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A QUASI-EQUIVALENCE BETWEEN BOREL SUMMABILITY AND CONVERGENCE FOR FOURIER-LAGUERRE SERIES AT THE END-POINT

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1. Background and objective

A suitable function $f(x)$, $0 \leq x < \infty$, can be expanded into a Fourier series of Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$, $n = 0, 1, \ldots, \alpha > -1$, whose interval of orthogonality is $0 \leq x < \infty$. The usual problems as to convergence and, lacking convergence, summability, and also the asymptotic behaviour of Lebesgue constants, arise for such developments. A summary of work on these convergence and summability problems, together with extensive references to the literature, can be found in the standard treatise by G. Szegö (**5**, especially Chapter IX) to whom many of these results are due.

Here we shall consider the classical case, $\alpha = 0$, for which the corresponding n-th polynomial is denoted simply by $L_n(x)$. The Laguerre series is defined by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n(x), \tag{1}$$

where

$$a_n = \int_0^\infty e^{-t} f(t) L_n(t)\, dt, \quad n = 0, 1, \ldots, \tag{2}$$

for functions $f(x)$ for which a_n exists.

The theory (indeed for all $\alpha > -1$) divides into two cases:

(i) $0 < x < \infty$, the interior of the interval of orthogonality, where conclusive results establish the equiconvergence of (1) with a trigonometric Fourier series (**5**, Theorem 9.1.5, p. 246) as well as convergence theorems (cf., e.g. (**3**, p. 88)), and

(ii) $x = 0$, the endpoint, where precise results are known for Cesàro summation (C, r).

Roughly speaking, in the case $\alpha = 0$, Szegö's Theorem 9.1.7 (**5**, p. 247) establishes that, for a function $f(x)$ subject to mild integrability conditions over $[0, \infty)$ and continuous at $x = 0$, the series (1) is (C, r) summable to $f(x)$ at $x = 0$ for $r > \frac{1}{2}$ but not necessarily for $r = \frac{1}{2}$.

In this note, we consider principally the application of Borel's exponential means (**2**, p. 79 and Chapters VIII and IX) to (1) at the endpoint $x = 0$. The class of functions $\mathfrak{F}$ to which we restrict ourselves consists of all locally Lebesgue integrable $f(x)$ such that

$$|f(x)| < K_M \exp(x - Mx^{\frac{1}{2}}), \quad \text{for all real } M, \quad 0 < x < \infty, \tag{3}$$

a condition satisfied, in particular, when

$$|f(x)| < Ke^{\beta x}, \quad \beta < 1, \quad 0 < x < \infty. \tag{3'}$$

(The constant K may have different values at different occurrences.) Without (3) or (3′) the Laguerre series might not even exist; there is no such series for e^x.

Our principal aim is to prove that *whenever* (1) *is Borel summable at* $x = 0$ *to* A, $-\infty \leq A \leq \infty$, *for* $f(x) \in \mathfrak{F}$, *then the Laguerre series for* $f(\frac{1}{2}x)$ *actually converges to* A *at* $x = 0$.

(The italicised statement is all the more true for the Euler (E, r), $r > 0$, transform of (1), since the Borel method includes all the Euler methods (**2**, Theorem 128, p. 183). The application of these methods is discussed in §8 below.)

This relation can be considered to exhibit a "*quasi-equivalence*" between Borel summability and convergence for the Laguerre series of functions in $\mathfrak{F}$ at $x = 0$. The existence of some sort of "quasi-equivalence" was suggested by unpublished calculations one of us made a number of years ago which indicated that the Lebesgue constants for Borel summation are of the same order of magnitude as that determined by G. Szegö (**6**) for convergence of Laguerre series. This was rather unexpected, since Borel summability is stronger than convergence, even when applied to the Laguerre expansion of a function in $\mathfrak{F}$: e.g., the Laguerre series for $e^{x/2}$ is $2\sum_0^\infty (-1)^n L_n(x)$, divergent at $x = 0$, since $L_n(0) = 1$, $n = 0, 1, 2, \ldots$, but Borel summable to 1 there. This function can therefore serve as an illustration of the italicised assertion above, since the series for e^{-ax} [3, (4.24.3), p. 90] clearly converges at $x = 0$ for $a = -\frac{1}{4}$, indeed for all $a > -\frac{1}{2}$.

