/n} n / H^{1/n(2n-1)}. \tag{2.19}$$

The inequality (2.19) is contrary to $H/t^n \to \infty$ as $t \to \infty$; thus the proof is complete.

THEOREM 2.1. *Under the conditions* (2.2)–(2.6), *the equation* $L(y) = \lambda y$ *has at most n linearly independent solutions in* $L_2[a, \infty)$.

Proof. Let $y \in V_1$, $z \in V_2$, and J_1 and J_2 be as in Lemma 2.2. We first show $J_1(\infty) < \infty$. From (2.11) and an integration by parts,

$$\begin{aligned} &\int_a^t \left[-\lambda |y|^2 + \sum_{i=0}^n p_{n-i} |y^{(i)}|^2 \right] (1 - s/t)^n (\rho^{4n}/p_0) ds \\ &\qquad = - \int_a^t \sum_{i=0}^{n-1} y^{[2n-i-1]} \bar{y}^{[i]} \{(1 - s/t)^n (\rho^{4n}/p_0)\}' ds + O(1). \end{aligned} \tag{2.20}$$

By part (ii) of Lemma 2.2, the right hand side of (2.20) is $O(J_1^{(2n-1)/2n})$. We have by (2.2) and (2.13) that for $0 < i < n$,

$$\begin{aligned} \int_a^t p_{n-i} |y^{(i)}|^2 (1 - s/t)^n (\rho^{4n}/p_0) ds &= O\left(\int_a^t \rho^{4i} |y^{(i)}|^2 ds \right) \\ &= O(J_1^{(n-1)/n}) = O(J_1^{(2n-1)/2n}). \end{aligned}$$

We also have by (2.3),

$$\operatorname{Re} \int_a^t |y|^2 (p_n - \lambda)(1 - s/t)^n (\rho^{4n}/p_0) ds \geqq -K \int_a^t |y|^2 ds.$$

These inequalities in (2.20) give

$$\int_a^t (t-s)^n \rho^{4n} |y^{(n)}|^2 ds = O(t^n J_1^{(2n-1)/2n}).$$

Lemma 2.3 with $F = \rho^{4n}|y^{(n)}|^2$ now applies to yield $J_1(\infty) < \infty$. Similarly, $J_2(\infty) < \infty$.

If Im $\lambda = 0$, then $V_1 = V_2$. If Im $\lambda \neq 0$, then the correspondence $y \to \bar{y}$ is one-one from V_1 onto V_2; thus dim $V_1 =$ dim V_2. Suppose now dim $V_1 > n$. By Lemma 2.1, we may choose $y \in V_1$ and $z \in V_2$ such that $[y, z] = 1$; hence

$$\int_a^t (1 - s/t)^{n-1}(\rho^{4n-2}/p_0)ds$$
$$= \int_a^t \sum_{i=0}^{n-1} [y^{(i)}\bar{z}^{[2n-i-1]} - y^{[2n-i-1]}\bar{z}^{(i)}](1 - s/t)^{n-1}(\rho^{4n-2}/p_0)ds$$
$$= O([J_1 J_2]^{1/2}),$$

by part (i) of Lemma 2.2. Now $J_1(\infty) < \infty$ and $J_2(\infty) < \infty$; thus the above yields

$$\limsup_{t\to\infty} \int_a^t (1 - s/t)^{n-1}(\rho^{4n-2}/p_0)ds < \infty,$$

contrary to (2.5), i.e.,

$$\int_a^\infty (\rho^{4n-2}/p_0)ds = \infty.$$

Therefore dim $V_1 \leqq n$ and the proof is complete. For Im $\lambda \neq 0$, we have in addition that dim $V_1 = n$.

COROLLARY 2.1. *If* $p_0(t) = t^\alpha$, $\alpha \leqq 2n$, $|p_i| = O(t^{\gamma_i})$,

$$\gamma_i = [4i + \alpha(4n - 4i - 2)]/(4n - 2) \qquad (i = 1, \dots, n - 1),$$

and $-p_n(t) \leqq Kt^{(4n-2\alpha)/(4n-2)}$ *for some* $K > 0$, *then* dim $V_1 \leqq n$.

