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LOCAL RADIAL BASIS FUNCTION APPROXIMATION ON THE SPHERE

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Abstract

In this paper we derive local error estimates for radial basis function interpolation on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. More precisely, we consider radial basis function interpolation based on data on a (global or local) point set $X \subset \mathbb{S}^2$ for functions in the Sobolev space $H^s(\mathbb{S}^2)$ with norm $\| \cdot \|_s$, where $s > 1$. The zonal positive definite continuous kernel ϕ, which defines the radial basis function, is chosen such that its native space can be identified with $H^s(\mathbb{S}^2)$. Under these assumptions we derive a local estimate for the uniform error on a spherical cap $S(\mathbf{z}; r)$: the radial basis function interpolant $\Lambda_X f$ of $f \in H^s(\mathbb{S}^2)$ satisfies $\sup_{\mathbf{x} \in S(\mathbf{z};r)} |f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \leq c h^{(s-1)/2} \|f\|_s$, where $h = h_{X,S(\mathbf{z};r)}$ is the local mesh norm of the point set X with respect to the spherical cap $S(\mathbf{z}; r)$. Our proof is intrinsic to the sphere, and makes use of the Videnskii inequality. A numerical test illustrates the theoretical result.

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1. Introduction

Radial basis function interpolation and approximation on the sphere is a topic that has been of great interest over the last decade (see [18, Chapter 17], [4, Chapters 5–7], and the references therein). Since it is essentially a meshless method that allows the handling of scattered data, it is of great interest for applications. The development of more powerful computers as well as progress in numerical methods have made it possible to perform radial basis function interpolation and approximation for very large sets of data.

In this paper we are concerned with establishing *local error estimates* for the maximal absolute pointwise error of a radial basis function interpolant on a *spherical cap* in terms of the local mesh norm of the interpolation point set with respect to this cap. More precisely, we consider the following scenario.

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Let ϕ be a zonal positive definite continuous kernel on $\mathbb{S}^2 \times \mathbb{S}^2$ such that the native space $\mathcal{N}_\phi$ associated with the kernel ϕ can be identified with the Sobolev space $H^s(\mathbb{S}^2)$, where $s > 1$, with the norm $\| \cdot \|_s$ (for definitions see Section 2). The space $H^s(\mathbb{S}^2)$ is intuitively the space of all those functions on $\mathbb{S}^2$ whose generalized derivatives up to the order s are square-integrable. Given the values of $f \in H^s(\mathbb{S}^2)$ on the point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$, the *radial basis function interpolant* $\Lambda_X f = \sum_{j=1}^N \alpha_j \phi(\cdot, \mathbf{x}_j)$ of f is obtained by solving the linear system

$$\Lambda_X f(\mathbf{x}_i) = \sum_{j=1}^N \alpha_j \phi(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i), \quad i = 1, 2, \ldots, N.$$

We estimate the maximal absolute pointwise error of the radial basis function interpolant $\Lambda_X f$ on a spherical cap $S(\mathbf{z}; r)$ of centre $\mathbf{z}$ and radius $r \leq \pi/2$. The error will be expressed in terms of the *local mesh norm* $h_{X,S(\mathbf{z};r)}$ of X with respect to the spherical cap $S(\mathbf{z}; r)$, defined by

$$h_{X,S(\mathbf{z};r)} := \sup_{\mathbf{x} \in S(\mathbf{z};r)} \inf_{\mathbf{y} \in X \cap S(\mathbf{z};r)} \operatorname{dist}(\mathbf{x}, \mathbf{y}), \tag{1.1}$$

where $\operatorname{dist}(\mathbf{x}, \mathbf{y}) := \arccos(\mathbf{x} \cdot \mathbf{y})$ is the spherical (geodesic) distance between $\mathbf{x}$ and $\mathbf{y}$. Under certain assumptions we show in Theorem 3.1 that

$$\sup_{\mathbf{x} \in S(\mathbf{z};r)} |f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \leq c h_{X,S(\mathbf{z};r)}^{(s-1)/2} \|f\|_s, \quad f \in H^s(\mathbb{S}^2). \tag{1.2}$$

If we compare (1.2) with the corresponding global error estimate

$$\sup_{\mathbf{x} \in \mathbb{S}^2} |f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \leq c h_X^{s-1} \|f\|_s, \quad f \in H^s(\mathbb{S}^2), \tag{1.3}$$

(see [7, Corollaries 2 and 3] with the improvement from [10, Theorem 13] and [6, Theorem 2.12]), we see that the local error estimate (1.2) has only half the powers of the local mesh norm $h_{X,S(\mathbf{z};r)}$ compared to the powers of the global mesh norm h_X. (The global mesh norm h_X of X is defined by replacing $S(\mathbf{z}; r)$ by $\mathbb{S}^2$ in (1.1), that is, $h_X = h_{X,\mathbb{S}^2}$.) This loss of convergence order is due to the fact that we use a local estimate for trigonometric polynomials, the *Videnskii inequality* (see [2, p. 243, E. 19 c]), in the proof. Our proof is intrinsic to the sphere, and does not make use of mapping the spherical cap onto $\mathbb{R}^2$ and using results for subsets of $\mathbb{R}^2$.

Apart from this main result, we derive as a by-product an interesting novel statement (see Theorem 3.2) which guarantees, under certain assumptions on the local mesh norm $h_{X,S(\mathbf{z};r)}$ of a point set X with respect to a spherical cap $S(\mathbf{z}; r)$, that we have a set of local functions (defined on this spherical cap) that provide a local polynomial reproduction on $S(\mathbf{z}; r)$.

An error estimate like (1.2) but with the power of the mesh norm doubled is implicit in a recent paper [8] for the case of $\mathbb{S}^d$: such a result can be deduced from the proofs

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and a comment in the introduction in [8]. The proof techniques in [8] are completely different from ours, since they map the sphere $\mathbb{S}^d$ with charts onto subsets of $\mathbb{R}^d$ and make use of results for the Euclidean case.

In [9] and [5], global error estimates for radial basis function interpolation on $\mathbb{S}^2$ and $\mathbb{S}^d$, respectively, were proved in terms of the global mesh norm. A closer inspection of the proof, however, shows that the same proof does also imply local error estimates on a spherical cap in terms of the local mesh norm with respect to a small neighbourhood of the spherical cap.

Why is the local error estimate (1.2) not only of theoretical but also of practical interest? If we have a point set X for which $h_{X,S(\mathbf{z};r)} \approx h_X$, then clearly the global estimate (1.3) always gives a better result than the local estimate (1.2), and nothing can be gained from (1.2). However, if we have a sequence of point sets X for which $h_{X,S(\mathbf{z};r)} = o(h_X^2)$, then the local estimate (1.2) gives asymptotically a better estimate than the global estimate (1.3).

In particular, the extreme case is of interest, where we have *only points in the spherical cap* $S(\mathbf{z}; r)$ (that is, $X \subset S(\mathbf{z}; r)$), or where we have only points in $S(\mathbf{z}; r + \varepsilon)$, with $\varepsilon > 0$ small compared to r. In both cases we have $h_X \approx \pi/2$ and so the global error estimate (1.3) gives *no* information, whereas the local one (1.2) still gives useful results.

The paper is organized as follows. In Section 2 we introduce all the necessary background material about the sphere, function spaces on the sphere, and radial basis functions, and in Section 3 we formulate and discuss the results. In Section 4 we present the proofs, and in Section 5 we prove an important lemma (Lemma 4.1 in Section 4), about the geometry of a point set Y on a spherical cap $S(\mathbf{z}; r)$, expressed in terms of the local mesh norm with respect to this cap. In Section 6 we present a numerical test that illustrates the theory.

2. Preliminaries

In this section we introduce all the necessary notation and background material. In Section 2.1 we shall discuss point sets on the unit sphere $\mathbb{S}^2$, spherical caps, and the (global and local) mesh norm of point sets on $\mathbb{S}^2$. In Section 2.2 we introduce function spaces on $\mathbb{S}^2$, spherical harmonics, the Sobolev spaces $H^s(\mathbb{S}^2)$, and related terminology. In Section 2.3, we finally briefly discuss zonal positive definite continuous kernels, radial basis functions, the radial basis function interpolation problem, and the native space associated with the zonal positive definite continuous kernel.

2.1. Point sets and geometry on the sphere Let $\mathbf{x} \cdot \mathbf{y}$ denote the Euclidean inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, and let $\|\mathbf{x}\| := (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denote the induced Euclidean norm of $\mathbf{x} \in \mathbb{R}^3$. The *unit sphere* $\mathbb{S}^2$ of the Euclidean space $\mathbb{R}^3$ is given by

$$\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}.$$

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For any two points $\mathbf{x}$ and $\mathbf{y}$ on the unit sphere $\mathbb{S}^2$, the *spherical distance* $\operatorname{dist}(\mathbf{x}, \mathbf{y})$ is defined to be the geodesic distance, that is, the length of a shortest geodesic arc connecting $\mathbf{x}$ and $\mathbf{y}$. (By a geodesic arc we mean an arc which is part of a great circle.) Since the unit sphere has radius one, this length is just the angle in $[0, \pi]$ between the two points, that is,

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) := \arccos(\mathbf{x} \cdot \mathbf{y}).$$

For $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$ with $\operatorname{dist}(\mathbf{x}, \mathbf{y}) < \pi$, the shortest geodesic arc connecting $\mathbf{x}$ and $\mathbf{y}$ is uniquely determined; we shall denote that shortest geodesic arc between $\mathbf{x}$ and $\mathbf{y}$ (which then has length $\operatorname{dist}(\mathbf{x}, \mathbf{y})$) by $[\mathbf{x}, \mathbf{y}]$ (or $[\mathbf{y}, \mathbf{x}]$, since here we usually do not care about orientation).

For a measurable set $\mathcal{A} \subset \mathbb{S}^2$, we denote the area of $\mathcal{A}$ by $|\mathcal{A}|$.

For a point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$ the *global mesh norm*, defined by

$$h_X := \sup_{\mathbf{x} \in \mathbb{S}^2} \inf_{\mathbf{x}_j \in X} \operatorname{dist}(\mathbf{x}, \mathbf{x}_j),$$

measures how far away a point $\mathbf{x} \in \mathbb{S}^2$ can be from the closest point of the point set X.

In this paper we are interested in local approximation on spherical caps. The (closed) *spherical cap* $S(\mathbf{z}; r)$ with centre $\mathbf{z} \in \mathbb{S}^2$ and radius r is defined by

$$S(\mathbf{z}; r) := \{\mathbf{x} \in \mathbb{S}^2 : \operatorname{dist}(\mathbf{x}, \mathbf{z}) \le r\}.$$

For a point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$, the *local mesh norm* $h_{X,S(\mathbf{z};r)}$ with respect to $S(\mathbf{z}; r)$ is defined by

$$h_{X,S(\mathbf{z};r)} := \sup_{\mathbf{x} \in S(\mathbf{z};r)} \inf_{\mathbf{x}_j \in X \cap S(\mathbf{z};r)} \operatorname{dist}(\mathbf{x}, \mathbf{x}_j).$$

2.2. Function spaces on the sphere Let $L_2(\mathbb{S}^2)$ denote the Hilbert space of square-integrable functions on the unit sphere $\mathbb{S}^2$ with the usual $L_2(\mathbb{S}^2)$-inner product

$$(f, g)_{L_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} f(\mathbf{x}) g(\mathbf{x})\, d\omega(\mathbf{x});$$

the corresponding induced norm is $\|f\|_{L_2(\mathbb{S}^2)} := (f, f)_{L_2(\mathbb{S}^2)}^{1/2}$. Here $d\omega$ denotes the surface element of $\mathbb{S}^2$.

The space $C(\mathbb{S}^2)$ is the vector space of continuous functions on the sphere $\mathbb{S}^2$ endowed with the supremum norm $\|f\|_{\mathbb{S}^2} := \sup_{\mathbf{x} \in \mathbb{S}^2} |f(\mathbf{x})|$, and likewise $C(S(\mathbf{z}; r))$ is the space of continuous functions $f : S(\mathbf{z}; r) \to \mathbb{R}$ on the spherical cap $S(\mathbf{z}; r)$, endowed with the (local) supremum norm $\|f\|_{S(\mathbf{z};r)} := \sup_{\mathbf{x} \in S(\mathbf{z};r)} |f(\mathbf{x})|$. For functions f defined on an interval $[a, b]$ we shall denote the supremum norm by $\|f\|_{[a,b]} := \sup_{t \in [a,b]} |f(t)|$.

