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KRULL SEMIGROUPS AND DIVISOR CLASS GROUPS

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1. Introduction and notation. R. Matsuda has shown that a group ring is a Krull domain if and only if the coefficient ring is a Krull domain and the group is a torsion-free abelian group satisfying the ascending chain condition (acc) on cyclic subgroups [**6**]. D. F. Anderson has used this to obtain a partial determination of when a semigroup ring is a Krull domain, and under certain circumstances to describe the divisor class group of such a ring ([**1**], [**2**]). Using some of Anderson's techniques, but taking a different approach, we arrive at a complete answer of a different nature to these questions. We call a semigroup satisfying the major new conditions arising a Krull semigroup, and define its divisor class group.

In particular, every abelian group is the divisor class group of such a ring, and it follows that every abelian group is the divisor class group of a quasi-local ring, which seems to be a new result. We also obtain a condition which can be used in theory to classify the isomorphism classes of semigroups with a given divisor class group. Although we will not pursue the point, from the results in this paper it also follows that there are algorithms to determine if a finitely generated cancellative commutative semigroup is Krull, and if it is, to determine its divisor class group. In a subsequent paper, we will show that if S is a Krull semigroup with torsion divisor class group, and D is a Dedekind domain, then finitely generated projective $D[S]$-modules are extended from D, thus generalizing the Quillen–Suslin result.

All rings are commutative with unit $1 \neq 0$, and semigroups are commutative, cancellative, with unit and have torsion-free total quotient group. Note that this poses no real restriction since for a semigroup ring to be an integral domain, the semigroups must have these properties. If S is a semigroup, $\langle S \rangle$ will denote its total quotient group.

Semigroups are written additively. If S is a semigroup we write $a|b$, or a divides b, if $b = a + x$ for some $x \in S$. An ideal of S is a subset closed under addition of the elements of S. If T is a subset of S, $S + (-T)$ denotes the subsemigroup of $\langle S \rangle$ generated by $S \cup (-T)$.

$\mathbf{Z}$ denotes the integers; if $F = \bigoplus_{\alpha \in I} \mathbf{Z}_{x\alpha}$ is a free abelian group with given basis $\{x_\alpha | \alpha \in I\}$, we let $\mathrm{pr}_\alpha \colon F \to \mathbf{Z}$ denote the projection onto the coefficient of x_α for any $\alpha \in I$, and set

$$F_+ = \{x \in F | \ \mathrm{pr}_\alpha (x) \geqq 0 \text{ for all } \alpha \in I\}.$$

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If R is a domain, q.f.(R) is the quotient field of R; if R is a Krull domain, $\mathrm{Cl}(R)$ denotes the divisor class group of R. When R is a ring and S is a semigroup, we use $R[S]$ for the semigroup ring of S with coefficients in R. We write the elements of $R[S]$ in the form $x = \sum_{s \in S} r_s X^s$, $r_s \in R$, all but finitely many $r_s = 0$. With x as above,

$$\mathrm{supp}\,(x) = \{s \in S \mid r_s \neq 0\};$$

note $\mathrm{supp}\,(x) = \emptyset$ if and only if $x = 0$.

2. Krull semigroups. It turns out that if care is taken, most of the results in VII. 1 nos. 1–4 and 6 of [**4**] have semigroup analogs, except for the last result in each of Sections 1, 4, and 6. As we shall need some of these analogs, in this section we present the definitions necessary to state these results together with a few other items.

If S is a semigroup, a principal fractional ideal of S is a subset of $\langle S \rangle$ of the form $\{t + s \mid s \in S\}$ for some $t \in \langle S \rangle$. A divisorial fractional ideal of S is a nonempty intersection of principal fractional ideals. There is, as with rings, a natural semigroup structure on $D(S)$, the set of divisorial fractional ideals. Furthermore, Prin (S), the set of principal fractional ideal of S, is a subgroup of $D(S)$, so we may define the divisor class semigroup of S to be $\mathrm{Cl}(S) = D(S)/\mathrm{Prin}\,(S)$.

