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REPRESENTATIONS OF GROUPS AS AUTOMORPHISMS ON ORTHOMODULAR LATTICES AND POSETS

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1. Introduction. In this paper we study the problem of representing groups as groups of automorphisms on an orthomodular lattice or poset. This problem not only has intrinsic mathematical interest but, as we shall see, also has applications to other fields of mathematics and also physics. For example, in the "quantum logic" approach to an axiomatic quantum mechanics, important parts of the theory can not be developed any further until a fairly complete study of the representations of physical symmetry groups on orthomodular lattices is accomplished [**1**].

We will consider two main topics in this paper. The first is the analogue of Schur's lemma and its corollaries in this general setting and the second is a study of induced representations and systems of imprimitivity. One will note that some of the results can be generalized to representations of groups on orthocomplemented lattices and posets and even to posets, but for simplicity we will consider only the richer structures stated above.

2. Definitions. Let L be an orthomodular poset and let G be a group. A map $G \rightarrow \text{aut}(L)$, $g \rightarrow U_g$, is a *representation* of G on L if $U_{g_1 g_2} = U_{g_1} U_{g_2}$ for all $g_1, g_2 \in G$. We define three notions of reducibility of representations. If a is in the centre $Z(L)$ of L, we can write $L = [0, a] \oplus [0, a']$ since every $c \in L$ has the form $c = c_1 \vee c_2$ where c_1 and c_2 are unique elements in $[0, a]$ and $[0, a']$ respectively. We say that a representation U of G is *strongly reducible* if there is a non-trivial (i.e., $\neq 0, 1$) element $a \in Z(L)$ such that $L = [0, a] \oplus [0, a']$ and representations U_1, U_2 of G on $[0, a]$ and $[0, a']$ respectively such that if $c = c_1 \vee c_2$, $c_1 \in [0, a]$, $c_2 \in [0, a']$ then $U_g c = U_{1g} c_1 \vee U_{2g} c_2$ for all $g \in G$. In this case we write $U = U_1 \oplus U_2$. We say that U is *reducible* if there is a non-trivial $a \in L$ such that $U_g a = a$ for all $g \in G$. Finally U is *weakly reducible* if there is a non-trivial (i.e., $\neq 0, 1$) sub-orthomodular poset $L_0 \subset L$, $L_0 \neq L$, such that $U_g L_0 \subset L_0$ for all $g \in G$. If L is a lattice we assume in the last definition that L_0 is a non-trivial sub-orthomodular lattice.

It is clear that strong reducibility implies reducibility which in turn implies weak reducibility. However, each type of reducibility is strictly stronger than its successor as can be easily seen by examples. For instance let

$$L = \{0, 1, a, a', b, b'\}$$

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where a and b are not related and let $T \in \text{aut}(L)$ be given by $T(a) = a'$, $T(b) = b'$. Let Z_2 be the group with two elements $\{0, 1\}$. Then $U_0 = I$, $U_1 = T$ is a representation of G which is weakly reducible since it leaves the sub-poset $L_0 = \{0, 1, a, a'\}$ invariant but it is not reducible. It is clear that there are representations that are reducible but not strongly reducible since the identity representation $U_g = I$ for all $g \in G$ is reducible but is not strongly reducible unless there is a non-trivial element in $Z(L)$. U is weakly irreducible, irreducible, strongly irreducible respectively if it is not weakly reducible, reducible, strongly reducible respectively. Of course, each of these implies its successor. We will deal mainly with weakly irreducible representations in the next section. However, it should be noted that strong reducibility brings up the important question of the decomposition of representations into strongly irreducible sub-representations which is so well developed in the usual theory of linear group representations. This latter question will not be treated in this paper.

Now let G be a topological group. When considering representations of topological groups on L we always assume that L is σ-orthocomplete and that L has a *full* set of states M. That is, if $m(a) \leqq m(b)$ for all $m \in M$ then $a \leqq b$. A representation U of G on L is a *Borel representation* (relative to M) if the functions $G \rightarrow R$ given by $g \rightarrow m(U_g a)$ is a real valued Borel function for all $a \in L$ and $m \in M$. The representation is *continuous* (relative to M) if the functions $g \rightarrow m(U_g a)$ are continuous for all $a \in L$ and $m \in M$. In the case of ordinary unitary representations, Borel representations are always continuous. This is not the case in this more general setting. For example, the Borel sets $B(G)$ of G form an orthomodular lattice and $U_g S = gS$, $S \in B(G)$, is a representation of G on $B(G)$. Let M be the set of point probability measures on $B(G)$. Then U is a Borel representation relative to M but is not in general continuous.

3. Schur's Lemma. In this section we assume that L_1, L_2 are orthomodular lattices with more than two elements, and that G is a group. We say that a map $h: L_1 \rightarrow L_2$ is a *morphism* if $h(a \vee b) = h(a) \vee h(b)$ for all $a, b \in L$, and if $a_1 \perp a_2$ implies that $h(a_1) \perp h(a_2)$. Notice that $h(0) = 0$ since $h(0) \perp h(0)$. Also, if $a \leqq b$ then $h(a) \leqq h(a) \vee h(b) = h(a \vee b) = h(b)$. Finally, since $h(a) \perp h(a')$ we have

$$h(1) \wedge h(a)' = [h(a) \vee h(a')] \wedge h(a)' = h(a') \wedge h(a)' = h(a').$$

A morphism h is a *monomorphism* if it is injective, an *epimorphism* if it is surjective, and an *isomorphism* if it is bijective. If $L_1 = L_2$, an isomorphism is called an *automorphism*. Notice that if $h: L_1 \rightarrow L_2$ is an epimorphism then there is an $a \in L$ such that $h(a) = 1$, so $h(1) \geqq h(a) = 1$ which gives $h(1) = 1$. In this case, $h(a') = h(a)'$ so $h(a \wedge b) = h(a) \wedge h(b)$. Thus h preserves all the lattice theoretic structure. This also holds for any morphism h

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such that $h(1) = 1$. We now show that $h(a \wedge b) = h(a) \wedge h(b)$ for any morphism.

LEMMA 3.1. *Let $h: L_1 \to L_2$ be a morphism.*

(i) *$h(a \wedge b) = h(a) \wedge h(b)$ for all $a, b \in L_1$.*

(ii) *h is a monomorphism if and only if $h(a) = 0$ implies that $a = 0$.*

Proof. (i) $h(a \wedge b) = h((a' \vee b')') = h(1) \wedge (h(a' \vee b'))'$
$= h(1) \wedge (h(a') \vee h(b'))' = h(1) \wedge (h(a'))' \wedge h(1) \wedge (h(b'))' = h(a) \wedge h(b)$.