The *space* $\mathbb{P}_L(\mathbb{S}^2)$ *of spherical polynomials on* $\mathbb{S}^2$ *of degree* at most L is obtained by restricting all polynomials on $\mathbb{R}^3$ of degree at most L to the sphere $\mathbb{S}^2$. The restriction to $\mathbb{S}^2$ of any homogeneous harmonic polynomial on $\mathbb{R}^3$ of exact degree ℓ is called a *spherical harmonic* of degree ℓ. The linear space $\mathbb{H}_\ell(\mathbb{S}^2)$ of all spherical harmonics of degree ℓ (and the zero polynomial) has the dimension $\dim(\mathbb{H}_\ell(\mathbb{S}^2)) = 2\ell + 1$, and by

$$\{Y_{\ell,k} : k = 1, 2, \ldots, 2\ell + 1\} \tag{2.1}$$

we shall in the following always denote an orthonormal basis of $\mathbb{H}_\ell(\mathbb{S}^2)$ (with respect to $(\cdot, \cdot)_{L_2(\mathbb{S}^2)}$). Since spherical harmonics of different degree are orthogonal to each other, and since $\mathbb{P}_L(\mathbb{S}^2) = \bigoplus_{\ell=0}^{L} \mathbb{H}_\ell(\mathbb{S}^2)$, we find $\dim \mathbb{P}_L(\mathbb{S}^2) = \sum_{\ell=0}^{L}(2\ell + 1) = (L + 1)^2$, and the set

$$\{Y_{\ell,k} : \ell = 0, 1, \ldots, L;\ \ k = 1, 2, \ldots, 2\ell + 1\}$$

forms an orthonormal basis for $\mathbb{P}_L(\mathbb{S}^2)$.

A point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ on $\mathbb{S}^2$ is said to be $\mathbb{P}_L(\mathbb{S}^2)$-*unisolvent* if the only polynomial p in $\mathbb{P}_L(\mathbb{S}^2)$ that satisfies $p(\mathbf{x}_j) = 0$ for all $j = 1, 2, \ldots, N$ is the zero polynomial.

The spherical harmonics of degree ℓ satisfy the *addition theorem* (see [4, Theorem 3.1.3] and [13, Lemma 4.5 and Theorem 4.7]): for any orthonormal basis (2.1) of $\mathbb{H}_\ell(\mathbb{S}^2)$, we have

$$\sum_{k=1}^{2\ell+1} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = \frac{2\ell + 1}{4\pi} P_\ell(\mathbf{x} \cdot \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{S}^2, \tag{2.2}$$

where P_ℓ denotes the *Legendre polynomial* (see [16, Chapter IV]) of degree ℓ. The Legendre polynomials P_ℓ, $\ell \in \mathbb{N}_0$, are the uniquely determined complete orthogonal set in the space $L_2([-1, 1])$ of square-integrable functions on $[-1, 1]$, endowed with the inner product $(f, g)_{L_2([-1,1])} = \int_{-1}^{1} f(t)g(t)\,dt$, with the following properties: (i) P_ℓ is a polynomial of exact degree ℓ, and (ii) $\|P_\ell\|_{[-1,1]} = 1$ and $P_\ell(1) = 1$. The normalization of the Legendre polynomials is such that $\int_{-1}^{1} |P_\ell(t)|^2\,dt = 2/(2\ell + 1)$ for $\ell \in \mathbb{N}_0$.

The union over $\ell \in \mathbb{N}_0$ of all the sets (2.1) of orthonormal spherical harmonics of degree ℓ forms a complete orthonormal system in $L_2(\mathbb{S}^2)$. Thus any function $f \in L_2(\mathbb{S}^2)$ can be expanded into a *Fourier series* (or *Laplace series*) with respect to this orthonormal system,

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \widehat{f}_{\ell,k} Y_{\ell,k}, \tag{2.3}$$

with the Fourier coefficients

$$\widehat{f}_{\ell,k} := (f, Y_{\ell,k})_{L_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} f(\mathbf{x}) Y_{\ell,k}(\mathbf{x})\,d\omega(\mathbf{x}),$$

and the series (2.3) converges in the $L_2(\mathbb{S}^2)$-sense to f.

For $s \geq 0$, the *Sobolev space* $H^s(\mathbb{S}^2)$ (see [4, Sections 5.1 and 5.2]) is the completion of $\bigcup_{L=0}^{\infty} \mathbb{P}_L(\mathbb{S}^2)$ with respect to the norm

$$\|f\|_s := \left(\sum_{\ell=0}^{\infty} (\ell+1)^{2s} \sum_{k=1}^{2\ell+1} |\widehat{f}_{\ell,k}|^2 \right)^{1/2}. \tag{2.4}$$

The linear space $H^s(\mathbb{S}^2)$ is a Hilbert space with the inner product

$$(f, g)_s := \sum_{\ell=0}^{\infty} (\ell+1)^{2s} \sum_{k=1}^{2\ell+1} \widehat{f}_{\ell,k} \widehat{g}_{\ell,k},$$

which also induces the norm $\|\cdot\|_s$. We observe that $H^0(\mathbb{S}^2) = L_2(\mathbb{S}^2)$.

For $s > 1$, the Sobolev space $H^s(\mathbb{S}^2)$ is embedded into the space $C(\mathbb{S}^2)$ of continuous functions on $\mathbb{S}^2$; that is, $H^s(\mathbb{S}^2)$ is a subset of $C(\mathbb{S}^2)$ and there exists a positive constant c such that $\|f\|_{\mathbb{S}^2} \leq c\|f\|_s$ for all $f \in H^s(\mathbb{S}^2)$. This implies that every point evaluation functional $\delta_{\mathbf{x}} : H^s(\mathbb{S}^2) \to \mathbb{R}$, defined by $\delta_{\mathbf{x}}(f) := f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{S}^2$ is fixed, is bounded, and hence that the space $H^s(\mathbb{S}^2)$, with $s > 1$, is a reproducing kernel Hilbert space (see [1]).

For more background information on spherical harmonics and functions on the sphere we refer the reader to [4] and [13, Chapters 4 and 6].

2.3. Zonal positive definite kernels, radial basis functions, and native spaces Let $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be a symmetric and continuous kernel on $\mathbb{S}^2$. The kernel is said to be *positive definite* if

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(\mathbf{x}_i, \mathbf{x}_j) \geq 0,$$

for any $N \in \mathbb{N}$, any vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N$, and any point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ of N distinct points on $\mathbb{S}^2$, *and* if equality holds only for $\boldsymbol{\alpha} = \mathbf{0}$. The kernel ϕ is said to be *zonal* if there exists some function $\Phi : [-1, 1] \to \mathbb{R}$ such that

$$\phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} \cdot \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^2. \tag{2.5}$$

In other words, a zonal kernel is essentially a function of one variable, since for arbitrary fixed $\mathbf{x} \in \mathbb{S}^2$ it is constant on any 'latitude' $\{\mathbf{y} \in \mathbb{S}^2 : \mathbf{y} \cdot \mathbf{x} = r\}$, $r \in (-1, 1)$, with respect to $\mathbf{x}$ as the north pole. For any zonal continuous kernel $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$, the function Φ in (2.5) is continuous and thus, in particular, is in $L_2([-1, 1])$, and can be expanded into a Legendre series

$$\Phi = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell, \tag{2.6}$$

with the Legendre coefficients a_ℓ defined by

$$a_\ell := 2\pi \int_{-1}^{1} \Phi(t) P_\ell(t)\, dt, \quad \ell \in \mathbb{N}_0.$$

Substituting (2.6) into (2.5) and then using the addition theorem (2.2) yields that in the L_2-sense

$$\phi(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\mathbf{x} \cdot \mathbf{y}) = \sum_{\ell=0}^{\infty} a_\ell \sum_{k=1}^{2\ell+1} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}). \tag{2.7}$$

We shall in the following only consider zonal positive definite continuous kernels ϕ. In addition, we shall assume that the coefficients a_ℓ, $\ell \in \mathbb{N}_0$, in the series representation (2.7) satisfy

$$\sum_{\ell=0}^{\infty} \ell |a_\ell| < \infty, \tag{2.8}$$

thus guaranteeing that the series in (2.6) converges uniformly to Φ, since the Legendre polynomials satisfy $\|P_\ell\|_{[-1,1]} = 1$.

In [3] (building on earlier results of Schoenberg [15] and Xu and Cheney [19]) a complete discussion of zonal positive definite continuous kernels satisfying (2.8) is given: a kernel of the form (2.7) with the property (2.8) is positive definite if and only if $a_\ell \geq 0$ for all $\ell \in \mathbb{N}_0$ with $a_\ell > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ.

From now on let $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be a fixed zonal positive definite continuous kernel of the form (2.7) satisfying (2.8). For a given point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ of N distinct points on $\mathbb{S}^2$, let the (N-dimensional) approximation space be

$$V_X := \operatorname{span}\{\phi(\cdot, \mathbf{x}_j) : j = 1, 2, \ldots, N\}.$$

The *radial basis function interpolation problem* can be formulated as follows: given the values $f(\mathbf{x}_i)$, $i = 1, 2, \ldots, N$, of a continuous function f on the point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$, find the function $\Lambda_X f \in V_X$ such that

$$\Lambda_X f(\mathbf{x}_i) = f(\mathbf{x}_i), \quad i = 1, 2, \ldots, N. \tag{2.9}$$

Writing (2.9) more explicitly yields the linear system

$$\Lambda_X f(\mathbf{x}_i) = \sum_{j=1}^{N} \alpha_j \phi(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i), \quad i = 1, 2, \ldots, N. \tag{2.10}$$

Owing to the positive definiteness of ϕ, the matrix $[\phi(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1,2,\ldots,N}$ of the linear system (2.10) is positive definite, and hence the linear system (2.10) always has a uniquely determined solution.

Consider the linear function space

$$F_\phi := \left\{ \sum_{j=1}^{N} \alpha_j \phi(\cdot, \mathbf{x}_j) : \alpha_j \in \mathbb{R},\ \mathbf{x}_j \in \mathbb{S}^2,\ j = 1, 2, \ldots, N;\ N \in \mathbb{N} \right\},$$

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endowed with the inner product $(g, h)_\phi$ of $g \in F_\phi$, given by $g = \sum_{j=1}^{N} \alpha_j\, \phi(\cdot, \mathbf{x}_j)$, and $h \in F_\phi$, given by $h = \sum_{i=1}^{M} \beta_i\, \phi(\cdot, \mathbf{z}_i)$, defined by

$$(g, h)_\phi := \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \beta_i \phi(\mathbf{x}_j, \mathbf{z}_i). \tag{2.11}$$

Owing to the positive definiteness of ϕ, (2.11) is indeed an inner product for F_ϕ and induces as usual a norm for F_ϕ via $\|f\|_\phi := (f, f)_\phi^{1/2}$, $f \in F_\phi$. The *native space* $\mathcal{N}_\phi$ is now defined to be the completion of F_ϕ with respect to this norm. The space $\mathcal{N}_\phi$ is (by construction) a Hilbert space. We shall denote its inner product also by $(\cdot, \cdot)_\phi$ and the induced norm by $\|\cdot\|_\phi$.