2. Results to be established

Stated more precisely, and also more completely, the results to be proved assert:

Theorem. *Let* $f(x) \in \mathfrak{F}$. *If* $s_n(x)$ *is the* n-th *partial sum for the Laguerre series of* $f(\frac{1}{2}x)$, *and* $B\{z, x\}$ *the* z-th *Borel exponential mean of the Laguerre series* (1) *of* $f(x)$, *then*

$$\lim_{n\to\infty} |s_n(0) - B\{2(n+1), 0\}| = 0, \tag{4}$$

and, more generally,

$$\lim_{n\to\infty} |s_n(0) - B\{2z, 0\}| = 0, \quad \textit{where} \quad |z - n| \leq 1. \tag{4'}$$

If, in addition, $f \in L^\infty[0, \infty)$, *then the* z-th *Lebesgue constant*

$$\begin{aligned}\mathfrak{B}(z) &\equiv \sup_{f\in L^\infty[0,\infty)} |B\{z, 0\}|/\|f\|_\infty \\ &= 2\Gamma(\tfrac{1}{4})\pi^{-3/2}z^{1/4} + O(1),\end{aligned} \tag{5}$$

as $z \to \infty$,

Corollary 1. *If, for* $f(x) \in \mathfrak{F}$, (1) *is Borel summable to* A, $-\infty \leq A \leq +\infty$ (*in*

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particular, if (1) *converges to A*), *at* $x=0$, *then the Laguerre series for* $f(2^{-\nu}x)$ *converges to A for* $\nu=1,2,\ldots$, *at* $x=0$.

Corollary 2. *The n-th Lebesgue constant*

$$L_n \equiv \sup_{f\in L^\infty[0,\infty)} |s_n(0)|/\|f\|_\infty = \\ = 2^{5/4}\Gamma(\tfrac{1}{4})\pi^{-3/2}n^{1/4}+O(1), \tag{5'}$$

as $n\to\infty$.

3. A generating function

To construct a convenient representation (12) for the Borel mean there will be needed the following generating function:

$$e^{-z}\sum_{k=0}^{\infty} L_k^{(1)}(t)\frac{z^k}{k!} = (z/t)^{\frac{1}{2}}J_1(2t^{\frac{1}{2}}z^{\frac{1}{2}}) + J_0(2t^{\frac{1}{2}}z^{\frac{1}{2}}), \tag{6}$$

both as written, and in the companion form

$$e^{-z}\sum_{k=0}^{\infty} L_k^{(1)}(-t)\frac{z^k}{k!} = (z/t)^{\frac{1}{2}}I_1(2t^{\frac{1}{2}}z^{\frac{1}{2}}) + I_0(2t^{\frac{1}{2}}z^{\frac{1}{2}}). \tag{7}$$

Here, as usual, J_ν denotes the Bessel function of first kind and order ν, I_ν the corresponding modified Bessel function.

Formulae (6) and (7) follow from the standard generating function (5, (5.1.16), p. 102)

$$\sum_{k=0}^{\infty} L_k(t)\frac{z^k}{k!} = e^zJ_0(2t^{\frac{1}{2}}z^{\frac{1}{2}}), \tag{8}$$

since

$$\frac{\partial^2}{\partial t\partial z}\left\{\sum_{k=0}^{\infty} L_k(t)\frac{z^k}{k!}\right\} = \sum_{k=0}^{\infty} L'_{k+1}(t)\frac{z^k}{k!}.$$

Now, applying this to (8),

$$\frac{\partial^2}{\partial t\partial z}\{e^zJ_0(2t^{\frac{1}{2}}z^{\frac{1}{2}})\} = e^zJ_0''(2t^{\frac{1}{2}}z^{\frac{1}{2}}) + e^z[(z/t)^{\frac{1}{2}}+\tfrac{1}{2}(tz)^{-1}]J_0'(2t^{\frac{1}{2}}z^{\frac{1}{2}}).$$

Furthermore (5, p. 15),

$$J_0'(v) = -J_1(v),\; J_0''(v) = v^{-1}J_1(v) - J_0(v),$$

and so (6) is proved, since $-L'_{k+1}(t) = L_k^{(1)}(t)$, and, with it, (7).