(ii) If $h(a) = h(b)$ then

$$\begin{aligned} h(a \wedge (a \wedge b)') &= h(a \wedge (a' \vee b')) = h(a) \wedge h(a' \vee b') \\ &= h(a) \wedge (h(a') \vee h(b')) = h(a) \wedge h(b') = h(1) \wedge h(a) \wedge h(b)' \\ &= h(1) \wedge h(a) \wedge h(a)' = 0. \end{aligned}$$

Therefore, $a \wedge (a \wedge b)' = 0$. Since $a \wedge b \leqq a$, by orthomodularity we have $a = a \wedge b$ so $a \leqq b$. Similarly, $b \leqq a$ so $a = b$.

We say that a morphism $h: L_1 \to L_2$ is *trivial* if $h(a) = 0$ or 1 for all $a \in L_1$.

THEOREM 3.2. *Let U_1 and U_2 be representations of G in L_1 and L_2 respectively and let $h: L_1 \to L_2$ be a morphism that satisfies $hU_1 = U_2h$. If U_2 is irreducible then $h \equiv 0$ or $h(1) = 1$. If U_2 is weakly irreducible then h is trivial or is an epimorphism. If U_1 is weakly irreducible and $h(1) = 1$, then h is either a monomorphism or is trivial. If U_1 and U_2 are weakly irreducible then h is trivial or an isomorphism.*

Proof. If U_2 is irreducible then $U_{2g}h(1) = hU_{1g}(1) = h(1)$ and hence $h(1)$ is 0 or 1. In the first case $h \equiv 0$. If U_2 is weakly irreducible then by the above $h \equiv 0$ or $h(1) = 1$. In the first case h is trivial, so assume that $h(1) = 1$. Then the set $L_0 = \{h(a): a \in L_1\}$ is a sub-orthomodular lattice of L_2. If $b \in L_0$ then $b = h(a)$ for some $a \in L_1$ and we have $U_{2g}b = U_{2g}h(a) = hU_{1g}a \in L_0$. Hence $U_2L_0 \subset L_0$ and $L_0 = \{0, 1\}$ or $L_0 = L_2$. In the first case h is trivial and in the second case h is an epimorphism. Now suppose that U_1 is weakly irreducible and that $h(1) = 1$. Let $N = \{a \in L_1: h(a) = 0 \text{ or } 1\}$. Then N is a sub-orthomodular lattice of L_1 and $\{0, 1\} = U_{2g}hN = hU_{1g}N$ so $U_{1g}N \subseteq N$ and hence $N = \{0, 1\}$ or L_1. In the first case h is a monomorphism by Lemma 3.1 and in the second case h is trivial.

COROLLARY 3.3. *If U is a weakly irreducible representation of G on L and $h: L \to L$ is a morphism such that $U_gh = hU_g$ for all $g \in G$ then h is trivial or an automorphism.*

We now show that Theorem 3.2 does not hold if we replace weak irreducibility by irreducibility. Let $L_1 = \{0, 1, a, a'\}$ and let $L_2 = \{0, 1, b, b', c, c'\}$, where b and c are not related. Let $h: L_1 \to L_2$ and $f: L_2 \to L_1$ be the morphisms given by $h(1) = 1$, $h(a) = b'$ and $f(1) = 1$, $f(b) = f(c) = a'$. Let Z_2 be the

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two element group $\{e, g\}$, and let U_1, U_2 be the representations of Z_2 on L_1 and L_2 respectively given by $U_{1g}(a) = a'$ and $U_{2g}(b) = b'$, $U_{2g}(c) = c'$. Then U_1 and U_2 are irreducible and we have $hU_1 = U_2h$, $fU_2 = U_1f$. Now clearly h is not trivial and h is not an epimorphism, and also f is not trivial and f is not a monomorphism.

Recall that an important form of Schur's lemma states that if a linear operator T on a complex vector space commutes with an irreducible unitary representation then $T = \lambda I$ for some complex number λ. Thus one might conjecture in our case that if a morphism h commutes with an irreducible (or weakly irreducible) representation then $h = I$. However this cannot hold as can be seen by the following example. Let.

$$U_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, t \in \mathbf{R}.$$

Then U_t is a unitary representation of the real line $\mathbf{R}$ in the vector space $\mathbf{R}^2$ and U_t induces a weakly irreducible representation of $\mathbf{R}$ as automorphisms on the subspaces of $\mathbf{R}^2$. Now if $t_0 \in \mathbf{R}$, $U_{t_0}U_t = U_tU_{t_0}$ for all $t \in \mathbf{R}$ and in general $U_{t_0} \neq I$. The problem here, of course, is that Schur's lemma need not hold in a real vector space. Thus if we want Schur's lemma in this form to be valid we must add a condition on L which eliminates such cases as the one just considered.

A map $\phi: L \to L$ is a *projection* if there is an $a \in L$ such that ϕ has the form $\phi_a(b) = (b \vee a') \wedge a$ for all $b \in L$. We say that ϕ is *non-trivial* if $a \neq 0$ or 1.

LEMMA 3.4. *If $h \in$ aut(L) and $a \in L$ the following are equivalent.*

(1) $ha \leqq a$,

(2) $h\phi_a = \phi_a h \phi_a$,

(3) $\phi_{a'}h = \phi_{a'}h\phi_{a'}$.

Proof. If $ha \leqq a$ then clearly $h\phi_a = \phi_a h\phi_a$. If (2) holds then $ha = h\phi_a a = \phi_a h \phi_a a \leqq a$ and thus (1) and (2) are equivalent. Now if $ha \leqq a$ then $h^{-1}a \geqq a$ so $h^{-1}a' \leqq a'$ and applying (2), $h^{-1}\phi_{a'} = \phi_{a'}h^{-1}\phi_{a'}$. Since h and ϕ_a are residuated maps, by taking * of both sides we have $\phi_{a'}h = \phi_{a'}h\phi_{a'}$. Conversely, if (3) holds, then $h^{-1}\phi_{a'} = h^*\phi_{a'} = \phi_{a'}h^{-1}\phi_{a'}$. Since (2) and (1) are equivalent, $h^{-1}a' \leqq a'$ and hence $a' \leqq ha'$ and $ha \leqq a$.