The kernel ϕ is symmetric, and F_ϕ contains $\phi(\cdot, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^2$. Furthermore, (2.11) with $g \in F_\phi$ and $h = \phi(\cdot, \mathbf{x})$ yields $(g, \phi(\cdot, \mathbf{x}))_\phi = g(\mathbf{x})$, and thus ϕ is a reproducing kernel for F_ϕ. This property extends from F_ϕ to its completion $\mathcal{N}_\phi$ with respect to the inner product $(\cdot, \cdot)_\phi$ of $\mathcal{N}_\phi$. This means that (i) ϕ is symmetric, (ii) $\phi(\cdot, \mathbf{x}) \in \mathcal{N}_\phi$ for all $\mathbf{x} \in \mathbb{S}^2$, and (iii) the reproducing property holds, that is,

$$(f, \phi(\cdot, \mathbf{x}))_\phi = f(\mathbf{x}) \quad \text{for all } f \in \mathcal{N}_\phi \text{ and all } \mathbf{x} \in \mathbb{S}^2. \tag{2.12}$$

The native space $\mathcal{N}_\phi$ is therefore a *reproducing kernel Hilbert space* (see [1]) with the reproducing kernel ϕ.

From (2.12) it is obvious that the *radial basis function interpolation problem* (2.9) for a function f in the native space $\mathcal{N}_\phi$ can also be written as follows: given the values of $f \in \mathcal{N}_\phi$ on a point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$, find $\Lambda_X f \in V_X$ such that

$$(\Lambda_X f, \phi(\cdot, \mathbf{x}_i))_\phi = (f, \phi(\cdot, \mathbf{x}_i))_\phi \quad \text{for } i = 1, 2, \ldots, N.$$

This means that $\Lambda_X : \mathcal{N}_\phi \to V_X$ is just the $\mathcal{N}_\phi$*-orthogonal projection* (with respect to $(\cdot, \cdot)_\phi$) *onto the finite-dimensional subspace* V_X.

It is well known that the native space $\mathcal{N}_\phi$ associated with a zonal positive definite continuous kernel ϕ, of the form (2.7) and satisfying (2.8) and $a_\ell > 0$ for all $\ell \in \mathbb{N}_0$, has the inner product

$$(f, g)_\phi = \sum_{\ell=0}^{\infty} \frac{1}{a_\ell} \sum_{k=1}^{2\ell+1} \widehat{f}_{\ell,k} \widehat{g}_{\ell,k}, \qquad f, g \in \mathcal{N}_\phi,$$

and the associated norm

$$\|f\|_\phi = \left(\sum_{\ell=0}^{\infty} \frac{1}{a_\ell} \sum_{k=1}^{2\ell+1} |\widehat{f}_{\ell,k}|^2 \right)^{1/2}, \qquad f \in \mathcal{N}_\phi. \tag{2.13}$$

For more background information on radial basis functions on the sphere refer to [4, Chapters 5–7] and [18, Chapter 17].

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3. Local radial basis function approximation

Let $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be a zonal positive definite continuous kernel of the form (2.7) satisfying

$$c_1(\ell+1)^{-2s} \le a_\ell \le c_2(\ell+1)^{-2s} \quad \text{for all } \ell \in \mathbb{N}_0, \tag{3.1}$$

with positive constants c_1 and c_2 and some $s > 1$. We observe that, since $s > 1$, (3.1) implies that (2.8) is automatically satisfied, and hence the series in (2.7) converge uniformly on $\mathbb{S}^2 \times \mathbb{S}^2$ to ϕ.

Under the assumption (3.1), the norm $\|\cdot\|_\phi$, given by (2.13), is equivalent to the $\|\cdot\|_s$-norm (see (2.4)), that is,

$$c_2^{-1/2}\|f\|_s \le \|f\|_\phi \le c_1^{-1/2}\|f\|_s, \quad f \in \mathcal{N}_\phi, \tag{3.2}$$

with the constants c_1 and c_2 from (3.1). Thus the native space $\mathcal{N}_\phi$ associated with ϕ can be identified with the Sobolev space $H^s(\mathbb{S}^2)$, and is, in particular, embedded in $C(\mathbb{S}^2)$.

The following theorem, which is the main result of the paper, gives *local error estimates* for radial basis function interpolation on the sphere. More precisely, for $f \in H^s(\mathbb{S}^2)$ we obtain an estimate for the maximal absolute pointwise error of the radial basis function interpolant $\Lambda_X f$ on the cap $S(\mathbf{z}; r)$ in terms of the local mesh norm $h_{X,S(\mathbf{z};r)}$ of X with respect to $S(\mathbf{z}; r)$.

THEOREM 3.1. *Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of N distinct points on $\mathbb{S}^2$, and let $S(\mathbf{z}; r)$ be the spherical cap with centre $\mathbf{z} \in \mathbb{S}^2$ and radius r, where $0 < r \le \pi/2$. Assume that the local mesh norm $h_{X,S(\mathbf{z};r)}$ of X satisfies the estimate*

$$h_{X,S(\mathbf{z};r)} \le \frac{\tan(r/4)}{4(1+2/[\sqrt{3}\cos(r/2)])}. \tag{3.3}$$

Let $s > 1$, and let $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be a zonal positive definite continuous kernel of the form (2.7) with a_ℓ, for $\ell \in \mathbb{N}_0$, satisfying (3.1). Then

$$\|f - \Lambda_X f\|_{S(\mathbf{z};r)} := \sup_{\mathbf{x} \in S(\mathbf{z};r)} |f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \le c_r h_{X,S(\mathbf{z};r)}^{(s-1)/2} \|f\|_s, \quad f \in H^s(\mathbb{S}^2), \tag{3.4}$$

with the positive constant c_r given by

$$c_r := \frac{3\sqrt{c_2}}{2\sqrt{\pi(s-1)c_1}} \left(\frac{4(1+2/[\sqrt{3}\cos(r/2)])}{\tan(r/4)} \right)^{(s-1)/2},$$

where the positive constants c_1 and c_2 are the constants from (3.1).

REMARK 1. Note that the constant c_r depends only on r, s, c_1, and c_2, and that $c_r \to \infty$ as $r \to 0^+$.

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For the proof of Theorem 3.1 we need the following theorem, which is of independent interest. It establishes a condition on the local mesh norm $h_{Y,S(\mathbf{z};r)}$ of a point set $Y \subset S(\mathbf{z}; r)$ for the $\mathbb{P}_L(\mathbb{S}^2)$-*unisolvency* of Y and for the existence of a set of local functions that provide *local polynomial reproduction on the cap* $S(\mathbf{z}; r)$. This theorem is motivated by a corresponding result which guarantees local polynomial reproduction on balls in $\mathbb{R}^{d+1}$ (see [18, Chapter 3]).

THEOREM 3.2. *Let $S(\mathbf{z}; r)$ be the spherical cap with centre $\mathbf{z} \in \mathbb{S}^2$ and radius r, where $0 < r \le \pi/2$, and let $L \ge 1$ be an integer. Let $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$ be a set of M distinct points on $S(\mathbf{z}; r)$ whose local mesh norm $h_{Y,S(\mathbf{z};r)}$ satisfies*

$$h_{Y,S(\mathbf{z};r)} \le \frac{\tan(r/4)}{4(1 + 2/[\sqrt{3}\cos(r/2)])L^2}. \tag{3.5}$$

Then Y is a $\mathbb{P}_L(\mathbb{S}^2)$-unisolvent set, and there exist functions $u_j : S(\mathbf{z}; r) \to \mathbb{R}$, $j = 1, 2, \ldots, M$, such that

$$\sum_{j=1}^{M} u_j(\mathbf{x}) p(\mathbf{y}_j) = p(\mathbf{x}) \quad \text{for all } p \in \mathbb{P}_L(\mathbb{S}^2) \text{ and all } \mathbf{x} \in S(\mathbf{z}; r), \tag{3.6}$$

and

$$\sum_{j=1}^{M} |u_j(\mathbf{x})| \le 2 \quad \text{for all } \mathbf{x} \in S(\mathbf{z}; r). \tag{3.7}$$

REMARK 2. Note that for a (global) finite point set $X \subset \mathbb{S}^2$ whose local mesh norm $h_{X,S(\mathbf{z};r)}$ with respect to $S(\mathbf{z}; r)$ satisfies (3.5), Theorem 3.2 can be applied to $Y := X \cap S(\mathbf{z}; r)$ and thus Y, and also X, is $\mathbb{P}_L(\mathbb{S}^2)$-unisolvent.

The proofs of Theorems 3.1 and 3.2 will be given in the next section.

4. Proofs

First we prove Theorem 3.2. This proof makes use of a lemma establishing a geometric property of a (local) finite point set on a spherical cap under some assumptions on the local mesh norm with respect to that cap. We also apply the Videnskii inequality, and a fundamental theorem about norming sets. The argument is inspired by arguments for $\mathbb{R}^{d+1}$ (see [18, Chapter 3]) that make use of the Markov inequality and domains that satisfy a cone condition.

LEMMA 4.1. *Let $S(\mathbf{z}; r)$ be the spherical cap on $\mathbb{S}^2$ with centre $\mathbf{z} \in \mathbb{S}^2$ and radius $r \le \pi/2$, and let $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$ be a point set on $S(\mathbf{z}; r)$ whose local mesh norm satisfies $h_{Y,S(\mathbf{z};r)} \le r/2$. For every $\mathbf{x} \in S(\mathbf{z}; r)$, there exists a point $\mathbf{y} \in Y$ with the following two properties:*

(i) $\operatorname{dist}(\mathbf{x}, \mathbf{y}) \le (1 + 2/[\sqrt{3}\cos(r/2)])h_{Y,S(\mathbf{z};r)}$;

(ii) *the geodesic arc starting at $\mathbf{x}$ and going through $\mathbf{y}$ and continuing up to the boundary of $S(\mathbf{z}; r)$ has length greater than or equal to r.*

The delicate proof of this lemma is given in Section 5. Here we just want to interpret the result. Since $h_{Y,S(\mathbf{z};r)}$ is the local mesh norm of the point set Y with respect to $S(\mathbf{z}; r)$, it is clear that for every $\mathbf{x} \in S(\mathbf{z}; r)$ there exists a point $\mathbf{y}' \in Y$ with $\operatorname{dist}(\mathbf{x}, \mathbf{y}') \le h_{Y,S(\mathbf{z};r)}$. However, we have in general no information about the length of the geodesic arc starting at $\mathbf{x}$ and going through $\mathbf{y}'$ up to the boundary of $S(\mathbf{z}; r)$. The lemma guarantees that there is a point $\mathbf{y} \in Y$ reasonably close to $\mathbf{x}$ with the property that the geodesic arc from $\mathbf{x}$ through $\mathbf{y}$ up to the boundary of the cap $S(\mathbf{z}; r)$ has length at least r.

REMARK 3. We also observe that the estimate in (i) easily gives an upper bound which is independent of r: since $r \le \pi/2$ we have $\cos(r/2) \ge 1/\sqrt{2}$, and thus $\operatorname{dist}(\mathbf{x}, \mathbf{y}) \le (1 + \sqrt{8/3})h_{Y,S(\mathbf{z};r)}$.

REMARK 4. The second term $2/(\sqrt{3}\cos(r/2))$ in the constant in the estimate in (i) is *asymptotically optimal*, in a sense that will become apparent in the proof of Lemma 4.1 in Section 5.

After some elementary transformations, the *Videnskii inequality* (see [2, p. 243, E. 19 c]) reads as follows.

LEMMA 4.2 (Videnskii inequality). *Let $\omega \in (0, 2\pi)$. If*

$$2L \ge \left(3\left(\tan\frac{\omega}{4}\right)^2 + 1\right)^{1/2}, \tag{4.1}$$

then, for every trigonometric polynomial s_L of degree $\le L$, the derivative s_L' satisfies

$$\|s_L'\|_{[0,\omega]} \le 2L^2 \cot\left(\frac{\omega}{4}\right) \|s_L\|_{[0,\omega]}. \tag{4.2}$$

Let V be a finite-dimensional real vector space with norm $\|\cdot\|_V$, and let V^* denote the dual space of V consisting of all linear and continuous functionals on V. A finite subset Z of V^* with cardinality M is called a *norming set for* V if the mapping $T : V \to \mathbb{R}^M$ defined by $T(v) = (z(v))_{z\in Z}$ is injective.

We shall use [18, Theorem 3.4] (see also [7] for the original source of Theorem 4.3 below), which we quote (with appropriate adaptation to our notation) for the reader's convenience.