We call S integrally closed if $x \in \langle S \rangle$, $a \in S$ and $nx = a$ for some $n \geqq 1$ implies $x \in S$. S is completely integrally closed if $x \in \langle S \rangle$, $a \in S$ and $a + nx \in S$ for all $n \geqq 1$ implies $x \in S$ (this is Anderson's condition (*) in [**1**], Section 7). As with domains, if S is completely integrally closed then it is integrally closed, and the two conditions are equivalent if S satisfies the acc on ideals, so in particular if S is finitely generated (the proofs are similar to those for domains). It follows that the condition "Γ is integrally closed" can be added to the equivalent conditions of Corollary 7.10 of [**1**]. More importantly, $D(S)$ and $\mathrm{Cl}(S)$ are groups if and only if S is completely integrally closed.

Next, define a discrete valuation of a group G to be a group map $v: G \rightarrow \mathbf{Z}$. The valuation semigroup of v is $\{x \in G \mid v(x) \geqq 0\}$ and the residue group of the valuation is ker (v). We say v is normed if it is surjective. Note that if v is nontrivial, the valuation semigroup of v is isomorphic to $(\ker v) \times \mathbf{Z}_+$ (since every nontrivial subgroup of $\mathbf{Z}$ is isomorphic to $\mathbf{Z}$).

Definition. The semigroup S is a *Krull* semigroup if there exists a family $(v_i)_{i \in I}$ of discrete valuations of $\langle S \rangle$ such that S is the intersection of the valuation semigroups of the v_i, and for every $x \in S$, $\{i \in I \mid v_i(x) > 0\}$ is finite.

The above definition is obviously a semigroup analogue to the usual definition of Krull domains. However, in this setting the definition can be reformulated in a simpler way.

PROPOSITION 1. *S is a Krull semigroup if and only if $S \cong G \times S_1$ where G is a group and S_1 is a subsemigroup of a free group $F = \bigoplus_{i \in I} \mathbf{Z} x_i$ such that $S_1 = \langle S_1 \rangle \cap F_+$.*

Proof. If S is Krull, let the indexing set for F equal that for a set of valuations on $\langle S \rangle$ which shows S to be Krull, and define a group map ψ from $\langle S \rangle$ to F by

$$\mathrm{pr}_i(\psi(x)) = v_i(x).$$

Then $S = \psi^{-1}(F_+)$, and

$$\langle S \rangle \cong \ker(\psi) \times \mathrm{im}(\psi)$$

(since $\mathrm{im}\,\psi$ is free), so setting $S_1 = \psi(S)$ and $G = \ker \psi$, we have the desired isomorphism. Conversely, if $S \cong G \times S_1$ as described, it is clear that the valuations $(v_i)_{i \in I}$ of $\langle S \rangle$ defined by

$$v_i(x) = \mathrm{pr}_i\,(\mathrm{pr}_{\langle S_1 \rangle}(x))$$

satisfy the requirements necessary to show S is a Krull semigroup.

The following result is proved similarly to Theorem 2 in [**4**], VII. 1.

PROPOSITION 2. *S is a Krull semigroup if and only if it is completely integrally closed and satisfies the* a.c.c. *on divisorial ideals in S. In that case, the set $P(S)$ of maximal proper divisorial ideals of S forms a $\mathbf{Z}$-basis for $D(S)$, such that under this basis $D(S)_+$ corresponds to the divisorial ideals in S.*

Proposition 2 allows us to define the essential valuations of the Krull semigroup S as with Krull domains. These essential valuations are in one to one correspondence with the elements of $P(S)$, which in turn can be identified with the set of minimal (nonempty) prime ideals of S.

Remark 1. It can be shown that for any semigroup map $\psi: S \rightarrow \mathbf{Z}$ with S a finitely generated semigroup, there exists a (calculable) finite generating set for the semigroup $\psi^{-1}(\mathbf{Z}_+)$. Using induction on the number of valuations, this implies that a Krull semigroup S is finitely generated if and only if $\langle S \rangle$ is finitely generated.

We shall state analogues to other results from VII. 1 of [**4**] as needed.

3. Semigroup rings which are Krull domains. We first observe that if $S \cong G \times S_1$ is a Krull semigroup with G and S_1 as in Proposition 1, then $\langle S \rangle$ satisfies the acc on cyclic subgroups if and only if G does.