COROLLARY 3.5. *If $h \in$ aut(L) then $ha = a$ if and only if $h\phi_a = \phi_a h$.*

Proof. For necessity, since $ha \leqq a$, we have by Lemma 3.4 that $h\phi_a = \phi_a h\phi_a$. Since $h^{-1}a \leqq a$ we also have $h^{-1}\phi_a = \phi_a h^{-1}\phi_a$ and hence, taking the * of both sides, $\phi_a h = \phi_a h \phi_a$. For sufficiency, $h\phi_a = h\phi_a\phi_a = \phi_a h\phi_a$ so by Lemma 3.4, $ha \leqq a$. Since $h^{-1}\phi_a = \phi_a h^{-1}$, we also have $h^{-1}a \leqq a$ so $a \leqq ha$ which completes the proof.

COROLLARY 3.6. *Let U be an irreducible representation of G on L. If ϕ is a projection such that $U_g\phi = \phi U_g$ for all $g \in G$, then $\phi = 0$ or I.*

We now introduce a condition that characterizes those L on which the stronger form of Schur's lemma holds.

AXIOM 1. If $I \neq h \in \mathrm{aut}(L)$ then there is a non-trivial projection that commutes with all automorphisms that commute with h.

We will show later that this axiom holds in the lattice of all closed subspaces of a complex Hilbert space.

THEOREM 3.7. *The following two statements are equivalent.*

(i) *L satisfies Axiom* 1.

(ii) *If U is an irreducible representation of G and $h \in \mathrm{aut}(L)$ satisfies $U_g h = hU_g$ for every $g \in G$, then $h = I$.*

Proof. Suppose that Axiom 1 is valid on L and that $I \neq h \in \mathrm{aut}(L)$ satisfies $U_g h = hU_g$ for every $g \in G$. Then there is a non-trivial projection ϕ such that $\phi U_g = U_g \phi$ for all $g \in G$. This contradicts Corollary 3.6 and hence $h = I$. Suppose that (ii) holds and that $I \neq h \in \mathrm{aut}(L)$. Let

$$A = \{f \in \mathrm{aut}(L): fh = hf\}.$$

Now A is a group of automorphisms and clearly gives a representation of itself on L. Since h commutes with A, by hypothesis A must be reducible. Thus there is a non-trivial $a \in L$ such that $Aa = a$. Corollary 3.5 implies that $A\phi_a = \phi_a A$.

We now consider weak irreducibility.

LEMMA 3.8. *Let U be a weakly irreducible representation of G on L and let $h: L \to L$ be a morphism such that $U_g h = hU_g$ for all $g \in G$. Then either h is trivial, $h = I$, or $h \in \mathrm{aut}(L)$ and there is no non-trivial $a \in L$ with $ha = a$.*

Proof. Applying Corollary 3.3, h is trivial or an automorphism. Suppose that $I \neq h \in \mathrm{aut}(L)$ and let $L_0 = \{a \in L: ha = a\}$. Then L_0 is a sub-orthomodular lattice of L. Now if $b \in L_0$ then $U_g b = U_g hb = hU_g b$ so $U_g b \in L_0$ for all $g \in G$ and hence $U_g L_0 \subseteq L_0$ for all $g \in G$. Since U is weakly irreducible, $L_0 = L$ or $L_0 = \{0, 1\}$. Since $h \neq I$ the first case is impossible, so $L_0 = \{0, 1\}$.

We shall eliminate the last possibility in Lemma 3.8 with the following axiom.

AXIOM 2. If $h \in \mathrm{aut}(L)$ then there is a non-trivial $a \in L$ with $ha = a$.

THEOREM 3.9. *Suppose that Axiom 2 holds in L and that U is a weakly irreducible representation of G on L. If $h: L \to L$ is a morphism such that $U_g h = hU_g$ for all $g \in G$ then h is trivial or $h = I$.*

Axiom 1 implies Axiom 2. Although the author conjectures that Axiom 2 is strictly weaker than Axiom 1, he has not been able to prove it. We now show that both axioms hold in a complex Hilbert space of dimension >2.

THEOREM 3.10. *Let L be the lattice of all orthogonal projections in a complex Hilbert space H of dimension >2. Then Axiom 1 and hence Axiom 2 hold in L.*

Proof. In a complex Hilbert space of dimension >2 any automorphism is implemented by a unitary or anti-unitary operator U [**4**]. If $I \neq U$ is unitary, any non-trivial projection in its spectral resolution will satisfy the requirement of Axiom 1. If U is anti-unitary, then U^2 is unitary. If $U^2 \neq I$ then any non-trivial projection in the spectral resolution of U^2 is a weak limit of polynomials in U^2 [**2**] and hence polynomials in U and will hence satisfy the requirement of Axiom 1. If $U^2 = I$ then U is isomorphic to the operator $U_1: f \rightarrow f^*$ on some L_2 space H_1 [**3**]. If $0 \neq f \in H_1$ then $g = f + f^*$ is an eigenvector of U_1 and the projection onto this vector will now satisfy the requirement of Axiom 1.

4. Induced representations. Let us first consider a simple example. Let L be the Boolean algebra 2^4; that is, L is the Boolean algebra with four atoms a, b, c, d. Let $G = \{u, g_1, g_2, g_3\}$ be the four group $G = Z_2 \times Z_2$. Let U be the representation of G on L given by $U_{g_1}a = c$, $U_{g_1}b = d$; $U_{g_2}a = b$, $U_{g_2}c = d$; $U_{g_3}a = d$, $U_{g_3}b = c$. Now let K be the subgroup $\{u, g_1\}$ of G and consider the quotient space G/K of right cosets $s_1 = \{u, g_1\}$, $s_2 = \{g_2, g_3\}$. Let E be a map from G/K to the set of projections on L given by $E(s_1) = \phi_{b\vee d}$, $E(s_2) = \phi_{a\vee c}$. Now E is a "projection valued measure" based on the subsets of G/K and we have $E(\Lambda g) = U_{g^{-1}}E(\Lambda)U_g$ for every $\Lambda \subset G/K$ and $g \in G$. We will later call E a "system of imprimitivity" for U based on the subsets of G/K. It will follow from the theory we develop in this section that U is equivalent to a representation that is "induced" from a representation of K and that under this equivalence E has a canonical form.