4. An expression for the Borel mean $B\{z, 0\}$

With (6) and (7) established, it is now possible to construct a convenient representation (12) for $B\{z,0\}$. The customary representation , which will be needed in §5 also, for the n-th partial sum $s_n(x)$ of the Fourier-Laguerre series of the function $f(\frac{1}{2}x)$ (**5** (9.5.1), p. 266) is

$$s_n(x) = \int_0^\infty e^{-t} f(\tfrac{1}{2}t) K_n(x, t)\, dt, \tag{9}$$

where $K_n(x, t)$ is the kernel function defined in (**5**, (5.1.11), p. 101) for which

$$K_n(0, t) = (n+1)\frac{L_n(t) - L_{n+1}(t)}{t} = -L'_{n+1}(t) = L_n^{(1)}(t). \tag{10}$$

By definition, the z-th Borel exponential mean of the Laguerre series of $f(x)$ at $x = 0$ is then

$$B\{z, 0\} = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^\infty e^{-t} f(t) L_k^{(1)}(t)\, dt. \tag{11}$$

To use (6), we must invert the order of summation and integration in (11). This interchange can be justified by the Lebesgue dominated convergence theorem (**7**, p. 187). To do so, we consider

$$\phi_n(t) = e^{-t} f(t) e^{-z} \sum_{k=0}^{n} L_k^{(1)}(t) \frac{z^k}{k!}, \quad t \geq 0.$$

Clearly, from (7), since the coefficients of $L_k^{(1)}(t)$ alternate in sign (**3**, (4.17.2), p. 77; **5**, (5.16), p. 101),

$$\begin{aligned} |\phi_n(t)| &\leq e^{-t}|f(t)|e^{-z} \sum_{k=0}^{n} |L_k^{(1)}(t)| \frac{z^k}{k!} \\ &\leq e^{-t}|f(t)|e^{-z} \sum_{k=0}^{n} L_k^{(1)}(-t) \frac{z^k}{k!} \\ &\leq e^{-t}|f(t)|e^{-z} \sum_{k=0}^{\infty} L_k^{(1)}(-t) \frac{z^k}{k!} \\ &= e^{-t}|f(t)|[(z/t)^{\frac{1}{2}} I_1(2t^{\frac{1}{2}}z^{\frac{1}{2}}) + I_0(2t^{\frac{1}{2}}z^{\frac{1}{2}})], \quad t \geq 0. \end{aligned}$$

Now (**8**, p. 203, (2)), as $w \to \infty$,

$$I_\nu(w) = (2\pi w)^{-1/2}[e^w + O(1/w)].$$

Hence,

$$|\phi_n(t)| < e^{-t}|f(t)|\{(4\pi)^{-\frac{1}{2}} t^{-\frac{1}{4}} z^{-\frac{1}{4}}[\exp(2t^{\frac{1}{2}}z^{\frac{1}{2}})][(z/t)^{\frac{1}{2}} + 1] + O(t^{-3/4}z^{-3/4})\}.$$

This is dominated by a function integrable over $(0, \infty)$, since $f(t) \in \mathfrak{F}$, for each $z > 0$.