THEOREM 1. *$A[S]$ is a Krull domain if and only if A is a Krull domain and S is a Krull semigroup such that $\langle S\rangle$ satisfies the* acc *on cyclic subgroups.*

Proof. Since $A[\langle S\rangle]$ can be regarded as a localization of $A[S]$, if $A[S]$ is Krull so is $A[\langle S\rangle]$ ([**4**], Proposition 6, VII. 1.4). Thus by Matsuda's result ([**6**], Proposition 3.3; or see [**1**], Theorem 2.7), A is a Krull domain and $\langle S\rangle$ satisfies the acc on cyclic subgroups. Moreover, if $\{v_\alpha|\alpha \in I\}$ is the set of essential valuations of $A[S]$, their restriction to $\langle S\rangle \subset$ q.f. $(A[S])$ show that S is Krull.

Conversely, if A and S satisfy the second part of the theorem, the result of Matsuda cited above proves that $A[\langle S\rangle]$ is a Krull domain. Thus it suffices to define discrete valuations v_α on $A[\langle S\rangle]$ such that each nonzero element of $A[\langle S\rangle]$ has nonzero value at only finitely many of the valuations, and such that if $0 \neq f \in A[\langle S\rangle]$, then $f \in A[S]$ if and only if $v_\alpha(f) \geqq 0$ for all such v_α. Let $(\bar{v}_i)_{i\in I}$ be a set of discrete valuations of the group $\langle S\rangle$ which show that S satisfies the definition of a Krull semigroup. For each $i \in I$, let v_i be the inf valuation of $A[\langle S\rangle]$ defined by $\bar{v}_i$, i.e., if $0 \neq f \in A[\langle S\rangle]$, let

$$v_i(f) = \inf\{\bar{v}_i(g)|g \in \operatorname{supp}(f)\}$$

([**3**]). Anderson and Ohm show that this is a valuation by noting that the partial order on $\langle S\rangle$ induced by $\bar{v}_i$ extends to at least one total order of $\langle S\rangle$, and examining the minimal elements under any such total order in supp (f_1) and supp (f_2), $0 \neq f_1, f_2 \in A[\langle S\rangle]$, it is easily seen that

$$v_i(f_1f_2) = v_i(f_1) + v_i(f_2) \text{ and}$$
$$v_i(f_1 + f_2) \geqq \inf(v_i(f_1), v_i(f_2)).$$

But clearly $v_i(f) \geqq 0$ for all $i \in I$ if and only if $\bar{v}_i(\operatorname{supp}(f)) \subseteq \mathbf{Z}_+$ for all $i \in I$ if and only if $f \in A[S]$, as desired.

4. The class group of a Krull semigroup.

LEMMA 1. *Let $A[S]$ be a Krull domain, so $S \cong G \times S_1$ as in Proposition 1. Then*

$$\mathrm{Cl}(A[S]) \cong \mathrm{Cl}(A) \oplus \mathrm{Cl}(S_1).$$

Proof. By Proposition 7.3 and Corollary 6.7 of [**1**],

$$\begin{aligned}\mathrm{Cl}(A[S]) &\cong \mathrm{Cl}(A) \oplus \mathrm{Cl}(K[S])\\ &\cong \mathrm{Cl}(A) \oplus \mathrm{Cl}(K[G]) \oplus \mathrm{Cl}(K_1[S_1])\\ &\cong \mathrm{Cl}(A) \oplus \mathrm{Cl}(K_1[S_1])\end{aligned}$$

where $K = \text{q.f.}(A)$ and $K_1 = \text{q.f.}(K[G])$, using the natural isomorphism $K[G \times S_1] \cong K[G][S_1]$. So we need only see that if K_1 is a