In this section we assume that L is a σ-orthocomplete orthomodular poset (or *logic*) with a full set of states M, and that G is a locally compact group. Let Ω be a set and L^Ω denote the set of maps $f: \Omega \rightarrow L$. If $f, g \in L^\Omega$ we write $f \leqq g$ if $f(\omega) \leqq g(\omega)$ for all $\omega \in \Omega$ and we define $f'(\omega) = f(\omega)'$ for all $\omega \in \Omega$. Then L^Ω is a logic under this partial order and orthocomplementation. Also L^Ω has a full set of states. If $m \in M$ and $\omega \in \Omega$, define m_ω on L^Ω by $m_\omega(f) = m(f(\omega))$. Then m_ω is a state on L^Ω and the set

$$M_\Omega = \{m_\omega: \omega \in \Omega, m \in M\}$$

is a full set of states on L^Ω; indeed, if $f \neq g$ there is an $\omega \in \Omega$ such that $f(\omega) \neq g(\omega)$ and there is an $m \in M$ such that $m(f(\omega)) \neq m(g(\omega))$ so $m_\omega(f) \neq m_\omega(g)$. This set of states can be generalized further. Suppose (Ω, F) is a measurable space and let $B(L^\Omega)$ be the $f \in L^\Omega$ such that $\omega \rightarrow m(f(\omega))$ is measurable for all $m \in M$. It is easy to see that $B(L^\Omega)$ is a logic. If μ is a probability measure on F, $f \in B(L^\Omega)$, and $m \in M$ then

$$m_\mu(f) = \int_\Omega m(f(\omega))\mu(d\omega)$$

is a state on L^Ω and the m_μ's give a full set of states on $B(L^\Omega)$. We can extend this even further if we let the m's vary over Ω and, in fact, we can find all the states on $B(L^\Omega)$, but we do not need this result in this paper.

Let K be a closed subgroup of G and let $k \to W_k \in \text{aut}(L)$ be a Borel representation of K on L. Let L_W be the set of functions $f: G \to L$ such that $g \to m(f(g))$ is Borel for all $m \in M$ and $f(kg) = W_k f(g)$ for all $g \in G$, $k \in K$. Then L_W is a logic. For $g_1 \in G$ define $U_{g_1}{}^W: L_W \to L_W$ by $(U_{g_1}{}^W f)(g) = f(gg_1)$. Then $U_{g_1}{}^W \in \text{aut}(L_W)$ for every $g_1 \in G$. For example, to show that $U_{g_1}{}^W f \in L_W$ if $f \in L_W$ we have $(U_{g_1}{}^W f)(kg) = f(kgg_1) = W_k f(gg_1) = W_k (U_{g_1}{}^W f)(g)$ and of course $g \to m(f(gg_1))$ is Borel. Now $g \to U_g{}^W$ is a representation of G on L_W since $(U_{g_1 g_2}{}^W f)(g) = f(gg_1g_2) = (U_{g_2}{}^W f)(gg_1) = (U_{g_1}{}^W U_{g_2}{}^W f)(g)$. U^W is Borel relative to the full set of canonical states of the previous paragraph since $g_1 \to m_{g_0}(U_{g_1}{}^W f) = m_{g_0}(f(gg_1)) = m(f(g_0g_1))$ is Borel. U^W is called the representation of G *induced* by the representation W of the subgroup K.

If U_1 and U_2 are representations of G on L_1 and L_2 respectively we say that U_1 and U_2 are *equivalent* ($U_1 \cong U_2$) if there is an isomorphism $h: L_1 \to L_2$ such that $hU_{1g} = U_{2g}h$ for all $g \in G$.

THEOREM 4.1. *If U is a Borel representation of G on L then U is equivalent to a representation induced by a representation of a subgroup of G.*

Proof. Let the subgroup be G itself and let the inducing representation be U. Then L_U is the set of maps $f: G \to L$ such that $g \to m(f(g))$ is Borel for all $m \in M$, and $f(g_1g_2) = U_{g_1}f(g_2)$ for all $g_1, g_2 \in G$. This is the set of maps $f: G \to L$ which satisfy $f(g) = U_g f(u)$ (u is the identity of G) for all $g \in G$. Define $h: L_U \to L$ by $h: f \to f(u)$. Now h is surjective since for an $a \in L$ define $f_a(g) = U_g a$; then $f_a \in L_u$ and $h: f_a \to a$. Also h is injective since if $f(u) = k(u)$ then $f(g) = U_g f(u) = U_g k(u) = k(g)$. It is now easy to see that h is an isomorphism. To show that $U \cong U^U$ we have

$$(h^{-1}U_{g_1}hf)(g) = U_g(U_{g_1}hf) = U_gU_{g_1}f(u) = U_{gg_1}f(u) = f(gg_1) = (U_{g_1}{}^U f)(g).$$

Let U be a representation of G on L and let L_0 be a sublogic of L such that $U_g L_0 \subseteq L_0$ for all $g \in G$. Then the restriction of U to L_0 denoted by $U|_{L_0}$ is called a *sub-representation* of U.

We have shown in Theorem 4.1 that any representation is equivalent to an induced representation. We now show that if K is a subgroup of G then any representation of G is equivalent to a sub-representation of a representation that is induced by a representation of K.

THEOREM 4.2. *Let U be a Borel representation of G on L and let K be a subgroup of G. Then there is a representation W of K and a sublogic L_1 of L_W such that $U \cong U^W|_{L_1}$.*

Proof. Let W be the restriction of U to K. Then L_W is the set of maps $f: G \to L$ such that $g \to m(f(g))$ is Borel for all $m \in M$, and

$$f(kg) = W_k f(g) = U_k f(g)$$

for all $k \in K, g \in G$. Let L_1 be the set of maps $f: G \to L$ of the form $f(g) = U_g a$ for some fixed $a \in L$. If $f \in L_1$ then $f(kg) = U_{kg}a = U_k U_g a = U_k f(g)$, so $f \in L_W$. We then see that L_1 is a sublogic of L_W. Let us now define the map $h: L \to L_1$ by $h: a \to U_g a$. Now h is an isomorphism and

$$h^{-1}U_{g_1}{}^W|_{L_1}ha = h^{-1}U_{g_1}{}^W|_{L_1}(U_g a) = h^{-1}(U_{gg_1}a) = h^{-1}U_g(U_{g_1}a) = U_{g_1}a.$$

Hence $U \cong U^W|_{L_1}$.