Thus, the order of integration and summation can be interchanged in (11). In view of (6), this gives the desired formula for the Borel mean

$$B\{z, 0\} = \int_0^\infty e^{-t} f(t)\{(z/t)^{\frac{1}{2}} J_1(2t^{\frac{1}{2}}z^{\frac{1}{2}}) + J_0(2t^{\frac{1}{2}}z^{\frac{1}{2}})\}\, dt. \tag{12}$$

5. Proof of (4)

With (12) established, we can proceed to the proof of (4). We write $N = n + 1$ and, recalling (9), we define

$$\begin{aligned}\Delta_n &\equiv s_n(0) - B\{2N, 0\}\\ &= \int_0^\infty e^{-t} f(\tfrac{1}{2}t) L_n^{(1)}(t)\, dt - \int_0^\infty e^{-t} f(t)\{J_1(2(2Nt)^{1/2})(2N/t)^{1/2}\\ &\quad + J_0(2(2Nt)^{\frac{1}{2}})\}\, dt.\end{aligned}$$

Changing the variable in the first integral from t to $2t$, this becomes

$$\Delta_n = \int_0^\infty f(t)\{2e^{-2t} L_n^{(1)}(2t) - e^{-t}[J_1(2(2Nt)^{1/2})(2N/t)^{1/2} + J_0(2(2Nt)^{1/2})]\}\, dt. \tag{13}$$

To express (13) in terms of Bessel functions, an asymptotic representation of $L_n^{(1)}(2t)$ is used (3, pp. 85–87; 5, p. 199, Theorem 8.22.4):

$$L_n^{(1)}(2t) = e^t\{2^{-1/2}(N/t)^{1/2} J_1(2(2Nt)^{1/2}) + R_n(2t)\}, \tag{14}$$

where the remainder term $R_n(2t)$ will be specified later.

Thus, (13) becomes

$$\Delta_n = \int_0^\infty e^{-t} f(t) J_0(2N^{\frac{1}{2}} t^{\frac{1}{2}})\, dt + 2\int_0^\infty e^{-t} f(t) R_n(2t)\, dt \tag{15}$$

whence the J_0-term contributes only $o(1)$ to (12), $z \to \infty$.

Now

$$\int_0^\infty e^{-t} f(t) J_0(2N^{\frac{1}{2}} t^{\frac{1}{2}})\, dt = o(1), \quad n \to \infty, \tag{16}$$

from the Lebesgue dominated convergence theorem, since $J_0(yt) \to 0$ as $y \to \infty$, $t \neq 0$.

To show that the other integral in (15) is also $o(1)$, $n \to \infty$, it is helpful to decompose that integral into two and use different forms of $R_n(t)$ in each.

Let $\omega > 0$ be a constant, and write

$$\int_0^\infty e^{-t} f(t) R_n(2t)\, dt = \int_0^\omega + \int_\omega^\infty \equiv I_1(n) + I_2(n).$$

In $I_1(n)$, it is convenient to use the estimate of $R_n(2t)$ given by (5, p. 199, Theorem 8.22.4), namely,

$$R_n(t) = t^{-1/2} O(n^{-1/4}), \tag{17}$$

$n \to \infty$, uniformly in $0 < t < \omega$. From this, it is immediate that

$$|I_1(n)| = O(n^{-1/4}), \quad n \to \infty. \tag{18}$$

In $I_2(n)$, we shall use the form of the remainder implied by (3, p. 87). There, however, some correction is required, as the Referee kindly called to our attention. Formulae (4.22.15) and (4.22.16), and the unnumbered formula preceding them, must be amended so that the factor $x^{\alpha/2+5/4}$ is replaced in each by $x^{5/4}$. These corrections arise from the Referee's observation that on p. 87, lines 8 and 9, where the Buniakovski-Schwarz inequality is applied, the factor y^α is improperly in the first integral on the right; it belongs in the second integral.

When this adjustment is made, then the corrected estimate (3, p. 87, (4.22.16)) becomes, for $\alpha = 1$, and $0 < x < \infty$,

$$|r_n(x)| < K(Nx)^{-3/4}[N^{-\frac{1}{4}} x^{5/4} + N^{-2} O(1)],$$

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which can be seen from the argument in (3, pp. 86–87) to be

$$|r_n(x)| < KN^{-1}x^{\frac{1}{2}}, \quad Nx > 1.$$

It is clear from (3, p. 86, (4.22.6)) that $R_n(x) = Nr_n(x)$ and so

$$|R_n(2t)| < Kt^{\frac{1}{2}}, \quad t > \omega. \tag{19}$$

Hence,

$$|I_2(n)| < K\int_\omega^\infty e^{-t}|f(t)|t^{\frac{1}{2}}\,dt.$$

Thus, for any fixed $\omega > 0$,

$$\limsup_{n\to\infty}|I_1(n) + I_2(n)| \leq K\int_\omega^\infty e^{-t}|f(t)|t^{\frac{1}{2}}\,dt,$$

where, as specified after (3′), K may have different values at different occurrences.