THEOREM 4.3. *Let V be a finite-dimensional real vector space with norm $\|\cdot\|_V$, and let $Z = \{z_1, z_2, \ldots, z_M\} \subset V^*$ be a norming set for V. Further, let $T : V \to \mathbb{R}^M$ be the sampling operator $T(v) := (z_1(v), z_2(v), \ldots, z_M(v))$, where $\mathbb{R}^M$ is equipped with the supremum norm. Then, for every $\Psi \in V^*$, there exists a vector $\mathbf{u} = \mathbf{u}_\Psi \in \mathbb{R}^M$, $\mathbf{u} = (u_1, u_2, \ldots, u_M)$, depending only on Ψ, such that*

$$\Psi(v) = \sum_{j=1}^{M} u_j z_j(v), \qquad v \in V,$$

and

$$\sum_{j=1}^{M} |u_j| \le \|T^{-1}\| \, \|\Psi\|_{V^*}.$$

PROOF OF THEOREM 3.2. Let $p \in \mathbb{P}_L(\mathbb{S}^2)$ be an arbitrary spherical polynomial such that $\|p\|_{S(\mathbf{z};r)} = 1$. Then there exists a point $\mathbf{x} \in S(\mathbf{z}; r)$ such that $|p(\mathbf{x})| = 1$.

Under the assumptions in Theorem 3.2 it is easy to see using (3.5) that $h_{Y,S(\mathbf{z};r)} \le r/2$; thus from Lemma 4.1 and the assumption (3.5) there exists a point $\mathbf{y} \in Y$ such that

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) \le \left(1 + \frac{2}{\sqrt{3}\cos(r/2)}\right) h_{Y,S(\mathbf{z};r)} \le \frac{\tan(r/4)}{4L^2}, \tag{4.3}$$

and such that the geodesic arc from $\mathbf{x}$ through $\mathbf{y}$ up to the boundary of the cap $S(\mathbf{z}; r)$ has length at least r. For simplicity of notation, let us from now on denote the point where this geodesic arc reaches the boundary of $S(\mathbf{z}; r)$ by $\mathbf{q}$, and let r' $(\ge r)$ denote the length of the geodesic arc $[\mathbf{x}, \mathbf{q}]$. The length of $[\mathbf{x}, \mathbf{q}]$ is bounded from above by the diameter of the cap, that is, $r' \le 2r \le \pi$.

The restriction of any polynomial in $\mathbb{P}_L(\mathbb{S}^2)$ to a great circle is a trigonometric polynomial of degree at most L. Let now $P := p|_{[\mathbf{x},\mathbf{q}]}$ denote the trigonometric polynomial $P : [0, r'] \to \mathbb{R}$ of degree at most L obtained by restricting p to the geodesic arc $[\mathbf{x}, \mathbf{q}]$. The parametrization is such that $P(0) = p(\mathbf{x})$ and $P(r') = p(\mathbf{q})$. Furthermore, let ρ be the value of the parameter that corresponds to $\mathbf{y}$, that is, $P(\rho) = p(\mathbf{y})$ and $\rho := \mathrm{dist}(\mathbf{x}, \mathbf{y})$. Since $L \ge 1$ and $r \le r' \le \pi$, the estimate

$$2L \ge 2 = \left(3\left(\tan\frac{\pi}{4}\right)^2 + 1\right)^{1/2} \ge \left(3\left(\tan\frac{r'}{4}\right)^2 + 1\right)^{1/2},$$

ensures that (4.1) is satisfied with $\omega = r'$, allowing us to use the Videnskii inequality (4.2) for the trigonometric polynomial P with $\omega = r'$. Then the mean value theorem, the Videnskii inequality (4.2) with $\omega = r'$, $\rho = \mathrm{dist}(\mathbf{x}, \mathbf{y})$, the estimate (4.3), $\|P\|_{[0,r']} = |P(0)| = |p(\mathbf{x})| = 1$, and $0 < r \le r' \le \pi$ imply that

$$\begin{aligned} |p(\mathbf{x}) - p(\mathbf{y})| = |P(0) - P(\rho)| &\le \rho \|P'\|_{[0,r']} \\ &\le \mathrm{dist}(\mathbf{x}, \mathbf{y}) 2L^2 \cot(r'/4) \|P\|_{[0,r']} \\ &\le \frac{\tan(r/4)}{4L^2} 2L^2 \cot(r'/4) \\ &\le \tfrac{1}{2}. \end{aligned} \tag{4.4}$$

From (4.4) and $|p(\mathbf{x})| = 1$, we obtain with the help of the lower triangle inequality for our spherical polynomial p of degree at most L

$$|p(\mathbf{y})| \ge |p(\mathbf{x})| - |p(\mathbf{x}) - p(\mathbf{y})| \ge 1 - \tfrac{1}{2} = \tfrac{1}{2}. \tag{4.5}$$

Now let $V := \mathbb{P}_L(\mathbb{S}^2)|_{S(\mathbf{z};r)} := \{p|_{S(\mathbf{z};r)} : p \in \mathbb{P}_L(\mathbb{S}^2)\}$ be equipped with the supremum norm $\|\cdot\|_{S(\mathbf{z};r)}$, and consider the sampling operator

$$T : \mathbb{P}_L(\mathbb{S}^2)|_{S(\mathbf{z};r)} \to \mathbb{R}^M, \qquad p \mapsto T(p) := (p(\mathbf{y}_1), p(\mathbf{y}_2), \ldots, p(\mathbf{y}_M)),$$

corresponding to point evaluation on $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$, where $\mathbb{R}^M$ is equipped with the supremum norm $\|\mathbf{v}\|_\infty := \max\{|v_j| : j = 1, 2, \ldots, M\}$ for $\mathbf{v} = (v_1, v_2, \ldots, v_M) \in \mathbb{R}^M$. The operator T satisfies $\|T\|_\infty \leq 1$ because $\|T(p)\|_\infty \leq \|p\|_{S(\mathbf{z};r)}$ for all $p \in \mathbb{P}_L(\mathbb{S}^2)$. From (4.5) it follows that T is injective, since for every non-zero $p \in \mathbb{P}_L(\mathbb{S}^2)$ we have $T(p) \neq \mathbf{0}$. By definition, this means that the subset $Z = \{\delta_{\mathbf{y}_j} : j = 1, 2, \ldots, M\} \subset V^*$ of point evaluation functionals $\delta_{\mathbf{y}_j} : V \to \mathbb{R}$, $\delta_{\mathbf{y}_j}(p) := p(\mathbf{y}_j)$, is a *norming set* for $V = \mathbb{P}_L(\mathbb{S}^2)|_{S(\mathbf{z};r)}$. The inverse operator of T exists on the range $T(V) \subset \mathbb{R}^M$ of T (since T is injective), and $T^{-1} : T(V) \to V$ satisfies

$$\begin{aligned}\|T^{-1}\|_\infty &:= \sup_{\mathbf{v}\in T(V)\setminus\{\mathbf{0}\}} \frac{\|T^{-1}\mathbf{v}\|_{S(\mathbf{z};r)}}{\|\mathbf{v}\|_\infty} = \sup_{p\in V\setminus\{0\}} \frac{\|p\|_{S(\mathbf{z};r)}}{\|T(p)\|_\infty} \\ &= \sup_{p\in\mathbb{P}_L(\mathbb{S}^2)\setminus\{0\}} \frac{\|p\|_{S(\mathbf{z};r)}}{\max\{|p(\mathbf{y}_j)| : j = 1, 2, \ldots, M\}} \leq 2, \end{aligned} \tag{4.6}$$

where the estimate in the last step follows from (4.5).

Now we apply Theorem 4.3 to the finite-dimensional space V and the norming set $Z = \{\delta_{\mathbf{y}_j} : j = 1, 2, \ldots, M\} \subset V^*$ for V. Theorem 4.3 guarantees that for every point evaluation functional $\delta_{\mathbf{w}} : V \to \mathbb{R}$, $\delta_{\mathbf{w}}(p) := p(\mathbf{w})$, where $\mathbf{w} \in S(\mathbf{z}; r)$, there exists $\mathbf{u} = \mathbf{u}(\mathbf{w}) = (u_1(\mathbf{w}), u_2(\mathbf{w}), \ldots, u_M(\mathbf{w})) \in \mathbb{R}^M$ such that

$$p(\mathbf{w}) = \delta_{\mathbf{w}}(p) = \sum_{j=1}^{M} u_j(\mathbf{w})\delta_{\mathbf{y}_j}(p) = \sum_{j=1}^{M} u_j(\mathbf{w})p(\mathbf{y}_j), \qquad p \in \mathbb{P}_L(\mathbb{S}^2)|_{S(\mathbf{z};r)}, \tag{4.7}$$

and

$$\sum_{j=1}^{M} |u_j(\mathbf{w})| \leq \|T^{-1}\|_\infty \, \|\delta_{\mathbf{w}}\|_{V^*} \leq 2.$$

In the last step we have used (4.6) and the fact that the norm of any point evaluation functional $\delta_{\mathbf{w}}$, with $\mathbf{w} \in S(\mathbf{z}; r)$, satisfies the estimate

$$\|\delta_{\mathbf{w}}\|_{V^*} = \sup_{p\in V\setminus\{0\}} \frac{|p(\mathbf{w})|}{\|p\|_{S(\mathbf{z};r)}} \leq 1.$$

Because (4.7) is valid for all $p \in V = \mathbb{P}_L(\mathbb{S}^2)|_{S(\mathbf{z};r)}$, it also holds true for all $p \in \mathbb{P}_L(\mathbb{S}^2)$, since the sets are in one-to-one correspondence. This proves that (3.6) and (3.7) are satisfied.

It remains to show that $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$ is $\mathbb{P}_L(\mathbb{S}^2)$-unisolvent. Consider $p \in \mathbb{P}_L(\mathbb{S}^2)$ with $p(\mathbf{y}_j) = 0$ for $j = 1, 2, \ldots, M$. Then (3.6) implies that $p \equiv 0$ on $\mathbb{S}^2$. Thus Y is $\mathbb{P}_L(\mathbb{S}^2)$-unisolvent. This concludes the proof of Theorem 3.2. □

Now we prove Theorem 3.1 with the help of Theorem 3.2.

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PROOF OF THEOREM 3.1 – PART I. The radial basis function interpolant $\Lambda_X f$ of any function $f \in \mathcal{N}_\phi$ can be written in the Lagrange representation as

$$\Lambda_X f = \sum_{j=1}^{N} f(\mathbf{x}_j)\ell_j, \tag{4.8}$$

where the Lagrangians ℓ_j, $j = 1, 2, \ldots, N$, are defined by $\ell_j \in V_X$ and $\ell_j(\mathbf{x}_i) = \delta_{i,j}$ for $i = 1, 2, \ldots, N$, with the Kronecker symbol $\delta_{i,j}$ defined by $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. From (4.8) and the reproducing property (2.12) of the reproducing kernel ϕ, we find for any f in the native space $\mathcal{N}_\phi$ that

$$\begin{aligned} f(\mathbf{x}) - \Lambda_X f(\mathbf{x}) &= (f, \phi(\cdot, \mathbf{x}))_\phi - \sum_{j=1}^{N}(f, \phi(\cdot, \mathbf{x}_j))_\phi \ell_j(\mathbf{x}) \\ &= \left(f, \phi(\cdot, \mathbf{x}) - \sum_{j=1}^{N} \phi(\cdot, \mathbf{x}_j)\ell_j(\mathbf{x})\right)_\phi, \quad \mathbf{x} \in \mathbb{S}^2. \end{aligned}$$

The Cauchy–Schwarz inequality then yields

$$|f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \le \|f\|_\phi \, P_X(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2, \tag{4.9}$$

with the *power function* $P_X : \mathbb{S}^2 \to \mathbb{R}_0^+$ given by

$$\begin{aligned} P_X(\mathbf{x}) &:= \left\| \phi(\cdot, \mathbf{x}) - \sum_{j=1}^{N} \phi(\cdot, \mathbf{x}_j)\ell_j(\mathbf{x}) \right\|_\phi \\ &= \left(\phi(\mathbf{x}, \mathbf{x}) - 2\sum_{j=1}^{N} \phi(\mathbf{x}, \mathbf{x}_j)\ell_j(\mathbf{x}) + \sum_{i=1}^{N}\sum_{j=1}^{N} \phi(\mathbf{x}_i, \mathbf{x}_j)\ell_i(\mathbf{x})\ell_j(\mathbf{x}) \right)^{1/2}. \end{aligned} \tag{4.10}$$

The equivalence (3.2) of the norms $\|\cdot\|_\phi$ of $\mathcal{N}_\phi$ and $\|\cdot\|_s$ of $H^s(\mathbb{S}^2)$ and (4.9) imply that for every $f \in H^s(\mathbb{S}^2)$

$$|f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \le c_1^{-1/2} \, \|f\|_s \, P_X(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2, \tag{4.11}$$

with the constant c_1 from (3.1). □

It remains to estimate the power function P_X uniformly on $S(\mathbf{z}; r)$. We use the following well-known lemma.