Although Theorem 4.1 and Theorem 4.2 are interesting, the important induced representations are those that are generated by a system of imprimitivity.

Let K be a closed subgroup of G, W a Borel representation of K on L, and U^W the representation of G induced by W. Let G/K be the quotient space of K right cosets and let $\psi: G \to G/K$ be the canonical surjection. Then G/K is a Hausdorff space under the strongest topology in which ψ is continuous. Denote the set of Borel subsets of G/K by $B(G/K)$. If $S \in B(G/K)$ define $E(S): L_W \to L_W$ by

$$(E(S)f)(g) = [(\chi_S \circ \psi) \wedge f](g) = \begin{Bmatrix} f(g) & \text{if } \psi(g) \in S \\ 0 & \text{if } \psi(g) \notin S \end{Bmatrix}.$$

Notice that if $k \in K$ and $g \in G$ we have

$$\begin{aligned}(E(S)f)(kg) &= (\chi_S \circ \psi)(kg) \wedge f(kg) \\ &= (\chi_S \circ \psi)(g) \wedge W_k f(g) \\ &= (W_k \chi_S \circ \psi)(g) \wedge W_k f(g) \\ &= W_k(\chi_S \circ \psi(g) \wedge f(g)) \\ &= W_k(E(S)f(g))\end{aligned}$$

and thus $E(S)f$ is indeed in L_W if $f \in L_W$. Also we have

$$\begin{aligned}(U_{g_1-1}{}^W E(S) U_{g_1}{}^W f)(g) &= (E(S)U_{g_1}{}^W f)(gg_1{}^{-1}) \\ &= \chi_S \circ \psi(gg_1{}^{-1}) \wedge U_{g_1}{}^W f(gg_1{}^{-1}) \\ &= \chi_S \circ \psi(gg_1{}^{-1}) \wedge f(g) \\ &= \chi_{Sg_1} \circ \psi(g) \wedge f(g) \\ &= (E(Sg_1)f)(g)\end{aligned}$$

and hence $U_{g_1-1}{}^W E(S) U_{g_1}{}^W = E(Sg_1)$ for all $S \in B(G/K)$, $g_1 \in G$. Furthermore, $E(S)$ satisfies the following conditions

(i) $E(G/K) = I$.

(ii) If $f \leqq h$ then $E(S)f \leqq E(S)h$ for all $S \in B(G/K)$.

(iii) If $S_1 \cap S_2 = \emptyset$ then $E(S_1)f \perp E(S_2)f$.

(iv) If S_i are mutually disjoint then $E(\bigcup S_i)f = \bigvee E(S_i)f$.

(v) $E(S_1 \cap S_2)f = E(S_1)E(S_2)f$ for all $f \in L_W$, $S_1, S_2 \in B(G/K)$.

Also we see that $S \to E(S)1$ is a σ-morphism from $B(G/K)$ to L_W.

Now let Ω be a topological space, let $B(\Omega)$ be the set of Borel subsets of Ω, and let L be an arbitrary logic. A map $E: B(\Omega) \times L \to L$ is called a *generalized σ-morphism* if

(1) $E(\Omega, a) = a$ for all $a \in L$,
(2) $E(S, a) \leqq E(S, b)$ if $a \leqq b$,
(3) if $S_1 \cap S_2 = \emptyset$ then $E(S_1, a) \perp E(S_2, a)$ for all $a \in L$,
(4) if S_i are mutually disjoint then $E(\bigcup S_i, a) = \bigvee E(S_i, a)$ for all $a \in L$,
(5) $E(S_1 \cap S_2, a) = E(S_1, E_2(S_2, a))$ for all $S_1, S_2 \in B(\Omega)$, $a \in L$.

We sometimes write $E(S)a$ for $E(S, a)$. Notice that $S \to E(S, 1)$ is a σ-morphism and if $\Omega = R$ then $S \to E(S, 1)$ is an observable. If $m \in M$ is a state on L then for every $a \in L$, $S \to m(E(S, a))$ is a measure on $B(\Omega)$ and from (2) this measure is absolutely continuous with respect to the probability measure $S \to m(E(S, 1))$.

If U is a representation of G on L, Ω a topological space on which G acts as a continuous transformation group, and $E: B(\Omega) \times L \to L$ is a generalized σ-morphism which satisfies $E(Sg, a) = U_{g^{-1}}E(S, U_g a)$ for every $S \in B(\Omega)$, $a \in L$, then E is a *system of imprimitivity* for U based on $B(\Omega)$. It is important to consider systems of imprimitivity for U based on $B(G/K)$ where K is a closed subgroup of G. Recall that a σ-finite measure ν on $B(G/K)$ is *quasi-invariant* if $\nu(\Lambda) = 0$ implies that $\nu(\Lambda g) = 0$ for all $g \in G$. Quasi-invariant measures exist on $B(G/K)$ and any two quasi-invariant measures are mutually absolutely continuous.

LEMMA 4.3. *Let U be a continuous representation of G on L, let E be a system of imprimitivity for U based on $B(G/K)$ where K is a closed subgroup of G, and let ν be a quasi-invariant measure on $B(G/K)$. If $m \in M$, $a \in L$, then the measure $S \to m(E(S, a))$ is absolutely continuous with respect to ν.*

Proof. It suffices to show that the measure $\lambda(S) = m(E(S, 1))$ is absolutely continuous with respect to ν. Let μ be a right Haar measure on G and let $\psi: G \to G/K$ be the canonical surjection. Suppose that $\nu(\Lambda) = 0$ for $\Lambda \in B(G/K)$. It is then a known result that $\mu[\psi^{-1}(\Lambda)] = 0$ and it follows that $\mu[(\psi^{-1}(\Lambda))^{-1}] = 0$. Now the function $g \to \chi_\Lambda(sg^{-1})$ on G is 1 when $sg^{-1} \in \Lambda$ and is 0 otherwise. Now $s \in \Lambda g$ if and only if $\psi^{-1}(s) \subset \psi^{-1}(\Lambda g) = \psi^{-1}(\Lambda)g$ which implies that $g \in [\psi^{-1}(\Lambda)]^{-1}\psi^{-1}(s)$. In this way we see that the set on which $g \to \chi_\Lambda(sg^{-1})$ equals 1 is $\{g \in G: g \in [\psi^{-1}(\Lambda)]^{-1}\psi^{-1}(s)\}$. Now every element in $\psi^{-1}(s)$ has the form kg_1, $k \in K$, $g_1 \in G$ and every element in $\psi^{-1}(\Lambda)$ has the form $k_1 g$, $k_1 \in K$, $g \in \Lambda_0$ where Λ_0 is a subset of G. Therefore, the elements of $[\psi^{-1}(\Lambda)]^{-1}\psi^{-1}(s)$ have the form $g^{-1}k_1^{-1}kg_1$, $g \in \Lambda_0$ and we see that $[\psi^{-1}(\Lambda)]^{-1}\psi^{-1}(s) = [\psi^{-1}(\Lambda)]^{-1}g_1$. Thus, $\mu([\psi^{-1}(\Lambda)]^{-1}\psi^{-1}(s)) = \mu([\psi^{-1}(\Lambda)]^{-1}g_1) = \mu([\psi^{-1}(\Lambda)]^{-1}) = 0$, since μ is an invariant measure. Using Fubini's theorem on $G \times G/K$ we have