Letting $\omega \to \infty$ completes the proof of (4).

Remark. In reading (3, pp. 86–87), it may be helpful to note in (4.22.13) that the constant implied in the O-estimates can be given numerically, essentially as described in connection with our formula (27) below.

6. Proof of (4′)

In order to prove (4′) we may assume without loss of generality that $f(0) = 0$, since only the differences between $s_n(0)$ and $B\{2z, 0\}$ enter the discussion.

It suffices to prove that $\delta(z, z') \to 0$ as $z \to \infty$, uniformly in $|z - z'| \leq 1$, where

$$\delta(z, z') \equiv \int_0^\infty e^{-t}|f(t)|\ |J_1[2(tz)^{\frac{1}{2}}](z/t)^{\frac{1}{2}} - J_1[2(tz')^{\frac{1}{2}}](z'/t)^{\frac{1}{2}}|\,dt. \tag{20}$$

Now, with $F(x) = x^{\frac{1}{2}}J_1(2x^{\frac{1}{2}})$,

$$\begin{aligned}(z/t)^{\frac{1}{2}}J_1[2(tz)^{\frac{1}{2}}] &- (z'/t)^{\frac{1}{2}}J_1[2(tz')^{\frac{1}{2}}]\\ &= (z - z')F'(\zeta)\\ &= (z - z')\{\tfrac{1}{2}\zeta^{-\frac{1}{2}}J_1(2\zeta^{\frac{1}{2}}) + J_1'(2\zeta^{\frac{1}{2}})\}\end{aligned} \tag{21}$$

for some ζ between tz and tz'. For small v,

$$J_1(v) = O(v), \quad J_1'(v) = O(1), \tag{22}$$

and for large v,

$$J_1(v) = O(v^{-\frac{1}{2}}), \quad J_1'(v) = O(v^{-\frac{1}{2}}), \tag{23}$$

where, e.g., the estimates for $J_\nu(v)$ are given in (5, p. 16) and those for $J_1'(v)$ follow from these and the recurrence formula (8, p. 17(20)) $2J_1'(x) = J_0(x) - J_2(x)$. Hence the expression (21) is

$$O(|z - z'|) \quad (0 \leq t \leq 1/z), \quad O(|z - z'|t^{-\frac{1}{4}}z^{-\frac{1}{4}}) \quad (t > 1/z).$$

Thus,

$$\delta(z, z') = O\{|z - z'|[\int_0^{1/z}|f(t)|\,dt + z^{-\frac{1}{4}}\int_{1/z}^\infty |f(t)|e^{-t}t^{-\frac{1}{4}}\,dt]\},$$

whence $\delta(z, z') \to 0$ as $z \to \infty$, uniformly in $|z - z'| \leq 1$.

7. Proof of (5) and (5′)

In view of (4), it suffices to prove (5).

From (12) it is clear that the Lebesgue constant $\mathfrak{B}(z)$ can be expressed as

$$\mathfrak{B}(z)=\int_0^{\infty} e^{-t}|J_1[2(tz)^{\frac{1}{2}}](z/t)^{\frac{1}{2}}+J_0[2(tz)^{\frac{1}{2}}]|\,dt. \tag{24}$$

Reasoning as in (16) we get the simpler form

$$\mathfrak{B}(z)=\int_0^{\infty} e^{-t}|J_1[2(tz)^{\frac{1}{2}}]|(z/t)^{\frac{1}{2}}\,dt+o(1),\quad z\to\infty, \tag{25}$$

and note that

$$\int_z^{\infty} e^{-t}|J_1[2(tz)^{\frac{1}{2}}]|(z/t)^{\frac{1}{2}}\,dt<\int_z^{\infty} e^{-t}\,dt=O(e^{-z})=o(1)$$

as $z\to\infty$.