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field and S_1 is Krull semigroup, then

$$\mathrm{Cl}(K_1[S_1]) \cong \mathrm{Cl}(S_1).$$

By the comments following the proof of Theorem 4.2 in [**1**], however,

$$\mathrm{Cl}(K_1[S_1]) \cong \mathrm{HDiv}(K_1[S_1])/\mathrm{HPrin}(K_1[S_1]),$$

where $\mathrm{HDiv}(K_1[S_1])$ is the subgroup of $\mathrm{Div}(K_1[S_1])$ generated by the homogeneous height 1 prime ideals of $K_1[S_1]$, while

$$\mathrm{HPrin}(K_1[S_1]) = \mathrm{Prin}(K_1[S_1]) \cap \mathrm{HDiv}(K_1[S_1]),$$

with $\mathrm{Prin}(K_1[S_1])$ the image of $\mathrm{q.f.}(K_1[S_1]) - \{0\}$ in $\mathrm{Div}(K_1[S_1])$. But there is a one to one correspondence from the homogeneous height 1 prime ideals of $K_1[S_1]$ to $P(S_1)$ defined by $p \mapsto p \cap S_1$ (again regarding S_1 as a subset of $K_1[S_1]$), setting up a natural isomorphism between $\mathrm{HDiv}(K_1[S_1])$ and $D(S_1)$. If, though, $x \in \mathrm{q.f.}(K_1[S_1]) - \{0\}$ which maps into $\mathrm{HDiv}(K_1[S_1])$, then in the localization $K_1[\langle S_1 \rangle]$, x must be a unit, since every surviving valuation corresponds to a nonhomogeneous height 1 prime of $K_1[S_1]$, where the value assigned x is by hypothesis 0. Since the units of $K_1[\langle S_1 \rangle]$ are exactly the nonzero homogeneous elements, x is homogeneous, i.e., $x = kX^g$, $k \in K_1$, $g \in \langle S_1 \rangle$. But then the image of x in $D(S_1)$ under the above isomorphism equals the image of X^g, which equals the image of g. This last equality can be checked at each element of $P(S_1)$ by first localizing S_1 (and $K_1[S_1]$) by the complement of that prime ideal of S_1, and noting that since the resulting semigroup must be a Krull semigroup with just one essential valuation, it has the form $G_1 \times \mathbf{Z}_+$, with G_1 a group. Thus $\mathrm{HPrin}(K_1[S_1])$ maps isomorphically onto $\mathrm{Prin}(S_1)$ under the given isomorphism from $\mathrm{HDiv}(K_1[S_1])$ to $D(S_1)$, and

$$\mathrm{Cl}(K_1[S_1]) \cong \mathrm{Cl}(S_1).$$

As a result of Lemma 1, to calculate $\mathrm{Cl}(A[S])$ it suffices to be able to calculate $\mathrm{Cl}(A)$ (which is often known) and $\mathrm{Cl}(S)$, which is a much easier problem in many cases. Furthermore, we may assume $S = S_1$, i.e.,

$$S \subseteq F = \textstyle\bigoplus_{i \in I} \mathbf{Z}x_i \text{ and } S = \langle S \rangle \cap F_+.$$

As Anderson points out in [**2**], $K[S]$ can then be regarded as a monomial generated subring of a polynomial ring. He showed in Theorem 3 there that if $F/\langle S \rangle$ is torsion and S is reduced in F (i.e., $\mathrm{pr}_i|_{\langle S \rangle}\colon \langle S \rangle \to \mathbf{Z}$ is surjective for all $i \in I$), then $\mathrm{Cl}(S) \cong F/\langle S \rangle$. We generalize this to eliminate the torsion requirement in the following result, which is the key to calculating class groups of Krull semigroups (note Anderson's hypotheses imply (2) below).

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THEOREM 2. *Let* $F = \bigoplus_{i \in I} \mathbf{Z}x_i$ *be a free abelian group*, S *a subsemigroup of* F *such that* $S = \langle S \rangle \cap F_+$. *The following are equivalent and imply* $\mathrm{Cl}(S) \cong F/\langle S \rangle$:

(1) *If* $\alpha, \beta \in I$, *then* $\mathrm{pr}_\alpha|_{\langle S \rangle}$ *is an essential valuation of* S, *and* $\mathrm{pr}_\alpha|_{\langle S \rangle} = \mathrm{pr}_\beta|_{\langle S \rangle}$ *implies* $\alpha = \beta$.

(2) S *is a reduced subsemigroup of* F, *and for every* $\alpha, \beta \in I$ *with* $\alpha \neq \beta$, *there exists an* $s \in S$ *such that* $\mathrm{pr}_\beta(s) > 0 = \mathrm{pr}_\alpha(s)$.