Proof. Choose $\rho(t) = t^{(\alpha-1)/(4n-2)}$.

For $n = 1$ and $\alpha = 0$, Corollary 2.1 has been given by E. C. Titchmarsh [**11**] and N. Levinson [**8**]. For $n = 1$ in Theorem 2.1, the conditions on ρ reduce to $-p_1\rho^4/p_0 \leqq K$, $\rho'\rho$ and $\rho^2 p_0'/p_0$ are $O(1)$ as $t \to \infty$, and

$$\int_a^\infty (\rho^2/p_0)dt = \infty.$$

This does not quite give the more general criterion of Levinson [**8**] that $L(y) = -(p_0 y')' + p_1 y$ is in the limit point at ∞ if for some continuous differentiable $M > 0$, $-p_1(t) \leqq M(t)$,

$$\int_a^\infty (p_0 M)^{-1/2}dt = \infty,$$

and $M' = O(M^{3/2}/p_0^{1/2})$. However, with $\rho = (p_0/M)^{1/4}$, we need only add $p_0' = O([p_0 M]^{1/2})$ to the above conditions on M in order to satisfy the hypothesis of Theorem 2.1.

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W. N. Everitt [6] has given the following criterion for the equation

$$y^{(iv)} - (p_1 y')' + p_2 y = \lambda y$$

to be in the limit point case at ∞:

$$|p_1(t)| = O(t^{2/3}) \quad \text{and} \quad -p_2(t) \leqq Kt^{4/3}, \qquad K > 0.$$

This criterion is the statement of Corollary 2.1 with $\alpha = 0$, $n = 2$.

The smoothness condition of n derivatives on p_0 can be relaxed in the following manner. Suppose $\hat{p}_0$ is a positive function with n continuous derivatives such that conditions (2.2) through (2.6) hold with p_0 replaced by $\hat{p}_0$, and there are positive numbers c and d such that $c \leqq \hat{p}_0(t)/p_0(t) \leqq d$ for $t \geqq a$. Then the conclusion of Theorem 2.1 holds. We have avoided this more general setting in order to simplify the proof. The generalization requires using $\hat{p}_0$ in place of p_0 in the transformation (2.8) and in Lemma 2.2. The proof of Lemma 2.2 with $\hat{p}_0$ requires the observation that for $m = 2n$ and $k = n$ in Theorem 1.1, part (i) still holds if the condition $b_{n,n+1} = \pm 1$ is replaced by $b_{n,n+1} = O(1)$.

3. Self-adjoint fourth-order equations. The methods in the preceding section are applicable to formally self-adjoint operators with complex coefficients. We consider in this section the fourth-order case. A general such operator can be written as a sum of even order operators with real coefficients and odd order operators with imaginary coefficients (cf. [9, Chapter I]). We define the formally self-adjoint operator L by

$$L(y) = (y^{[3]})' + (ip_2/2)y' + qy, \tag{3.1}$$

where $y^{[2]} = ry'' + (ip_1/2)y'$ and $y^{[3]} = (y^{[2]})' + py' + (ip_1/2)y'' + (ip_2/2)y$. The coefficients r, p, q, p_1, and p_2 are assumed to be real, measurable functions on $[a, \infty)$ with $r > 0$ having two continuous derivatives and the other coefficients Lebesgue integrable on compact interval. The equation $L(y) = \lambda y$ has the vector formulation

$$Y' = AY, \tag{3.2}$$

where $Y = (y, y', y^{[2]}, y^{[3]})^T$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -ip_1/2r & 1/r & 0 \\ -ip_2/2 & -(p + p_1^2/2r) & -ip_1/2r & 1 \\ \lambda - q & -ip_2/2 & 0 & 0 \end{bmatrix}.$$