On the other hand, it is easy to estimate the part of the integral in (25) close to zero, bearing in mind that $|J_1(u)|=O(u)$, $u<1$. Thus,

$$\int_0^{1/(4z)} e^{-t}|J_1[2(tz)^{\frac{1}{2}}]|(z/t)^{\frac{1}{2}}\,dt=O(1)\int_0^{1/(4z)} e^{-t}(tz)^{\frac{1}{2}}(z/t)^{\frac{1}{2}}\,dt$$
$$=O(z)\int_0^{1/(4z)} e^{-t}\,dt=O(1),\quad \text{as } z\to\infty.$$

Hence,

$$\mathfrak{B}(z)=\int_{1/(4z)}^{z} e^{-t}|J_1[2(tz)^{\frac{1}{2}}]|(z/t)^{\frac{1}{2}}\,dt+O(1). \tag{26}$$

In (26), it is helpful to use the following expression for the Bessel function:

$$J_1(x)=(2/\pi)^{\frac{1}{2}}x^{-\frac{1}{2}}\cos(x-3\pi/4)+R_1(x)x^{-3/2}+R_2(x)x^{-5/2},\quad x>0, \tag{27}$$

where $|R_1(x)|\leq C$, $|R_2(x)|\leq C$ and C is a constant.

This is established by using the exact representation for $J_1(x)$, valid for all positive real values of the argument, given in (**8**, p. 206 (1)), taking p to be 1 in the formula for $P(x,1)$ and 0 in the formula for $Q(x,1)$, in the notation employed there. This gives (27) with $C=1$. (A still smaller value for C can be found (**1**).) Thus,

$$\mathfrak{B}(z)=\int_{1/(4z)}^{z} e^{-t}|(z/t)^{\frac{1}{2}}|\,\left|\left[\frac{1}{\pi(tz)^{1/2}}\right]^{\frac{1}{2}}\cos\left(2(tz)^{\frac{1}{2}}-\frac{3\pi}{4}\right)\right.$$
$$\left.+O((tz)^{\frac{1}{2}})^{-3/2}+O((tz)^{\frac{1}{2}})^{-5/2}\right|\,dt+O(1).$$

The two remainder terms are treated easily:

$$\int_{1/(4z)}^{z} e^{-t}(z/t)^{\frac{1}{2}}(tz)^{-3/4}\,dt=z^{-\frac{1}{4}}\int_{1/(4z)}^{z} e^{-t}t^{-5/4}\,dt=O(1),$$

and

$$\int_{1/(4z)}^{z} e^{-t}(z/t)^{\frac{1}{2}}(tz)^{-5/4}\,dt=z^{-3/4}\int_{1/(4z)}^{z} e^{-t}t^{-3/4}\,dt=O(1).$$

Hence

$$\mathfrak{B}(z)=\int_{1/(4z)}^{z} e^{-t}(z/t)^{\frac{1}{2}}\left[\frac{1}{\pi(tz)^{1/2}}\right]^{\frac{1}{2}}\left|\cos\left(2(tz)^{\frac{1}{2}}-\frac{3\pi}{4}\right)\right| dt+O(1)$$

$$=\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\int_{1/(4z)}^{z} e^{-t}t^{-3/4}\left|\cos\left(2(tz)^{\frac{1}{2}}-\frac{3\pi}{4}\right)\right| dt+O(1).$$

A simple application of the results in (4) gives

$$\mathfrak{B}(z)=2\pi^{-3/2}z^{\frac{1}{4}}\int_{1/(4z)}^{z} e^{-t}t^{-3/4}\,dt+O(1),$$

from which (5) follows readily. Clearly, this gives (5′) as well.