(3) *The set* $T_\alpha = \{x_\beta + \langle S \rangle | \alpha \neq \beta \in I\}$ *generates* $F/\langle S \rangle$ *as a semigroup for every* $\alpha \in I$.

Proof. If (1) holds, since essential valuations are always surjective, it follows that S is a reduced subsemigroup of F. However, analogously to Proposition 6 of [**4**], VII. 1, we have that for any Krull semigroup S, if $P \in P(S)$ then P is prime and the semigroup of v_P is $S + (-T)$ where T is the complement of P in S. Note also that

$$P = \{x \in S | \, v_P(x) > 0\}.$$

Thus if $\mathrm{pr}_\beta(s) > 0$ implies $\mathrm{pr}_\alpha(s) > 0$,

$$\{s \in S | \, \mathrm{pr}_\alpha(s) > 0\} \supseteq \{s \in S | \, \mathrm{pr}_\beta(s) > 0\},$$

so the valuation semigroup for $\mathrm{pr}_\beta|_{\langle S \rangle}$ contains that for $\mathrm{pr}_\alpha|_{\langle S \rangle}$, and they are equal. Thus the valuations are equal, a contradiction. So (1) implies (2).

To show (2) implies (1), we note that (2) implies the $\mathrm{pr}_\alpha|_{\langle S \rangle}$ are clearly distinct normed valuations on $\langle S \rangle$. But analogously to Corollary 2 to Theorem 3 of [**4**], VII. 1, if S is a Krull semigroup and v a normed valuation of $\langle S \rangle$, then v is an essential valuation of S if and only if $\{s \in S | v(s) > 0\}$ is a minimal (nonempty) prime ideal of S and $v(s) \geqq 0$ for all $s \in S$. Thus in our case it suffices to show that if $\alpha \in I$, $s_\alpha \in S$ such that $\mathrm{pr}_\alpha(s_\alpha) > 0$, then every element of S divides some element of the semigroup generated by $\ker(\mathrm{pr}_\alpha|_S) \cup \{s_\alpha\}$. However, since $S = \langle S \rangle \cap F_+$, divisibility in S agrees with divisibility in F_+ for elements of S. But $\mathrm{pr}_\alpha(s_\alpha) > 0$ and by (2) elements can be chosen from $\ker(\mathrm{pr}_\alpha|_S)$ which make any other projection positive, so any element of F_+ divides an appropriate product of elements from $\ker(\mathrm{pr}_\alpha|_S)$ and a power of s_α.

To prove (2) implies (3), we note that if $\alpha \neq \beta$ and $s \in S = \langle S \rangle \cap F_+$ such that $\mathrm{pr}_\beta(s) > 0 = \mathrm{pr}_\alpha(s)$, then $x_\beta | s$ and $x_\alpha \nmid s$, so

$$-x_\beta + s = \sum_{\gamma \neq \alpha} n_\gamma x_\gamma \in -x_\beta + \langle S \rangle, \, n_\gamma \geqq 0 \text{ for all } \gamma,$$

so $-x_\beta + \langle S \rangle$ is in the subsemigroup of $F/\langle S \rangle$ generated by T_α, and this subsemigroup is a subgroup. But since $S \subseteq F$ is reduced, $\langle S \rangle$ contains an element of the form $x_\alpha + \sum_{\gamma \neq \alpha} n_\gamma x_\gamma$, which demonstrates $-x_\alpha + \langle S \rangle$ is in the subsemigroup generated by T_α. Reversing this argument shows that (3) implies (2).