$$\begin{aligned} 0 &= \int \mu[(\psi^{-1}(\Lambda))^{-1}\psi^{-1}(s)]\lambda(ds) \\ &= \int [\int \chi_\Lambda(sg^{-1})\mu(dg)]\lambda(ds) \\ &= \int [\int \chi_\Lambda(sg^{-1})\lambda(ds)]\mu(dg) = \int \lambda(\Lambda g)\mu(dg). \end{aligned}$$

Since the function $g \to \lambda(\Lambda g) \geqq 0$ we have $\lambda(\Lambda g) = 0$ a.e. $[\mu]$. Let N_α, $\alpha \in A$, be a neighbourhood basis for the identity $u \in G$. We can assume that the N_α's form a net; that is, A is a directed set and $N_\alpha \subset N_\beta$ if $\alpha > \beta$. Since non-empty open sets have positive Haar measure there is a point $g_\alpha \in N_\alpha$ such that $\lambda(\Lambda g_\alpha) = 0$. We thus get a net $g_\alpha \to u$ such that $\lambda(\Lambda g_\alpha) = 0$. Since the representation U is continuous, the function $g \to m(U_{g^{-1}}E(\Lambda, 1))$ is continuous and using the imprimitivity relation, the function

$$g \to m(U_{g^{-1}}E(\Lambda, U_g 1)) = m(E(\Lambda g, 1)) = \lambda(\Lambda g)$$

is continuous. Hence $0 = \lim \lambda(\Lambda g_\alpha) = \lambda(\Lambda u) = \lambda(\Lambda)$ and we have proved that $\lambda \ll \mu$.

THEOREM 4.4. *Let U be a continuous representation of G on L and suppose that there is a system of imprimitivity E: $B(G/K) \times L \to L$ for U where K is a closed subgroup of G. Then there is a continuous representation W of K on a logic L_0 such that U is equivalent to a sub-representation of U^W and $E(S)$ is equivalent to multiplication by $\chi_{Sg^{-1}}$.*

Proof. Fix a quasi-invariant measure ν on G/K. Since by Lemma 4.3 the measure $S \to m(E(S, a))$ is absolutely continuous with respect to ν, applying the Radon–Nikodym Theorem there is a non-negative measurable function $f_{m,a}$ on G/K such that $m(E(S, a)) = \int_S f_{m,a} d\nu$. For $a \in L$ define a function F_a: $M \times G/K \to R$ given by $F_a(m, s) = f_{m,a}(s)$. Notice that the map $a \to F_a$ is injective since if $F_a = F_b$ then $f_{m,a}(s) = f_{m,b}(s)$ for all $m \in M$, $s \in G/K$ and we have

$$\begin{aligned} m(a) &= m(E(G/K, a)) \\ &= \int_{G/K} f_{m,a} d\nu \\ &= \int_{G/K} f_{m,b} d\nu \\ &= m(E(G/K, b)) \\ &= m(b). \end{aligned}$$

Define $F_a \leqq F_b$ if $F_a(m, s) \leqq F_b(m, s)$ for all m, s. Notice that $F_a \leqq F_b$ if and only if $a \leqq b$. Define $F_a' = F_{a'}$. In this way $L_0 = \{F_a: a \in L\}$ is a logic isomorphic to L. Define W_k: $L_0 \to L_0$, $k \in K$, by $W_k F_a = F_{U_k a}$. Then W is a continuous representation of K on L_0. Now the logic L_{0W} of the induced representation is the set of maps J: $G \to L_0$ such that $J(kg) = W_k J(g)$ and $g \to m(J(g))$ is Borel for every $m \in M$ and U_g^W on L_{0W} is given by $U_{g_1}^W J(g) = J(gg_1)$. Now let L_1 be the set of maps J: $G \to L_0$ which satisfy $J(g)(m, s) = f_{m, U_g a}(s)$ for some $a \in L$. If $J \in L_1$, $k \in K$, $g \in G$, we see that

$$\begin{aligned} J(kg)(m, s) &= f_{m, U_{kg}a}(s) \\ &= f_{m, U_k U_g a}(s) \\ &= W_k f_{m, U_g a}(s) \\ &= W_k J(g)(m, s). \end{aligned}$$

Also, for $J \in L_1$, since $L \cong L_0$, $m(J(g)) = m(U_g a)$ so J is continuous and hence Borel. Thus $L_1 \subseteq L_{0W}$ and it is easy to see that L_1 is a sublogic of L_{0W}. It is also clear that $U^W: L_1 \to L_1$ so $U^W|_{L_1}$ is a subrepresentation of U^W. Define $V: L \to L_1$, $a \to V_a$, by $V_a(g)(m, s) = f_{m, U_g a}(s)$. Now V is an isomorphism and we now show that $U \cong U^W|_{L_1}$ using V. Indeed,

$$\begin{aligned} V^{-1} U_{g_1}{}^W|_{L_1} V_a &= V^{-1} U_{g_1}{}^W|_{L_1} f_{m, U_g a}(s) \\ &= V^{-1} f_{m, U_{gg_1}a}(s) \\ &= V^{-1} f_{m, U_g U_{g_1} a}(s) \\ &= U_{g_1} a. \end{aligned}$$