8. Euler summation and some consequences

Corollary 1 (§1) states that the Borel summability of Laguerre series of $f(t)$ at $t=0$ implies the convergence of the corresponding series for $f(2^{-\nu}t)$ at $t=0$, $\nu=1,2,\ldots$, for $f(t)\in\mathfrak{F}$. The Theorem implies a converse as well, in the sense that convergence of the Laguerre series of $f(\frac{1}{2}t)$ for $f(t)\in\mathfrak{F}$ at $t=0$ implies Borel summability of the Laguerre series of $f(2^{-\nu}t)$ at $t=0$, $\nu=0,1,\ldots$.

This aspect of the Theorem can be strengthened if Borel summability is replaced by the (weaker) Euler method (E,r), $r\geqslant 0$, $(E,0)$ being convergence. To do so, using calculations similar to the ones in the previous sections, we use the generating function (5, p. 387, problem 67)

$$\sum_{\nu=0}^{n}\binom{n}{\nu}L_\nu(x)y^{n-\nu}=(y+1)^n L_n\left(\frac{x}{y+1}\right)$$

instead of (8).

The corresponding result states that the Laguerre series for $f\left(\frac{r+1}{2r+1}x\right)$ converges to A at $x=0$ if and only if the series for $f(x)$ is summable (E,r) to A at $x=0$, $r\geqslant 0$, for

$$f(x)\in\mathfrak{F}_r\equiv\left\{f\in\mathfrak{F}, |f(x)|\leqslant K\exp\left[\beta\frac{2r+1}{2r+2}x\right], \beta<1,\quad 0<x<\infty\right\}.$$

More precisely, let $\{t_n\}_1^\infty$ denote the (E,r) transforms of the Laguerre series for $f(x)$ and $\{\sigma_n\}_1^\infty$ the sequence of partial sums of the Laguerre series for $f\left(\frac{r+1}{2r+1}x\right)$. Then, for $f(x)\in\mathfrak{F}_r$, we have $\sigma_N-t_n\to 0$, where N, n tend to ∞ in such a way that $N=n/(2r+1)+O(1)$.

Furthermore, the asymptotic behaviour of the Lebesgue constants $L_n(r)$ for (E,r) summation can be ascertained much as in (5), with

$$L_n(r)=\frac{2^{5/4}}{\pi^{3/2}}\frac{r^{1/4}}{(2+r)^{1/4}}\Gamma(\tfrac{1}{4})n^{1/4}+O(1),\quad r>0.$$

Inasmuch as the methods (E,r) increase in strength as r increases (and are consistent with one another) and are all included in Borel's method, Corollary 1 can be strengthened.

Thus,

(i) For $f(x)\in\mathfrak{F}_r$, $r\geq 0$, if the Laguerre series for $f(x)$ is (E, r) summable to A at $x=0$, then the corresponding series for $f(s_1 s_2 \ldots s_q x)$ converges to A at $x=0$ for $\frac{1}{2}\leq s_j\leq\frac{r+1}{2r+1}$, $1\leq j\leq q$, $1\leq q$.

For the Laguerre series of $e^{x/2}$ this implies more than was inferred at the end of §1. That series is (E, r) summable at $x=0$ to 1 for every $r>0$ (but not for $r=0$). Hence, (i) implies that the corresponding series for e^{bx} converges to 1 for $0\leq b<\frac{1}{2}$, a result which can be confirmed readily (even for all $b<\frac{1}{2}$) from the explicit expansion for e^{-ax} (**3**, p. 90, (4.24.3)).

Next, convergence of the series for $f(s_0x)$ at $x=0$, $s_0=\frac{r+1}{2r+1}$, implies (E, r) summability of the series for $f(x)$ at $x=0$, $r\geq 0$. Hence,

(ii) For $f(x)\in\mathfrak{F}_r$, $0\leq r\leq\frac{1}{2}\sqrt{2}$, the convergence of the Laguerre series of $f(x)$ to A at $x=0$ implies the convergence of the corresponding series for $f(ax)$ to A at $x=0$, for $0<a\leq\frac{r+1}{2r+1}$.

In particular, if $f(x)\in\mathfrak{F}_0$, then $f(ax)$ will converge to A for $0<a\leq 1$. Again $f(x)=e^{bx}$ is illustrative, here with $b<\frac{1}{2}$.

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