It remains to be seen that (1) implies $\mathrm{Cl}(S) \cong F/\langle S\rangle$. First, we claim that $(\mathrm{pr}_i)_{i\in I}$ contains all the essential valuations of S. For there is a one to one correspondence between the essential valuations of S and the elements of $P(S)$, and also a one to one correspondence between the elements of $P(S)$ and the height one homogeneous primes of $K[S]$, K a field. But our proof of Theorem 1 shows that $K[S]$ is the intersection of $K[\langle S\rangle]$ with the valuation rings of the inf valuations corresponding to the $(\mathrm{pr}_i)_{i\in I}$, so by Corollary 2 to Proposition 5 in [**4**], VII. 1.4, every essential valuation of $K[S]$ must be equivalent to one of these, so every minimal prime of $K[S]$ must arise from one of these. But the minimal prime corresponding to a valuation from $K[\langle S\rangle]$ is nonhomogeneous, while the prime in $K[S]$ corresponding to the inf valuation from pr_i is the prime obtained by extending to $K[S]$ the prime of S corresponding to pr_i. Thus if any essential valuation of S were missing, there would also be an essential valuation of $K[S]$ missing, which is impossible. (This also serves to establish the analogue of Corollary 2 to Proposition 5 in [**4**] for Krull semigroups.) Now, if the identification between $(\mathrm{pr}_i)_{i\in I}$ and the set of essential valuations of S has pr_i corresponding to $P_i \in P(S)$, then

$$\mathrm{Prin}(S) = \{\textstyle\sum \mathrm{pr}_i(g)P_i |\, g \in \langle S\rangle\}$$

while

$$\langle S\rangle = \{\textstyle\sum \mathrm{pr}_i(g)x_i |\, g \in \langle S\rangle\}$$

so the isomorphism $\eta: F \rightarrow D(S)$ such that $\eta(x_i) = P_i$ takes $\langle S\rangle$ to $\mathrm{Prin}(S)$, and

$$\mathrm{Cl}(S) = D(S)/\mathrm{Prin}(S) \cong F/\langle S\rangle.$$

Remark 2. If H is a subgroup of F such that (3) holds with $\langle S\rangle$ replaced by H, then $H = \langle H \cap F_+\rangle$. If further H is a finitely generated group, then $H \cap F_+$ is a finitely generated Krull semigroup by Remark 1, and satisfies $\mathrm{Cl}(H \cap F_+) \cong F/H$.

Remark 3. Statement (3) in principle determines the isomorphism classes of Krull semigroups in which only the identity is invertible, and which have a given divisor class group G. By (3) and Remark 2, such semigroups are exactly those of the form $F_+ \cap \ker(\phi)$, where $\phi: F = \bigoplus_{i\in I} \mathbf{Z}x_i \rightarrow G$ is any group map such that for every $\beta \in I$, $\phi(\{x_\alpha|\alpha \neq \beta\})$ generates G as a semigroup. If $\phi': F' = \bigoplus_{j\in J} \mathbf{Z}y_j \rightarrow G$ satisfies similar conditions, the corresponding Krull semigroups are isomorphic if and only if there exists a bijection $\tau: J \rightarrow I$ and an automorphism σ of G such that

$$\phi'(y_j) = \sigma(\phi(x_{\tau(j)})) \text{ for all } j \in J.$$

Note that since $F_+ \cap \ker \phi$ must also satisfy (2), it cannot be finitely generated unless F is, while by Remark 2 if F is finitely generated, $\ker(\phi) \cap F_+$ must be also.

Example 1. Let $F = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3$, and define $\phi: F \to \mathbf{Z}_{30} = \mathbf{Z}/(30\mathbf{Z})$ by

$$\phi(x_1) = \bar{2}, \phi(x_2) = \bar{3}, \phi(x_3) = \bar{5}, S = F_+ \cap \ker(\phi).$$

Then S is a Krull semigroup with $\mathrm{Cl}(S) \cong \mathbf{Z}_{30}$. As a semigroup, S requires 21 generators. If instead we had set $\phi(x_3) = -\bar{5}$, S would have required only 10 generators.

Remark 4. To determine the generators of $F_+ \cap \ker \phi$, as in the example, is the same problem as to determine the minimal nonzero elements of $F_+ \cap \ker \phi$ under the partial ordering on F with positive cone $F_+ - \{0\}$. When the class group is torsion and F is finitely generated, we can immediately bound the coefficients of each basis element of F above by the smallest positive multiple of that basis element which sits in $\ker \phi$, and thus reduce the problem to a finite search. Thus in the example above, we know that any generator x of S will satisfy

$$0 \leqq \mathrm{pr}_1(x) \leqq 15, 0 \leqq \mathrm{pr}_2(x) \leqq 10, \text{ and } 0 \leqq \mathrm{pr}_3(x) \leqq 6.$$

Just starting with this and a basis such as $(15, 0, 0)$, $(-3, +2, 0)$, $(-1, -1, +1)$ for $\ker \phi$, it is not hard to determine the minimal elements of $S - \{0\}$ by hand. If the divisor class group is not torsion, determining the minimal nonzero elements of $F_+ \cap \ker \phi$ can still be done algorithmically, as indicated in Remark 1, but the only general algorithm the author is aware of uses the result mentioned in that remark, which is often awkward to use. A better algorithm which does not require an inductive procedure based on the number of valuations would be desirable.