LEMMA 4.4. *Let $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be a zonal positive definite continuous kernel of the form (2.7) with $a_\ell > 0$ for all $\ell \in \mathbb{N}_0$ and satisfying (2.8). Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of N distinct points on $\mathbb{S}^2$, and for fixed $\mathbf{x} \in \mathbb{S}^2$ define the quadratic functional $\mathcal{L}_\mathbf{x} : \mathbb{R}^N \to \mathbb{R}$ by*

$$\mathcal{L}_\mathbf{x}(\boldsymbol{\alpha}) = \phi(\mathbf{x}, \mathbf{x}) - 2\sum_{j=1}^{N} \phi(\mathbf{x}, \mathbf{x}_j)\alpha_j + \sum_{i=1}^{N}\sum_{j=1}^{N} \phi(\mathbf{x}_i, \mathbf{x}_j)\alpha_i\alpha_j,$$

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where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N$. *The unique global minimum of* $\mathcal{L}_{\mathbf{x}}$ *is achieved by the vector* $(\ell_1(\mathbf{x}), \ell_2(\mathbf{x}), \ldots, \ell_N(\mathbf{x}))$, *where* ℓ_j, $j = 1, 2, \ldots, N$, *is the Lagrangian corresponding to* $\mathbf{x}_j$, *defined by* $\ell_j \in V_X$ *and* $\ell_j(\mathbf{x}_i) = \delta_{i,j}$ *for* $i = 1, 2, \ldots, N$.

PROOF. Computation of the partial derivative $\partial \mathcal{L}_{\mathbf{x}}(\boldsymbol{\alpha})/\partial \alpha_j$ yields

$$\frac{\partial \mathcal{L}_{\mathbf{x}}(\boldsymbol{\alpha})}{\partial \alpha_j} = -2\phi(\mathbf{x}, \mathbf{x}_j) + 2\sum_{i=1}^{N} \phi(\mathbf{x}_i, \mathbf{x}_j)\alpha_i.$$

Thus $\nabla \mathcal{L}_{\mathbf{x}}(\boldsymbol{\alpha}) = 0$ is equivalent to

$$\phi(\mathbf{x}, \mathbf{x}_j) = \sum_{i=1}^{N} \phi(\mathbf{x}_i, \mathbf{x}_j)\alpha_i, \quad j = 1, 2, \ldots, N,$$

which has a unique solution since the kernel ϕ is positive definite. This unique solution is $\alpha_i = \ell_i(\mathbf{x})$, $i = 1, 2, \ldots, N$, since $\sum_{i=1}^{N} \phi(\mathbf{x}_i, \mathbf{x}_j)\ell_i$ is the Lagrange representation of $\phi(\cdot, \mathbf{x}_j) \in V_X$. At this critical point $\mathcal{L}_{\mathbf{x}}$ assumes a local minimum, because the Hessian of $\mathcal{L}_{\mathbf{x}}$ is given by the positive definite matrix $[2\phi(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1,2,\ldots,N}$. Since the local minimum at $\boldsymbol{\alpha} = (\ell_1(\mathbf{x}), \ell_2(\mathbf{x}), \ldots, \ell_N(\mathbf{x}))$ is the only local minimum, it is the global minimum of $\mathcal{L}_{\mathbf{x}}$. □

PROOF OF THEOREM 3.1 – PART II. Now we estimate the power function P_X uniformly on $S(\mathbf{z}; r)$. From Lemma 4.4, we know that an upper bound on the power function is obtained when the $\ell_j(\mathbf{x})$, $j = 1, 2, \ldots, N$, are replaced in (4.10) by the components of any vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N$. To obtain a suitable vector $\boldsymbol{\alpha}$ we apply Theorem 3.2. Consider the point set $Y := X \cap S(\mathbf{z}; r)$. For simplicity, we denote the M $(\leq N)$ points in Y by $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M$, that is, $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$, and, after renumbering the points in X, we may assume that $\mathbf{y}_i = \mathbf{x}_i$ for $i = 1, 2, \ldots, M$. We observe that $h_{Y,S(\mathbf{z};r)} = h_{X,S(\mathbf{z};r)}$, and we define

$$L := \left\lfloor \left(\frac{\tan(r/4)}{4(1 + 2/[\sqrt{3}\cos(r/2)])h_{X,S(\mathbf{z};r)}} \right)^{1/2} \right\rfloor,$$

which is greater than or equal to 1 by the assumption (3.3) in Theorem 3.1. From this it follows that

$$L^2 \leq \frac{\tan(r/4)}{4(1 + 2/[\sqrt{3}\cos(r/2)])h_{X,S(\mathbf{z};r)}},$$

which is equivalent to the assumption (3.5) in Theorem 3.2. Thus we know that Y is a $\mathbb{P}_L(\mathbb{S}^2)$-unisolvent set and that there exist functions $u_j : S(\mathbf{z}; r) \to \mathbb{R}$, $j = 1, 2, \ldots, M$, such that

$$\sum_{j=1}^{M} u_j(\mathbf{x})p(\mathbf{y}_j) = p(\mathbf{x}) \quad \text{for all } p \in \mathbb{P}_L(\mathbb{S}^2) \text{ and all } \mathbf{x} \in S(\mathbf{z}; r), \tag{4.12}$$

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and

$$\sum_{j=1}^{M} |u_j(\mathbf{x})| \le 2 \quad \text{for all } \mathbf{x} \in S(\mathbf{z}; r). \tag{4.13}$$

Now we fix $\mathbf{x} \in S(\mathbf{z}; r)$, and choose the vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N$ to be

$$\alpha_j := \begin{cases} u_j(\mathbf{x}), & j = 1, 2, \ldots, M, \\ 0, & j = M+1, M+2, \ldots, N. \end{cases}$$

(Remember that we have renumbered the points in X so that $\mathbf{x}_j = \mathbf{y}_j$ for $j = 1, 2, \ldots, M$.) From Lemma 4.4 we obtain then

$$P_X(\mathbf{x}) \le \left(\phi(\mathbf{x}, \mathbf{x}) - 2 \sum_{j=1}^{M} \phi(\mathbf{x}, \mathbf{x}_j) u_j(\mathbf{x}) + \sum_{i=1}^{M} \sum_{j=1}^{M} \phi(\mathbf{x}_i, \mathbf{x}_j) u_i(\mathbf{x}) u_j(\mathbf{x}) \right)^{1/2}. \tag{4.14}$$

From the local polynomial reproduction property (4.12) we have, for all $\ell \le L$,

$$\begin{aligned} &P_\ell(\mathbf{x} \cdot \mathbf{x}) - 2 \sum_{j=1}^{M} P_\ell(\mathbf{x} \cdot \mathbf{x}_j) u_j(\mathbf{x}) + \sum_{i=1}^{M} \sum_{j=1}^{M} P_\ell(\mathbf{x}_i \cdot \mathbf{x}_j) u_i(\mathbf{x}) u_j(\mathbf{x}) \\ &\quad = \left(P_\ell(\mathbf{x} \cdot \mathbf{x}) - \sum_{j=1}^{M} P_\ell(\mathbf{x} \cdot \mathbf{x}_j) u_j(\mathbf{x}) \right) \\ &\qquad - \sum_{j=1}^{M} \left(P_\ell(\mathbf{x} \cdot \mathbf{x}_j) - \sum_{i=1}^{M} P_\ell(\mathbf{x}_i \cdot \mathbf{x}_j) u_i(\mathbf{x}) \right) u_j(\mathbf{x}) = 0. \end{aligned}$$

Thus the terms up to (and including the) degree L in the series expansion (2.7) of ϕ cancel on the right-hand side of (4.14), and we can replace ϕ in (4.14) by

$$\phi^L(\mathbf{x}, \mathbf{y}) := \sum_{\ell=L+1}^{\infty} \frac{2\ell+1}{4\pi} a_\ell P_\ell(\mathbf{x} \cdot \mathbf{y}) = \sum_{\ell=L+1}^{\infty} a_\ell \sum_{k=1}^{2\ell+1} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}).$$

This, together with $|P_\ell(t)| \le 1$ for all $t \in [-1, 1]$, the estimate (4.13), and the assumption (3.1) on the coefficients a_ℓ, $\ell \in \mathbb{N}_0$, yield

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$$
\begin{aligned}
P_X(\mathbf{x}) &\leq \Big(\phi^L(\mathbf{x},\mathbf{x}) - 2\sum_{j=1}^{M}\phi^L(\mathbf{x},\mathbf{x}_j)u_j(\mathbf{x}) + \sum_{i=1}^{M}\sum_{j=1}^{M}\phi^L(\mathbf{x}_i,\mathbf{x}_j)u_i(\mathbf{x})u_j(\mathbf{x})\Big)^{1/2} \\
&= \bigg(\sum_{\ell=L+1}^{\infty}\frac{2\ell+1}{4\pi}a_\ell\bigg[P_\ell(1) - 2\sum_{j=1}^{M}P_\ell(\mathbf{x}\cdot\mathbf{x}_j)u_j(\mathbf{x}) \\
&\qquad + \sum_{i=1}^{M}\sum_{j=1}^{M}P_\ell(\mathbf{x}_i\cdot\mathbf{x}_j)u_i(\mathbf{x})u_j(\mathbf{x})\bigg]\bigg)^{1/2} \\
&\leq \bigg(\sum_{\ell=L+1}^{\infty}\frac{2\ell+1}{4\pi}a_\ell[1+4+4]\bigg)^{1/2} \qquad (4.15)\\
&\leq \frac{3\sqrt{c_2}}{\sqrt{2\pi}}\bigg(\sum_{\ell=L+1}^{\infty}(\ell+1)^{1-2s}\bigg)^{1/2} \\
&\leq \frac{3\sqrt{c_2}}{\sqrt{2\pi}\sqrt{2(s-1)}}(L+1)^{1-s}. \qquad (4.16)
\end{aligned}
$$

Finally, since

$$
L+1 > \bigg(\frac{\tan(r/4)}{4(1+2/[\sqrt{3}\cos(r/2)])h_{X,S(\mathbf{z};r)}}\bigg)^{1/2},
$$

the estimate (4.16) implies

$$
P_X(\mathbf{x}) \leq \frac{3\sqrt{c_2}}{2\sqrt{\pi(s-1)}}\bigg(\frac{4(1+2/[\sqrt{3}\cos(r/2)])}{\tan(r/4)}\bigg)^{(s-1)/2} h_{X,S(\mathbf{z};r)}^{(s-1)/2}, \qquad (4.17)
$$

for all $\mathbf{x} \in S(\mathbf{z}; r)$. Combining (4.11) and (4.17) yields (3.4). □

5. Proof of Lemma 4.1

The proof of Lemma 4.1 is done by a geometric construction. Note that in all our illustrations $S(\mathbf{z}; r)$ is depicted as a ball in $\mathbb{R}^2$, since this is easier to draw and visualize. However, this ignores the curvature of the sphere. In particular, this means that geodesic arcs appear as straight lines in our illustrations and that any (seemingly planar) triangle in our illustrations is in fact a spherical triangle, that is, a triangle given by connecting the corner points with geodesic arcs.

For purposes of explanation and description in the proof, we shall sometimes use *polar coordinates* for $\mathbb{S}^2$, given by

$$
\mathbf{x}(\vartheta,\varphi) = (\sin\vartheta\ \cos\varphi, \sin\vartheta\ \sin\varphi, \cos\vartheta), \quad \vartheta\in[0,\pi],\ \varphi\in[-\pi,\pi]. \qquad (5.1)
$$

We shall also use the following well-known fact.