We finally show that $(VE(S_1)V^{-1}J)(g)(m, s) = \chi_{S_g^{-1}}(s)J(g)(m, s)$. Indeed, since E is a system of imprimitivity, $U_g E(S_1, a) = E(S_1 g^{-1}, U_g a)$ and we have

$$\begin{aligned} &\int_S (VE(S_1)V^{-1}J)(g)(m, s)d\nu(s) \\ &= \int_S f_m U_g E(S_1) V^{-1} J(s) d\nu(s) \\ &= m(E(S) U_g E(S_1) V^{-1} J) \\ &= m(E(S) E(S_1 g^{-1}) U_g V^{-1} J) \\ &= m(E(S \cap S_1 g^{-1}) U_g V^{-1} J) \\ &= \int_{S \cap S_1 g^{-1}} f_{m, U_g V^{-1} J}(s) d\nu(s) \\ &= \int_{S \cap S_1 g^{-1}} V_{V^{-1}J}(g)(m, s) d\nu(s) \\ &= \int_{S \cap S_1 g^{-1}} J(g)(m, s) d\nu(s) \\ &= \int_S \chi_{S_g^{-1}}(s) J(g)(m, s) d\nu(s). \end{aligned}$$

It then follows that $(VE(S_1)V^{-1}J)(g)(m, s) = \chi_{S_1 g^{-1}}(s)J(g)(m, s)$ a.e. $[\nu]$ for every $g \in G$, $m \in M$.

We say that a group G is *discrete relative to a subgroup* K if there are only countably many right K cosets.

THEOREM 4.5. *With the same hypotheses as Theorem* 4.4, *suppose that G is discrete relative to K. Then there is a continuous representation W of K on a* *logic L_0 such that U is equivalent to U^W and $E(S)$ is equivalent to $(\chi_S \circ \psi) \wedge (\cdot)$; that is, there exists an isomorphism $V: L \to L_{0W}$ such that $VU_gV^{-1} = U_g^W$ for all $g \in G$ and $(VE(S)V^{-1}f)(g) = (\chi_S \circ \psi)(g) \wedge f(g)$ for all $f \in L_{0W}$, $g \in G$.*

Proof. We use the notation established in Theorem 4.4. Let L_0 be the set of functions $u: M \to R$ which have the form $u(m) = f_{m,a}(K)$ for some fixed $a \in L$. Notice that a need not be unique; in fact, $f_{m,a}(K) = f_{m,b}(K)$ for all $m \in M$ if and only if $E(K)a = E(K)b$. Thus we have $u(m) = f_{m,a}(K) = f_{m,E(K)a}(K)$. Define $u_1 \leqq u_2$ if $u_1(m) \leqq u_2(m)$ for all $m \in M$. If $u(m) = f_{m,a}(K)$ define $u'(m) = f_{m,(E(K)a)'}(K)$. It can then be shown that L_0 is a logic. Define $W: K \to \text{aut}(L_0)$ by $W_k u(m) = f_{m,U_k a}(K)$ for all $k \in K$. This gives a continuous representation of K on L_0. Let L_2 be the set of maps $H: G \to L_0$ such that $H(g)(m) = f_{m,U_g a}(K)$ for some $a \in L$. Let $V: L \to L_2$ be defined by $V_a = f_{m,U_g a}(K)$. Now V is injective. Suppose $f_{m,U_g a}(K) = f_{m,U_g b}(K)$ for all $g \in G$, $m \in M$. Then $m(E(K)U_g a) = m(E(K)U_g b)$ which implies that $m(U_g E(Kg)a) = m(U_g E(Kg)b)$ for all $g \in G$, $m \in M$. Thus $U_g E(Kg)a = U_g E(Kg)b$ so $E(Kg)a = E(Kg)b$ for all $g \in G$. Now if Kg_i are the right K cosets, $i = 1, 2, \ldots$, then we have $a = \bigvee E(Kg_i)a = \bigvee E(Kg_i)b = b$. Thus V is an isomorphism. As in the previous theorem, L_2 is a sublogic of L_{0W} and $VU_gV^{-1} = U_g^W|_{L_2}$ for all $g \in G$. Now if $H \in L_2$ then $H(g)(m) = f_{m,U_g a}(K) = J(g)(m, K)$ where $J \in L_1$ and L_1 is defined as in Theorem 4.4. As in Theorem 4.4 we have

$$
\begin{aligned}
(VE(S)V^{-1}H)(g)(m) &= (VE(S)V^{-1}J)(g)(m, K) \\
&= \chi_{S_{g^{-1}}}(K)J(g)(m, K) \\
&= \chi_{S_{g^{-1}}}(K)H(g)(m) \\
&= \chi_S(Kg)H(g)(m) \\
&= [(\chi_S \circ \psi) \wedge H](g)(m).
\end{aligned}
$$

We now show that $L_2 = L_{0W}$. Suppose that $J \in L_{0W}$ and suppose that $J(g_1) = f_{m,a}(K)$. Let $H \in L_3$ be defined as $H(g) = f_{m,U_g a}(K)$. Then if we define $H_0(g) = U_{g_1^{-1}}{}^W H(g)$ we have

$$H_0(g) = H(gg_1^{-1}) = f_{m,U_{gg_1^{-1}}}(K) = f_{m,U_g U_{g_1^{-1}} a}(K) \in L_2$$

and $H_0(g_1) = f_{m,a}(K) = J(g_1)$. Now for every $k \in K$, $H_0(kg_1) = W_k H_0(g_1) = W_k J(g_1) = J(kg_1)$. Thus $H = J$ on the coset Kg_1. Now from the above $H_1 = (\chi_{Kg_1} \circ \psi) \wedge H_0 \in L_2$ and $H_1 = J$ on the coset Kg_1. Similarly, if $g_2 \notin Kg_1$ there is an $H_2 \in L_2$ such that $H_2 = J$ on Kg_2 and

$$H_2 = (\chi_{Kg_2} \circ \psi) \wedge H_2.$$

Continuing, we get $H_3, H_4, \ldots$ with the above properties. Now $H_i \perp H_j$ for $i \neq j$ and hence $\bigvee H_i \in L_2$ and $\bigvee H_i = J$.

This last theorem includes the finite groups as an important special case. For example, let us consider the representation U of $G = \{u, g_1, g_2, g_3\}$ at the beginning of this section. Since U had a system of imprimitivity based on $B(G/K)$ we know that U is equivalent to U^W for some representation W of K on L_0 and that $E(S)$ is equivalent to $(\chi_S \circ \psi) \wedge (\cdot)$. One can check that $L_0 = \{0, 1, a_1, a_1'\}$ and that $W_\beta = I$, $W_{g_1}a = a'$. We thus have that the L of that example is equivalent to the logic of maps $f: G \to L_0$ such that $f(kg) = W_k f(g)$ for all $k \in K$, $g \in G$, that U_{g_1} is equivalent to $f(g) \to f(gg_2)$, and that $E(S)$ is equivalent to $f(g) \to (\chi_S \circ \psi)(g) \wedge f(g)$.