Remark 5. The comment in the proof of Theorem 2 that the semigroup analogue of Corollary 2 to Proposition 5 in [**4**], VII. 1.4 holds can also be used to assist in determining the divisor class group of a semigroup $S \subseteq F = \bigoplus_{i \in I} \mathbf{Z}x_i$ such that $S = \langle S \rangle \cap F_+$ but not satisfying (1)–(3). Namely, by replacing basis elements of F by appropriate multiples we may change to the case where $\langle S \rangle$ is a reduced subsemigroup of F. Next eliminate duplication on the projections, i.e., if for every element of S, the coefficients of x_i and x_j agree, eliminate one of $\{i, j\}$ from I, and continue eliminating coordinates until no two agree on S. Finally, eliminate any $\alpha \in I$ which does not satisfy condition (2) in Theorem 2 for every $\beta \in I$. This eliminates the valuations that correspond to non-minimal primes of S. What remains is a free group $F' = \bigoplus_{i \in I'} \mathbf{Z}(n_i x_i)$ such that S embeds in F' so that $S = \langle S \rangle \cap F_+'$ and the embedding satisfies (2) in Theorem 2.

Example 2. Let S be the subsemigroup of $F = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3$ generated by $(1, 0, 1)$ and $(0, 5, 2)$. It is easily checked that $S = \langle S \rangle \cap F_+$. We first make S reduced in F by dividing the second coordinate by 5 (which is equivalent to replacing x_2 by $5x_2$). Thus S is now generated by $(1, 0, 1)$ and $(0, 1, 2)$. Since the third coordinate cannot be made 0 while the first (or second) coordinate is positive, we eliminate it. Now S is generated by $(1, 0)$ and $(0, 1)$. This embedding satisfies Theorem 2, and we see $\mathrm{Cl}(S) \cong 0$.

COROLLARY 1. *Every abelian group is the class group of a Krull semigroup. The semigroup can be chosen to be finitely generated if and only if the group is finitely generated.*

Proof. If G is the abelian group, let $\{z_j | j \in J\}$ be a set of semigroup generators for G. Note J can be chosen to be finite if and only if G is finitely generated as a group. Let $I = \{0, 1\} \times J$ and define

$$\phi: F = \bigoplus_{i \in I} \mathbf{Z}_{xi} \rightarrow G$$

by

$$\phi(x_{(\delta, j)}) = z_j \text{ for any } (\delta, j) \in I.$$

By Remark 3, $S = F_+ \cap \ker(\phi)$ has class group isomorphic to G. By Remark 2, if J is finite, this semigroup is finitely generated.

In view of Claborn's result [**5**] that every abelian group is the divisor class group of a Dedekind domain, the above does not seem too exciting. Indeed, since S can only be chosen finitely generated if $\mathrm{Cl}(S)$ is to be so, the resulting semigroup rings are often non-Noetherian. However, following [**2**], Theorem 10, the above implies a more interesting result.

COROLLARY 2. *If G is any abelian group, there exists a quasi-local Krull domain A such that* $\mathrm{Cl}(A) \cong G$.

Proof. Let S be as in the proof of Corollary 1, K a field, and $A = K[S]_M$ where M is the maximal ideal of $K[S]$ consisting of elements whose coefficient of the identity is 0. Since only the identity is invertible in S, $K[S]$ is a Krull domain by Theorem 1. Grading $K[S]$ by assigning to each $s \in S$ the degree of the sum of its essential valuations, Proposition 7.4 of [**7**] implies that

$$\mathrm{Cl}(A) \cong \mathrm{Cl}(K[S]) \cong G.$$

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