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LEMMA 5.1. *Any spherical cap $S(\mathbf{z}; r)$ with $r < \pi/2$ is convex. That is, the shortest geodesic arc between any two points of $S(\mathbf{z}; r)$ lies in $S(\mathbf{z}; r)$, and is unique.*

PROOF OF LEMMA 4.1. First we discuss the trivial case $\mathbf{x} = \mathbf{z}$. In this case we know that there exists a point $\mathbf{y}$ in Y with $\text{dist}(\mathbf{x}, \mathbf{y}) \leq h_{Y,S(\mathbf{z};r)}$, and the geodesic arc from $\mathbf{x} = \mathbf{z}$ through $\mathbf{y}$ up to the boundary of $S(\mathbf{z}; r)$ has exactly the length r. So properties (i) and (ii) are clearly satisfied.

For the case $\mathbf{x} \neq \mathbf{z}$ the proof is much more complicated. First we need a geometric construction of a 'spherical cone' $C(\mathbf{x}, \mathbf{z})$ which is a subset of $S(\mathbf{z}; r)$ having a vertex at $\mathbf{x}$, and which contains $\mathbf{z}$, and which is guaranteed to contain a point $\mathbf{y} \in Y$ satisfying conditions (i) and (ii) in the lemma.

Without loss of generality, we may assume in the following that $\mathbf{x}$ and $\mathbf{z}$ both lie on the equator, and that $\mathbf{z}$ has the polar coordinates $(\pi/2, 0)$ (that is, $\mathbf{z} = (1, 0, 0)$) and that $\mathbf{x}$ lies west of it, that is, $\mathbf{x}$ has the polar coordinates $(\pi/2, -\rho)$ with $\rho := \text{dist}(\mathbf{x}, \mathbf{z}) \leq r$. In particular, this means that the uniquely determined great circle through $\mathbf{x}$ and $\mathbf{z}$ is the equator, and any great circle perpendicular to the equator passes through the north pole $\mathbf{n} = (0, 0, 1)$ and the south pole $\mathbf{s} = (0, 0, -1)$. We denote the point given by the intersection of the equator and the boundary of $S(\mathbf{z}; r)$ to the east of $\mathbf{z}$ by $\mathbf{q}$; this point has the polar coordinates $(\pi/2, r)$.

From now on we consider only $r < \pi/2$, and comment at the end on the case $r = \pi/2$. Since $r < \pi/2$, the cap $S(\mathbf{z}; r)$ contains neither the north nor the south pole, and from Lemma 5.1 $S(\mathbf{z}; r)$ is convex.

For $\mathbf{x} \in S(\mathbf{z}; r) \setminus \{\mathbf{z}\}$ consider a second spherical cap $S(\mathbf{x}; r)$, with centre $\mathbf{x}$ and radius r. Since $\mathbf{x} \neq \mathbf{z}$, there are exactly two points $\mathbf{p}_1$ and $\mathbf{p}_2$ where the boundaries of the two caps intersect, and the geodesic arc $[\mathbf{p}_1, \mathbf{p}_2]$ is perpendicular to the geodesic arc $[\mathbf{x}, \mathbf{z}]$, cutting $[\mathbf{x}, \mathbf{z}]$ in half (see Figure 1(a)). Since the two spherical caps $S(\mathbf{z}; r)$ and $S(\mathbf{x}; r)$ are not identical, their union has an area which is larger than that of each individual cap. Also $[\mathbf{p}_1, \mathbf{p}_2]$ cuts the union of the two caps into two mirror-symmetric regions of equal size, and each of these two regions has an area which is strictly larger than $|S(\mathbf{z}; r)|/2$. From now on we focus on the region containing $\mathbf{z}$, and denote it by $A(\mathbf{x}, \mathbf{z})$. The set $A(\mathbf{x}, \mathbf{z})$ can be described as the part of the cap $S(\mathbf{z}; r)$ to the east of the geodesic arc $[\mathbf{p}_1, \mathbf{p}_2]$. Since its area is larger than $|S(\mathbf{z}; r)|/2$, it has to contain the 'hemi-cap' $\{\mathbf{v} = \mathbf{v}(\vartheta, \varphi) \in S(\mathbf{z}; r) : \varphi \geq 0\}$ which is the part of $S(\mathbf{z}; r)$ to the east of the longitude through $\mathbf{z}$.

Now we construct our spherical cone $C(\mathbf{x}, \mathbf{z})$ as shown in Figure 1(b): the two geodesic arcs $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$ cut the original cap $S(\mathbf{z}; r)$ into a (smaller) part which does not contain $\mathbf{z}$ and a (larger) part which contains $\mathbf{z}$. We denote the part that contains $\mathbf{z}$ by $C(\mathbf{x}, \mathbf{z})$. We shall call $C(\mathbf{x}, \mathbf{z})$ the spherical cone with vertex at $\mathbf{x}$ and 'arms' $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$.

Since $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$ are geodesic arcs, and since $\mathbf{x}$ lies outside $A(\mathbf{x}, \mathbf{z})$, it follows that $C(\mathbf{x}, \mathbf{z})$ contains $A(\mathbf{x}, \mathbf{z})$. Hence $C(\mathbf{x}, \mathbf{z})$ contains in particular the 'hemi-cap' $\{\mathbf{v} = \mathbf{v}(\vartheta, \varphi) \in S(\mathbf{z}; r) : \varphi \geq 0\}$. Inside this 'hemi-cap' we can place a unique spherical cap with radius $r/2$ so that its centre is on the equator half-way between $\mathbf{z}$

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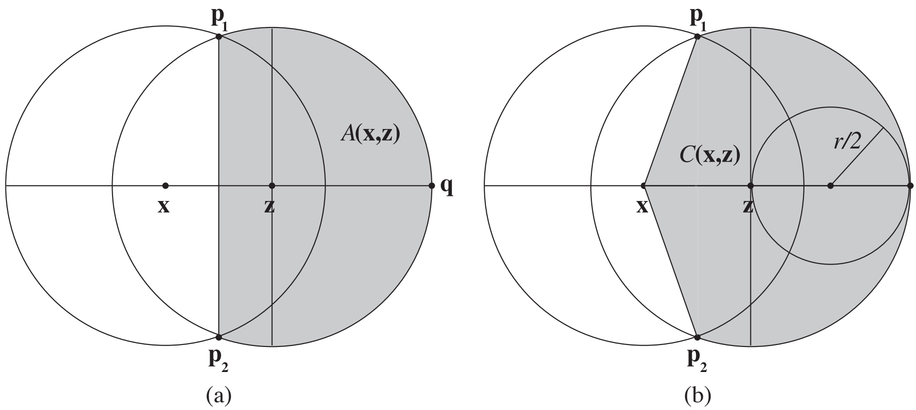

FIGURE 1. (a) The region $A(\mathbf{x}, \mathbf{z})$ and (b) the cone $C(\mathbf{x}, \mathbf{z})$ with the spherical cap of radius $r/2$ with centre in the middle of $[\mathbf{z}, \mathbf{q}]$, for $r < \pi/2$.

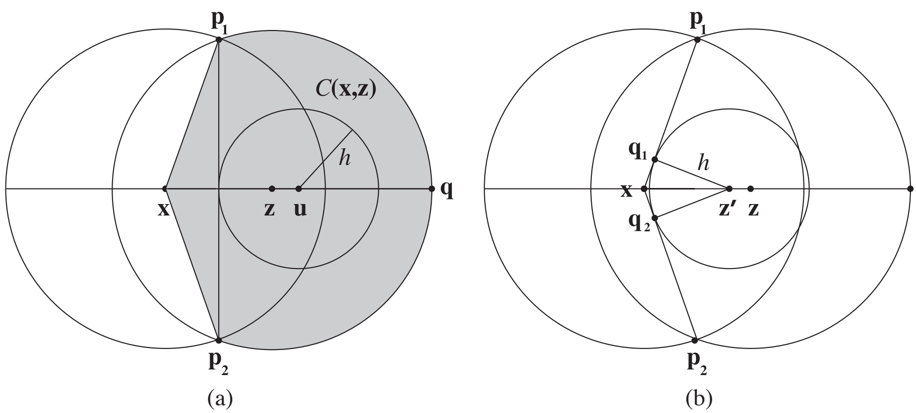

FIGURE 2. (a) The cone $C(\mathbf{x}, \mathbf{z})$ with the spherical cap $S(\mathbf{u}; h)$ and (b) the cone $C(\mathbf{x}, \mathbf{z})$ with the spherical cap $S(\mathbf{z}'; h)$. In both pictures $h = r/2$ and $r < \pi/2$.

and $\mathbf{q}$, as shown in Figure 1(b). It follows that any spherical cap with the same centre and radius $h \le r/2$ is also contained in the 'hemi-cap' and hence in $C(\mathbf{x}, \mathbf{z})$.

Now we move the centre of this spherical cap with radius $h \le r/2$ westwards along the equator subject to the condition that the cap stays inside $C(\mathbf{x}, \mathbf{z})$. At some point the west side of the boundary of the cap just touches $[\mathbf{p}_1, \mathbf{p}_2]$ as shown in Figure 2(a); we denote the centre of the cap of radius $h \le r/2$ at this position by $\mathbf{u}$. We observe that

$$\operatorname{dist}(\mathbf{x}, \mathbf{u}) = \frac{\operatorname{dist}(\mathbf{x}, \mathbf{z})}{2} + h \le \frac{r}{2} + \frac{r}{2} = r < \pi/2. \tag{5.2}$$

We now continue to move the centre of the cap with radius h further along the equator to the west until the cap just touches the arms $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$ of the cone $C(\mathbf{x}, \mathbf{z})$. This is as far as we can go to the west while keeping the cap with radius h completely inside $C(\mathbf{x}, \mathbf{z})$.

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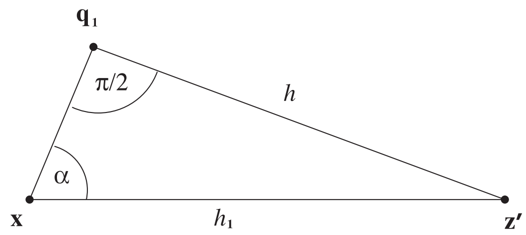

FIGURE 3. The triangle $\Delta(\mathbf{x}, \mathbf{z}', \mathbf{q}_1)$.

Let us denote by $\mathbf{z}'$ the centre of this final spherical cap. Thus $\mathbf{z}'$ is the centre of the spherical cap with centre on the equator and radius h, lying inside $C(\mathbf{x}, \mathbf{z})$ and just touching the arms $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$ of the cone $C(\mathbf{x}, \mathbf{z})$ (see Figure 2(b)). As in the figure, we denote the points where $S(\mathbf{z}'; h)$ touches $[\mathbf{x}, \mathbf{p}_1]$ and $[\mathbf{x}, \mathbf{p}_2]$ by $\mathbf{q}_1$ and $\mathbf{q}_2$, respectively.