5. Remarks. In the theory of general quantum mechanics one assumes that the system of experimental propositions forms a logic L with a full set of states M. The *symmetries* of the physical system are given by a locally compact group G. The effect of this group on the experimental propositions is given by a continuous representation U of G as automorphisms on L. If the system is *localizable* (for example, an "elementary particle") then there is a closed subgroup K of G and a system of imprimitivity E for U based on $B(G/K)$. Physically, $E(\Lambda, a)$ is the new proposition that is obtained from the proposition a when the system is located in the set $\Lambda \in B(G/K)$. The imprimitivity relation $E(\Lambda g, a) = U_g^{-1}E(\Lambda, U_g a)$ is physically an *invariance principle*. It follows from Theorem 4.4 that the logic of general quantum mechanics for a localizable system is isomorphic to a set of density functions $F: M \times G/K \to \mathbf{R}$. We now show that a kind of converse holds. Now one can show that there always exists a quasi-invariant measure ν on $B(G/K)$ such that $\nu(G/K) = 1$ if we assume that G satisfies the second axiom of countability. Let D be the set of measurable real functions f on G/K which satisfy $0 \leqq f \leqq 1$. Let M be a non-empty set and let L be a set of functions $F: M \times G/K \to \mathbf{R}$ such that $F(m, \cdot) \in D$ for all $m \in M$. Order D by defining $F_1 \leqq F_2$ if $F_1(m, s) \leqq F_2(m, s)$ for all $m \in M$, $s \in G/K$. We now make the following assumptions.

(1) If $F \in L$ then $F' = 1 - F \in L$.
(2) $0(m, s) \equiv 0$ is in L.
(3) If $F_i \in L$ and $\sum_{i=1}^{\infty} F_i(m, \cdot) \in D$ for all $m \in M$, then $\sum F_i \in L$ and $\sum F_i = \bigvee F_i$.
(4) If $\int_{\mathbf{R}} F(m, s)\nu(ds) \leqq \int_{\mathbf{R}} H(m, s)\nu(ds)$, $F, H \in L$ for every $m \in M$, then $F \leqq H$.

The following theorem is easily proved.

THEOREM 5.1. *If the above assumptions hold, then L is a logic with a full set of states M.*

Let us consider ordinary continuous unitary representations of G. If we let $K = \{u\}$ and assume that W is the irreducible representation of K then the induced representation is the *regular representation* of G. The Hilbert space becomes $L_2(G, \mu)$ where μ is Haar measure and $U_{g_1}{}^W f(g) = f(gg_1)$. Suppose now that we have a Borel representation U of G on a logic L_W which is induced by the irreducible representation W of $K = \{u\}$ on L. We call this the *regular Borel representation* of G. Now since W is irreducible we must have $L = \{0, 1\}$. Thus L_W is the set of Borel functions $f: G \to \{0, 1\}$. We thus see that L_W is just the set of all Borel subsets $B(G)$ of G and since

$$U_{g_1}{}^W \chi_S(g) = \chi_S(gg_1) = \chi_{Sg_1^{-1}}(g)$$

we see that the representation can be written $U_g S = Sg^{-1}$. Since G acts transitively on G, the regular representation is irreducible.

Let us generalize the regular representation a little. Let Ω be a topological space and suppose that G acts on Ω as a continuous transformation group. Furthermore, let us assume that there is an invariant measure μ on Ω. Let $B(\Omega)$ be the Boolean σ-algebra of Borel sets with sets which differ by a set of μ measure zero identified. Then $U_g S = gS$, $g \in G$, $S \in B(\Omega)$ gives a Borel representation of G on $B(\Omega)$. Now U is irreducible if and only if for $S \in B(\Omega)$, $U_g S = S$ for all $g \in G$ implies $S = \phi$ or Ω. In ergodic theory, the action of such a transformation group G is called *ergodic* or *metrically transitive*. In ergodic theory one frequently forms the Hilbert space $L_2(\Omega, \mu)$ and studies the unitary representation $U_g f(s) = f(g^{-1}s)$ on this Hilbert space. However it seems more natural to study the more general representation $U_g S = gS$ of U on $B(\Omega)$.

Let us now consider the case in which G is a compact group and U is a continuous representation of G on L. If μ is a Haar measure on G then $\mu(G) < \infty$ and we may assume that $\mu(G) = 1$. If $m \in M$ define $\hat{m}(a) = \int_G m(U_g a)\mu(dg)$, $a \in L$. Now $\hat{m}(1) = 1$ and if a_i are mutually disjoint then

$$\hat{m}(\vee a_i) = \int m(U_g \vee a_i)\mu(dg) = \int m(\vee U_g a_i)\mu(dg) = \int \sum m(U_g a_i)\mu(dg) = \sum \int m(W_g a_i)\mu(dg) = \sum \hat{m}(a_i)$$

so $\hat{m}$ is a state on L. Also, we see that $\hat{m}$ is an invariant state since using the invariance of μ we obtain:

$$\begin{aligned}\hat{m}(U_{g_1}a) &= \int_G m(U_g U_{g_1} a)\mu(dg) \\ &= \int m(U_{gg_1} a)\mu(dg) \\ &= \int m(U_g a)\mu(dg) = \hat{m}(a).\end{aligned}$$

We thus see that any state m generates an invariant state $\hat{m}$. Now suppose that U is a weakly irreducible representation of G on L and that m is an invariant state. Let $L_0 = \{a \in L: m(a) = 0 \text{ or } 1\}$. Then L_0 is a sublogic of L and $U_g L_0 \subseteq L_0$ for all $g \in G$. Hence L_0 is L or $\{0, 1\}$. Thus m is either *dispersion free* (that is, has only the values 0 and 1) or $m(a) > 0$ for $a \neq 0$. Notice that if there was an $\epsilon > 0$ such that $m(a) > \epsilon$ for all $a \in L$ then every set of mutually disjoint elements of L would be finite. This might lead one to conjecture that if G is compact and if U is a weakly irreducible representation of G on L then L is finite dimensional. This would correspond to a kind of weak lattice theoretic Peter–Weyl Theorem.

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