The Lagrange identity for (3.1) is

$$L(y)\bar{z} - y\overline{L(z)} = [y, z]',$$

where

$$[y, z] = y^{[3]}\bar{z} - y^{[2]}\bar{z}' + y'\bar{z}^{[2]} - y\bar{z}^{[3]}.$$

For $L(y) = \lambda y$, we have the quadratic expression

$$\{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}' = (\lambda - q)|y|^2 - r|y''|^2 + p|y'|^2 + (ip_1/2)[y''\bar{y}' - y'\bar{y}''] + (ip_2/2)[y\bar{y}' - y'\bar{y}]. \tag{3.3}$$

The analysis is similar to that of § 2. We transform (3.2) by $X = MY$ where M is the matrix

$$M = \text{diagonal } [\rho, \rho^3, \rho^5/r, \rho^7/r]$$

and ρ is a positive, twice continuously differentiable function.

THEOREM 3.1. *Suppose that ρ and the coefficients of* (3.1) *satisfy the following conditions.*

(i) $|p_1|\rho^2/r$, $|p_2|\rho^6/r$, $|p|\rho^4/r$ *are* $O(1)$ *as* $t \to \infty$.

(ii) *For some* $K > 0$, $-q\rho^8/r \leqq K$.

(iii) $\rho'\rho$, $r'\rho^2/r$, $\rho^3\rho''$, $r''\rho^4/r$ *are* $O(1)$ *as* $t \to \infty$.

(iv) $\int_a^\infty (\rho^6/r)dt = \infty$.

Then the equation $L(y) = \lambda y$ (Re $\lambda = 0$) *has at most two linearly independent solutions in* $L_2[a, \infty)$.

Proof. The proof is analogous to Theorem 2.1 and is only sketched. The elements b_{ij}, $i \leqq 2$, of $X' = (1/\rho^2)BX$ are bounded with $b_{12} = b_{23} = 1$. Let V_1 and V_2 be defined as in § 2. It can then be shown that Lemma 2.2 (with $n = 2$) holds for L of (3.1) (as before, $y^{[0]} = y$, $y^{[1]} = y'$). The condition (2.6) is fulfilled by (iii) above. The proof of Lemma 2.2, part (i) is the same as before except for consideration of integrals of p_1y', py', p_1y'', and p_2y times $\bar{z}^{[j]}\{(1 - s/t)^{n-1}\rho^{4n-2}/r\}^{[k]}$ with $j + k = 0$ for py', p_1y'', and p_2y and $j + k = 1$ for p_1y' (and with y and z interchanged). Condition (i) above implies each of these integrals is $O([J_1J_2]^{1/2})$. Similar integrals arise in the proof of part (ii).

We multiply the quadratic expression (3.3) by $(1 - s/t)^2\rho^8/r$ and integrate. The integrals of $p_1(y''\bar{y}' - y'\bar{y}'')$ and $p_2(\bar{y}y' - y'\bar{y})$ times $(1 - s/t)^2\rho^8/r$ are $O(J_1^{1/2}J_1^{1/4}) = O(J_1^{3/4})$ and $O(J_1^{1/4}) = O(J_1^{3/4})$ respectively; hence as before (when $n = 2$) as $t \to \infty$,

$$\int_a^t (t - s)^2\rho^8|y''|^2 ds = O(t^2J_1^{3/4}).$$

By Lemma 2.3, $J_1(\infty) < \infty$ for $y \in V_1$. Similarly, $J_2(\infty) < \infty$ for $z \in V_2$.

It has been proved by W. N. Everitt [**2**; **3**; **4**] that if Im $\lambda \neq 0$, then $2 \leqq \dim V_i \leqq 4$ $(i = 1, 2)$, although in general, $\dim V_1 \neq \dim V_2$. As before $V_1 = V_2$ if λ is real; hence we may prove as in Lemma 2.1 that if $\dim V_1 > 2$, then $[y, z] = 1$ for some $y \in V_1$, $z \in V_2$. (Define the linear transformation T here by $T(y)$ is the unique w with initial values

$$(w^{[0]}, w^{[1]}, w^{[2]}, w^{[3]})(a) = (y^{[3]}, -y^{[2]}, y^{[1]}, -y^{[0]})(a).)$$

The remainder of the proof is the same as that of Theorem 2.1.