We are interested in obtaining an estimate for the distance

$$h_1 := \text{dist}(\mathbf{x}, \mathbf{z}').$$

Obviously, from (5.2) one bound is $h_1 \leq \text{dist}(\mathbf{x}, \mathbf{u}) \leq r < \pi/2$, but we need a bound of the form $h_1 \leq \gamma h$, with a constant γ independent of h. For this purpose we consider the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}', \mathbf{q}_1)$ with corners given by $\mathbf{x}$, $\mathbf{z}'$, and $\mathbf{q}_1$ and sides given by the shortest geodesic arcs connecting the corner points (see Figures 2(b) and 3). Since $[\mathbf{x}, \mathbf{q}_1]$ is a sub-arc of the geodesic arc $[\mathbf{x}, \mathbf{p}_1]$ and since $S(\mathbf{z}'; h)$ just touches $[\mathbf{x}, \mathbf{p}_1]$ at $\mathbf{q}_1$, the angle at the corner $\mathbf{q}_1$ is $\pi/2$. Let us denote the angle at the corner $\mathbf{x}$ by α. We know that $\text{dist}(\mathbf{z}', \mathbf{q}_1) = h$. From the spherical sine theorem (see [12, Theorem 2.5.2]) we obtain

$$\sin h_1 = \frac{\sin h_1}{\sin(\pi/2)} = \frac{\sin h}{\sin \alpha}. \tag{5.3}$$

The angle α satisfies $\pi/3 < \alpha < \pi/2$. (This will be shown in Appendix A.) Since $\pi/3 < \alpha < \pi/2$, we have $\sin \alpha > \sqrt{3}/2$, and thus from (5.3)

$$\sin h_1 < \frac{2}{\sqrt{3}} \sin h. \tag{5.4}$$

Since $h_1 < \pi/2$, we can use $\sin h_1 > 2h_1/\pi$ and $\sin h \leq h$, yielding with (5.4)

$$\frac{2h_1}{\pi} < \frac{2h}{\sqrt{3}} \quad \Leftrightarrow \quad h_1 < \frac{\pi}{\sqrt{3}} h. \tag{5.5}$$

Since $\pi/\sqrt{3} \approx 1.8138$, (5.5) provides the upper bound $h_1 < 2h$ which we now use to improve the upper bound (5.5). Since we have $h_1 < 2h < \pi/2$ (with the last inequality following from $h \le r/2$ and $r < \pi/2$), from the monotonically decreasing nature of $\sin x/x$ on $[0, \pi/2]$ it follows that

$$\frac{\sin(2h)}{2h} < \frac{\sin h_1}{h_1}. \tag{5.6}$$

Using $\sin(2h) = 2\sin h \cos h$ we obtain from (5.6) and (5.4) that

$$\sin h \cos h \frac{h_1}{h} < \frac{2}{\sqrt{3}} \sin h \quad \Leftrightarrow \quad h_1 < \frac{2}{\sqrt{3}\cos h} h.$$

To obtain a constant that is independent of h, we use $\cos h \ge \cos(r/2)$ (from $h \le r/2 < \pi/4$), and thus find that

$$h_1 < \frac{2}{\sqrt{3}\cos(r/2)} h. \tag{5.7}$$

Before we continue with the proof we want to explain why (5.7) is in a certain sense *asymptotically optimal*. If we consider $r \to 0^+$ then the area of the triangle $\Delta(\mathbf{x}, \mathbf{z}', \mathbf{q}_1)$ shrinks and the geometry of the triangle becomes more and more Euclidean. In the limit case $r = 0$ we would have a planar triangle. For a planar triangle as in Figure 3, we have $h = h_1 \sin\alpha$. Since $\pi/3 < \alpha < \pi/2$ we have $h_1 = h/\sin\alpha < 2h/\sqrt{3}$, with equality as $\alpha \to \pi/3^+$. This is exactly the estimate that we obtain from (5.7) for $r \to 0^+$. Moreover, the limit $\alpha \to \pi/3^+$ is achieved in the Euclidean case when $\mathbf{x}$ is on the boundary of $S(\mathbf{z}; r)$ as we then have a triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ with three sides of equal length r. Thus (5.7) cannot be improved when r is small.

Now we continue with the proof. Let h in the above construction be the local mesh norm $h_{Y,S(\mathbf{z};r)}$ of Y with respect to $S(\mathbf{z}; r)$. We have established that the distance $h_1 = \text{dist}(\mathbf{x}, \mathbf{z}')$ satisfies the estimate (5.7) with $h = h_{Y,S(\mathbf{z};r)}$. Moreover, by the definition of the local mesh norm $h_{Y,S(\mathbf{z};r)}$, the spherical cap $S(\mathbf{z}'; h_{Y,S(\mathbf{z};r)})$ contains a point $\mathbf{y}$ from the point set Y. The distance of this point $\mathbf{y}$ from $\mathbf{x}$ is bounded by

$$\begin{aligned}
\text{dist}(\mathbf{x}, \mathbf{y}) &\le \text{dist}(\mathbf{x}, \mathbf{z}') + \text{dist}(\mathbf{z}', \mathbf{y}) \\
&\le h_1 + h_{Y,S(\mathbf{z};r)} \\
&< \left(1 + \frac{2}{\sqrt{3}\cos(r/2)}\right) h_{Y,S(\mathbf{z};r)},
\end{aligned}$$

and thus $\mathbf{y}$ satisfies property (i) in Lemma 4.1. The geodesic arc from $\mathbf{x}$ through $\mathbf{y}$ up to the boundary of $C(\mathbf{x}, \mathbf{z})$ lies inside $C(\mathbf{x}, \mathbf{z})$ and intersects the common boundary of $C(\mathbf{x}, \mathbf{z})$ and $S(\mathbf{z}; r)$. Since this geodesic arc crosses the boundary of $S(\mathbf{x}; r)$, it is clear from the construction that it has length $\ge r$.

It remains to briefly discuss the case when $r = \pi/2$. For this case the proof follows along the same lines as for $r < \pi/2$, but it is much simpler due to the particular geometric scenario. We briefly sketch the essential steps. For $r = \pi/2$, $S(\mathbf{z}; \pi/2)$

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is a hemisphere whose boundary is the union of the longitudes through (0, 1, 0) and (0, −1, 0); this boundary contains both the north pole $\mathbf{n} = (0, 0, 1)$ and the south pole $\mathbf{s} = (0, 0, -1)$. Now we consider the spherical cap $S(\mathbf{x}; \pi/2)$. It is also a hemisphere, and its boundary intersects the boundary of $S(\mathbf{z}; \pi/2)$ in $\mathbf{n}$ and $\mathbf{s}$. The union of the geodesic arcs $[\mathbf{x}, \mathbf{n}]$ and $[\mathbf{x}, \mathbf{s}]$ is just the longitude $\{\mathbf{v} = \mathbf{v}(\vartheta, \varphi) : \varphi = -\rho\}$ through $\mathbf{x}$. Now we choose $C(\mathbf{x}, \mathbf{z})$ to be the part of $S(\mathbf{z}; \pi/2)$ to the east of this longitude. Because in this case the angle at the vertex at $\mathbf{x}$ formed by the arms $[\mathbf{x}, \mathbf{n}]$ and $[\mathbf{x}, \mathbf{s}]$ of the cone is π (a degenerate case), the construction is now much simpler. For fixed $0 < h \le r/2 = \pi/4$, $\mathbf{z}'$ is now the point with polar coordinates $(\pi/2, -\rho + h)$, so that $S(\mathbf{z}'; h)$ lies in $C(\mathbf{x}, \mathbf{z})$ and touches the boundary of $C(\mathbf{x}, \mathbf{z})$ just in the point $\mathbf{x}$, and $h_1 = h$. From here onwards the proof continues analogously to that for the case $0 < r < \pi/2$. □

6. Numerical results

In our numerical experiment we consider the function

$$f(x, y, z) := \exp(x + y + z) + 25x_+^3, \quad (x, y, z) \in \mathbb{R}^3 \cap \mathbb{S}^2,$$

where x_+ is x if $x > 0$ and zero otherwise. The function f consists of the infinitely often differentiable component $f_1(x, y, z) := \exp(x + y + z)$ and the (local) component $f_2(x, y, z) := x_+^3$. The latter function has support on the hemisphere $S((1, 0, 0); \pi/2)$, is twice continuously differentiable, and is in the Sobolev space $H^s(\mathbb{S}^2)$ for any $s < 3.5$ (see [4, pp. 123–126]). Thus $f \in H^s(\mathbb{S}^2)$ for any $s < 3.5$.

As radial basis function (RBF) we use the zonal positive definite function of Wendland type (see [17] and [18, p. 129]) given by

$$\phi(\mathbf{x}, \mathbf{y}) := \psi\left(\sqrt{2 - 2\mathbf{x} \cdot \mathbf{y}}\right), \quad \psi(r) := (1 - r)_+^4 (4r + 1).$$

The asymptotic behaviour of the Fourier coefficients of this RBF is (see [11])

$$c_1(\ell + 1)^{-5} \le a_\ell \le c_2(\ell + 1)^{-5},$$

and hence the native space can be identified with $H^{2.5}(\mathbb{S}^2)$.

We compute the RBF interpolant $\Lambda_X f$ of the function f from the values of f given on a set X of uniformly distributed points on the polar cap $S(\mathbf{n}; 0.1)$, where $\mathbf{n} = (0, 0, 1)$ is the north pole. These points were generated by modifying Saff's algorithm for partitioning the sphere into regions of equal area and approximately equal diameter (see [14]), and then restricting the point set to $S(\mathbf{n}; 0.1)$; from now on we shall refer to these points as 'Saff points'. We stress that we do not use any values of the function at points outside the cap $S(\mathbf{n}; 0.1)$. The arc $x = 0$, along which the function x_+^3 'lacks smoothness', cuts the north polar cap $S(\mathbf{n}; 0.1)$ in half. We may expect the lack of smoothness along this arc to be reflected in the pointwise error of the RBF interpolant.

TABLE 1. The local mesh norm $h_{X,S(\mathbf{n};0.1)}$ of the set X of N Saff points, and the maximal absolute pointwise error err(X, $S(\mathbf{n}; 0.1)$) of the RBF interpolant on the cap $S(\mathbf{n}; 0.1)$. (Note that $X \subset S(\mathbf{n}; 0.1)$.)

N	$h_{X,S(\mathbf{n};0.1)}$	err(X, $S(\mathbf{n}; 0.1)$)
500	0.007353	0.001596
1000	0.005220	6.9838×10^{-4}
2000	0.003676	2.8189×10^{-4}
3000	0.003005	1.7838×10^{-4}
4000	0.002603	1.2306×10^{-4}
5000	0.002332	9.4106×10^{-5}

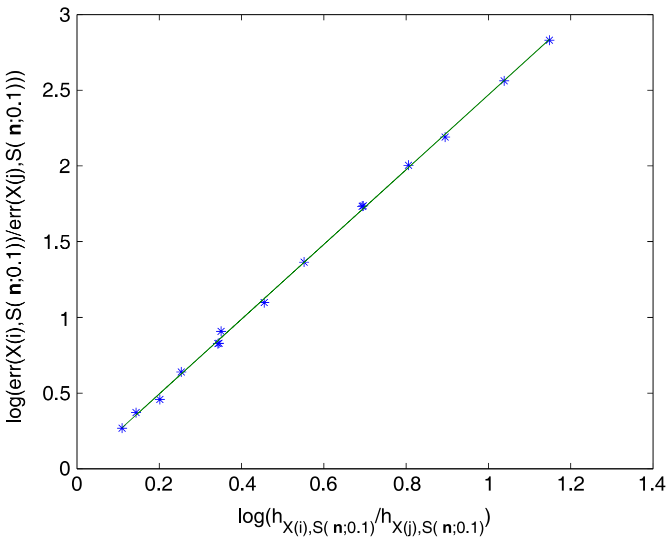
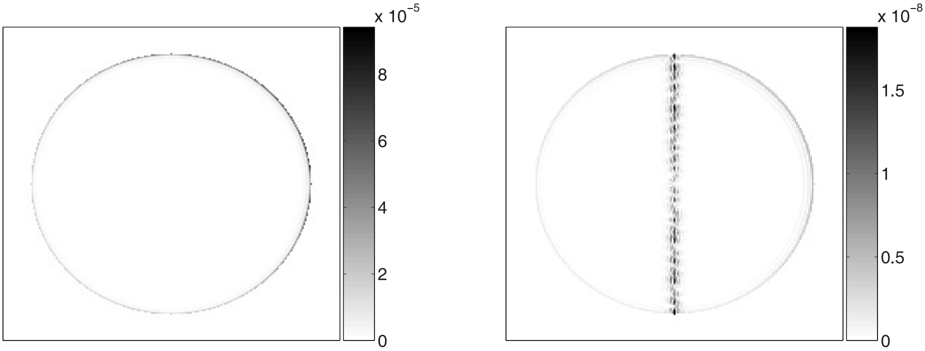


FIGURE 4. Plot of the logarithm of the ratio of the maximal absolute pointwise errors of the RBF interpolants on $S(\mathbf{n}; 0.1)$ against the logarithm of the ratio of the respective local mesh norms with respect to $S(\mathbf{n}; 0.1)$. All possible combinations of the data in Table 1 have been used.