4. Nonself-adjoint equations. The technique of §§ 2 and 3 may be applied to nonself-adjoint equations if the one-sided condition on the coefficient of y is replaced by an absolute value condition. We consider the operator L defined by (2.1), but now allow the coefficients to be complex valued. However, p_0 is still required to be real and positive. The operator L and its formal adjoint L^+ are then given by

$$L(y) = (-1)^n(p_0y^{(n)})^{(n)} + (-1)^{n-1}(p_1y^{(n-1)})^{(n-1)} + \ldots + p_ny \tag{4.1}$$

and

$$L^+(z) = (-1)^n(\bar{p}_0z^{(n)})^{(n)} + (-1)^{n-1}(\bar{p}_1z^{(n-1)})^{(n-1)} + \ldots + \bar{p}_nz. \tag{4.2}$$

For λ a complex number, define

$$V_1 = V_1(\lambda) = \{y|L(y) = \lambda y, y \in L_2[a, \infty)\}$$

and

$$V_2 = V_2(\lambda) = \{z|L^+(z) = \bar{\lambda}z, z \in L_2[a, \infty)\}.$$

For (4.1) define the quasi-derivatives $y^{[i]}$ by $y^{[i]} = y^{(i)}$ $(i = 0, \ldots, n-1)$, $y^{[n]} = p_0y^{(n)}$, and $y^{[n+i]} = p_iy^{(n-i)} - (y^{[n+i-1]})'$ $(i = 1, \ldots, n)$; for (4.2) define the quasi-derivatives as for y, but replacing p_i by $\bar{p}_i$. Then the Lagrange identity is

$$L(y)\bar{z} - y\overline{L^+(z)} = [y, z]',$$

where $[y, z]$ is the same as (2.10). Thus $L(y) = \lambda y$ and $L^+(z) = \bar{\lambda}z$ implies that $[y, z]' = 0$. Let ρ be a positive, continuously differentiable function.

THEOREM 4.1. *Suppose ρ and the coefficients of L satisfy the conditions* (2.2), (2.4), (2.5), *and for some $K > 0$, $|(p_n - \lambda)\rho^{4n}/p_0| \leqq K$. Then*

$$\dim V_1(\lambda) = \dim V_2(\lambda) \leqq n.$$

Proof. Let X, M be as in § 2. The elements of B are all bounded on $[a, \infty)$, and $b_{i,i+1} = \pm 1$ for each i. By Theorem 1.1 and (2.9), we have for $y \in V_1$,

$$\begin{aligned} &\int_a^t \rho^{4i-4}|y^{(i-1)}|^2 ds < \infty \qquad (i = 1, \ldots, n), \\ &\int_a^t (\rho^{4i-4}/p_0^2)|y^{[i-1]}|^2 ds < \infty \quad (i = n+1, \ldots, 2n). \end{aligned} \tag{4.3}$$

Similar considerations yield that for $z \in V_2$, the integrals (4.3) are finite with y replaced by z.

By taking complex conjugates in $L(y) = \lambda y$ and $L^+(z) = \bar{\lambda}z$, it follows that the correspondence $y \rightarrow \bar{y}$ is one-one from V_1 onto V_2; hence $\dim V_1 = \dim V_2$. If $\dim V_1 > n$, we may repeat the argument of Lemma 2.1 to find that for some $y \in V_1$ and $z \in V_2$, $[y, z] = 1$. The integrals (4.3) and their analogs for z imply that

$$\sum_{i=0}^{n-1} \{y^{[i]}\bar{z}^{[2n-i-1]} - y^{[2n-i-1]}\bar{z}^{[i]}\}\rho^{4n-2}/p_0 = \rho^{4n-2}/p_0 \tag{4.4}$$

is integrable over $[a, \infty)$, contrary to (2.5). The proof is now complete.

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