Table 1 lists the cardinality N of the set X of Saff points in the first column, and its local mesh norm $h_{X,S(\mathbf{n};0.1)}$ with respect to $S(\mathbf{n}; 0.1)$ in the second column. The third column lists the maximal absolute pointwise error err(X, $S(\mathbf{n}; 0.1)$) of $\Lambda_X f$ on the spherical cap $S(\mathbf{n}; 0.1)$. The maximal absolute pointwise error on the spherical cap $S(\mathbf{n}; 0.1)$ was estimated by determining the maximum of the absolute pointwise error at those points of a 241×241 point evaluation grid that lie inside the cap $S(\mathbf{n}; 0.1)$. This grid is a rectangular region of a polar coordinate grid which has been rotated such that its centre is mapped from the equator onto the north pole $\mathbf{n}$.

We see that the maximal absolute pointwise error declines as the cardinality N of the set X of Saff points inside the cap $S(\mathbf{n}; 0.1)$ increases and the local mesh norm $h_{X,S(\mathbf{n};0.1)}$ with respect to $S(\mathbf{n}; 0.1)$ decreases.



(a) (b)

FIGURE 5. The absolute pointwise error on (a) $S(\mathbf{n};\, 0.1)$ and (b) $S(\mathbf{n};\, 0.08)$ of the RBF interpolant of f, computed from data at 5000 Saff points located inside the cap $S(\mathbf{n};\, 0.1)$. In each picture the error has been set to zero outside the respective cap. Note the very different scale for the error on the smaller cap.

We are interested in monitoring the decay rate of the maximal absolute pointwise error on $S(\mathbf{n};\, 0.1)$ as the local mesh norm $h_{X,S(\mathbf{n};0.1)}$ of the set X of Saff points declines. From (3.4) in Theorem 3.1 we know that

$$\operatorname{err}(X,\, S(\mathbf{n};\, 0.1)) \approx \sup_{\mathbf{x}\in S(\mathbf{n};0.1)} |f(\mathbf{x}) - \Lambda_X f(\mathbf{x})| \le c h_{X,S(\mathbf{n};0.1)}^{(s-1)/2} \|f\|_s,$$

where the X is the set of N Saff points inside $S(\mathbf{n};\, 0.1)$. Treating the upper bound as an equality, and taking the quotient for any two different sets $X(i)$ and $X(j)$ of Saff points with corresponding local mesh norms $h_{X(i),S(\mathbf{n};0.1)}$ and $h_{X(j),S(\mathbf{n};0.1)}$ with respect to $S(\mathbf{n};\, 0.1)$, we find that

$$\frac{\operatorname{err}(X(i),\, S(\mathbf{n};\, 0.1))}{\operatorname{err}(X(j),\, S(\mathbf{n};\, 0.1))} \approx \left(\frac{h_{X(i),S(\mathbf{n};0.1)}}{h_{X(j),S(\mathbf{n};0.1)}}\right)^{(s-1)/2}.$$

Thus we plot $\log[\operatorname{err}(X(i),\, S(\mathbf{n};\, 0.1))/\operatorname{err}(X(j),\, S(\mathbf{n};\, 0.1))]$ on the vertical axis against $\log(h_{X(i),S(\mathbf{n};0.1)}/h_{X(j),S(\mathbf{n};0.1)})$ on the horizontal axis, for all possible combinations of data from Table 1, and we expect that the points lie approximately on a straight line with slope $\alpha \ge (s-1)/2 = (2.5-1)/2 = 0.75$. This is indeed the case, but with a value of α much bigger than 0.75: the linear fit of the data has the slope $\alpha \approx 2.47$. We observe that in Table 1 the local mesh norm condition (3.3) in Theorem 3.1 is satisfied for $N > 3000$ points but not for smaller N.

In Figure 5(a) we show the absolute pointwise error on $S(\mathbf{n};\, 0.1)$ of the RBF interpolant of f constructed from the 5000 Saff points. In the picture the error outside the cap $S(\mathbf{n};\, 0.1)$ has been set to zero. We observe that the absolute pointwise error at the boundary of $S(\mathbf{n};\, 0.1)$ is much larger than that inside $S(\mathbf{n};\, 0.1)$ away from the boundary, a phenomenon that might be expected but is not predicted by the theory. To explore the evident faster convergence in the interior, we evaluate the absolute

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TABLE 2. The local mesh norm $h_{X,S(\mathbf{n};0.08)}$ of the set X of N Saff points inside $S(\mathbf{n}; 0.1)$, and the maximal absolute pointwise error $\text{err}(X, S(\mathbf{n}; 0.08))$ of the RBF interpolant on the cap $S(\mathbf{n}; 0.08)$.

N	$h_{X,S(\mathbf{n};0.08)}$	$\text{err}(X, S(\mathbf{n}; 0.08))$
500	0.005492	2.6626×10^{-5}
1000	0.003893	4.1698×10^{-6}
2000	0.002757	3.2151×10^{-7}
3000	0.002266	7.5403×10^{-8}
4000	0.001963	2.1394×10^{-8}
5000	0.001756	1.8779×10^{-8}

pointwise error of the computed RBF interpolant on the smaller cap $S(\mathbf{n}; 0.08)$. The absolute pointwise error on the smaller cap is shown, on a very different scale, in Figure 5(b), using the same data as before but now setting to zero the error outside $S(\mathbf{n}; 0.08)$. Figure 5(b) can be considered to be obtained by zooming in to the smaller cap $S(\mathbf{n}; 0.08)$ in Figure 5(a). The very different scale of the second plot (differing by more than three orders of magnitude from the first) reveals a pattern of local error on the arc $x = 0$, arising of course from the lack of smoothness of x_+^3 along this arc.

In Table 2 we show the maximal absolute pointwise error on $S(\mathbf{n}; 0.08)$ and the local mesh norm $h_{X,S(\mathbf{n};0.08)}$ of the set X of N Saff points with respect to $S(\mathbf{n}; 0.08)$ for the same six values of N as in Table 1. (Note that the values of the local mesh norm differ from those in Table 1, because the local mesh norm in Table 2 is the local mesh norm of the Saff points with respect to the smaller cap $S(\mathbf{n}; 0.08)$.) The maximal absolute pointwise error of $\Lambda_X f$ on the smaller cap $S(\mathbf{n}; 0.08)$ is again seen to be much smaller than the maximal absolute pointwise error on $S(\mathbf{n}; 0.1)$. It also declines much faster as the number N of Saff points increases. From this we conclude that, to obtain a good RBF approximation of a function on a spherical cap from local data, it is advisable to take data of the function on a slightly larger cap. This purely empirical observation is not reflected by the present theory.

We have also computed the RBF interpolant of f for $N = 5000$ randomly chosen points inside the spherical cap $S(\mathbf{n}; 0.1)$. The maximal absolute pointwise error on the cap $S(\mathbf{n}; 0.1)$ was found to be comparable to the maximal absolute pointwise error for RBF interpolation with respect to the 5000 Saff points, but the error in the interior showed more variability.

Appendix A. Proof of the restrictions on the angle α in the proof of Lemma 4.1

Now we show that the angle α in the proof of Lemma 4.1 satisfies the estimate $\pi/3 < \alpha < \pi/2$. Remember that we assume here that $r < \pi/2$ and $\mathbf{x} \neq \mathbf{z}$.

PROOF OF $\pi/3 < \alpha < \pi/2$. Consider the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ with the corners $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{p}_1$ (see Figure A1). We have $\text{dist}(\mathbf{x}, \mathbf{p}_1) = \text{dist}(\mathbf{z}, \mathbf{p}_1) = r$ and $\rho = \text{dist}(\mathbf{x}, \mathbf{z}) \leq r$, and thus we have a triangle with two equal sides (and hence also

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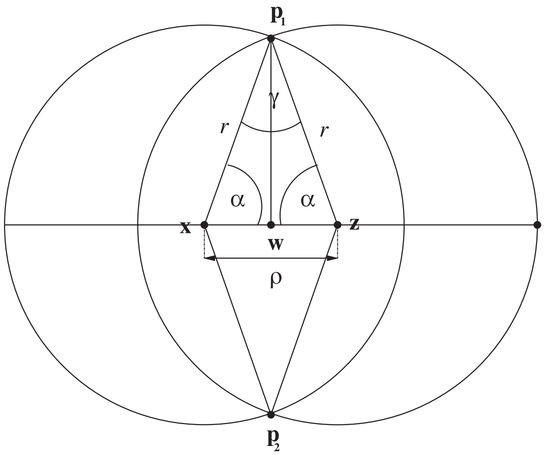

FIGURE A1. The triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$.

two equal angles). Furthermore, since $[\mathbf{x}, \mathbf{q}_1]$ is a sub-arc of $[\mathbf{x}, \mathbf{p}_1]$, and $[\mathbf{x}, \mathbf{z}']$ is a sub-arc of $[\mathbf{x}, \mathbf{z}]$ (or vice versa), the corner $\mathbf{x}$ of $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ has the same angle α as the corner $\mathbf{x}$ of the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}', \mathbf{q}_1)$ (see Figures 2(b) and 3). Hence the corner $\mathbf{z}$ of $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ also has the angle α. We denote the angle of the corner $\mathbf{p}_1$ by γ (see Figure A1).

The point $\mathbf{z}$ lies to the east of $\mathbf{x}$, and therefore the point $\mathbf{p}_1$, which lies on the longitude which cuts the geodesic arc $[\mathbf{x}, \mathbf{z}]$ in half, is also to the east of the longitude through $\mathbf{x}$. For this reason, the angle α formed by $[\mathbf{x}, \mathbf{p}_1]$ and by $[\mathbf{x}, \mathbf{z}]$ satisfies $\alpha < \pi/2$.

It remains to show that $\alpha > \pi/3$. To do this we shall make use of the fact that the area of the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ is given by (see [12, Theorem 2.5.5])

$$|\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)| = 2\alpha + \gamma - \pi. \tag{A.1}$$

Let $\mathbf{w}$ denote the point on the equator half-way between $\mathbf{x}$ and $\mathbf{z}$ (see Figure A1). The spherical sine theorem applied to the spherical triangle $\Delta(\mathbf{x}, \mathbf{w}, \mathbf{p}_1)$ yields

$$\sin r = \frac{\sin r}{\sin(\pi/2)} = \frac{\sin(\rho/2)}{\sin(\gamma/2)},$$

or equivalently

$$\sin(\gamma/2) = \frac{\sin(\rho/2)}{\sin r} = \frac{\sin(\rho/2)}{2\sin(r/2)\cos(r/2)} \quad \Rightarrow \quad \sin(\gamma/2) \leq \frac{1}{2\cos(r/2)},$$

since $\rho \leq r < \pi/2$. Moreover, because $r < \pi/2$ it follows that $\cos(r/2) > \cos(\pi/4) = 1/\sqrt{2}$, and hence (since clearly $\gamma/2 \leq \pi/2$, or $\gamma \leq \pi$)

$$\sin(\gamma/2) < \frac{1}{\sqrt{2}} = \sin(\pi/4) \quad \Rightarrow \quad \frac{\gamma}{2} < \frac{\pi}{4}.$$

Thus both α and γ are strictly less than $\pi/2$.

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Now apply the spherical sine theorem to the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$, to obtain (using $\rho \le r < \pi/2$)

$$\frac{\sin \rho}{\sin \gamma} = \frac{\sin r}{\sin \alpha} \quad \Rightarrow \quad \sin \alpha = \frac{\sin r}{\sin \rho} \sin \gamma \ge \sin \gamma.$$

Because of the monotonicity of $\sin x$ on $[0, \pi/2]$, it follows that $\alpha \ge \gamma$. From this and (A.1) and the fact the spherical triangle $\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)$ has positive area, we now obtain $0 < |\Delta(\mathbf{x}, \mathbf{z}, \mathbf{p}_1)| \le 3\alpha - \pi$, that is, $3\alpha > \pi$ or equivalently $\alpha > \pi/3$. This concludes the proof. □

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