Sums of roots

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Introduction

We are all familiar with the formulae for finding the sums of the powers of the first n natural numbers. For example,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \text{ and } 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

What about sums like

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \text{ or } \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n}?$$

Are there relatively simple formulae for those types of sums?

In this Article, we find an (almost) elementary way to approximate sums such as these using series techniques from calculus.

The technique

Let us first consider the sum

$$S(n) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}. \qquad (1)$$

There is no tidy summing approach like the one Gauss used when he was a boy to find the sum of the first 100 numbers. To find the sum

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$

he paired the numbers 1 and 100, 2 and 99, 3 and 98, etc. to get 50 pairs of 101, which yielded a sum of 5050 [1]. Further, the sum in (1) cannot be obtained by the inductive, telescoping approach that one uses to find the sums of the natural numbers raised to integral powers 2 and higher [1].

To begin finding a formula for $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$, we can gain some initial insight from the integral $\int_0^n \sqrt{x}\, dx$. Why the integral? Because the corresponding Riemann sum, with a dx length of 1, is precisely the sum $S(n)$. Since the value of the integral is $\frac{2}{3}n^{3/2}$, we get

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \sim \tfrac{2}{3}n^{3/2}.$$

The symbol $\sim$ means 'is asymptotic to' in the sense that the ratio of $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$ divided by $\frac{2}{3}n^{3/2}$ approaches 1 as $n \to \infty$. To see that we are on the right track, let us try some values of n:

n	$S(n)$	$2n^{3/2}/3$	$S(n)/\left(2n^{3/2}/3\right)$
10	22.4683	21.0819	1.06576
100	671.463	666.667	1.00719
1000	21 097.5	21 081.9	1.00074
10000	666 716	666 667	1.00007

TABLE 1

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There is clearly some promise in our initial insight to connect the sum with the integral. However, the difference of the actual sums from their integral approximations grows as n increases, and there is a persistent error term in the ratio of around $7/10n$. These observations suggest that we create an asymptotic series $A(n)$ which incorporates $\frac{2}{3}n^{3/2}$, along with some correction terms, to approximate the sum $S(n)$. Formally, we define $A(n)$ so that it satisfies

$$S(n) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \sim A(n),$$

with $A(n)$ given by

$$A(n) = \sqrt{n}\left(\tfrac{2}{3}n + a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \ldots\right) + z. \qquad (2)$$

The first term arises from the integral $\left(\frac{2}{3}n^{3/2}\right)$, the coefficients a_k are yet to be determined, and the reason for naming the last term z will be revealed towards the end of the analysis.

Finding the coefficients

We start with the ready observation that

$$S(n) - S(n-1) = \left(1 + 2 + \ldots + \sqrt{n-1} + \sqrt{n}\right) - \left(1 + 2 + \ldots + \sqrt{n-1}\right)$$
$$= \sqrt{n},$$

which translates to

$$A(n) - A(n-1) \sim \sqrt{n}. \qquad (3)$$

We will use this equation, along with (2), to determine the first few coefficients of $A(n)$. This will then give accurate answers for the sum of $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}$.

The first complication in computing $A(n) - A(n-1)$ is that $A(n)$ is a series in powers of n whereas

$$A(n-1) = \sqrt{n-1}\left(\tfrac{2}{3}(n-1) + a_0 + \frac{a_1}{n-1} + \frac{a_2}{(n-1)^2} + \frac{a_3}{(n-1)^3} + \ldots\right) + z$$

is a series in powers of $n - 1$, so the terms do not automatically match up. To circumvent this, we expand each of the terms in $A(n-1)$ in terms of powers of n, considering n to be large. As examples, for large n, the Taylor series for $\sqrt{n-1}$ is $\sqrt{n}\sqrt{1 - \frac{1}{n}} \sim \sqrt{n}\left(1 - \frac{1}{2n} - \frac{1}{8n^2} - \frac{1}{16n^3}\ldots\right)$. Also, $\frac{1}{n-1} = \frac{1}{n}\left(\frac{1}{1 - \frac{1}{n}}\right) \sim \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}\ldots$, and $\frac{1}{(n-1)^2} \sim \frac{1}{n^2} + \frac{2}{n^3} + \frac{3}{n^4}\ldots$. The higher order terms of $A(n-1)$ are found similarly via Taylor expansions.

Using these series, the equation for the difference $A(n) - A(n-1)$ becomes

$$A(n) - A(n-1) \sim \sqrt{n} \times$$
$$\left(1 + \frac{2a_0 - 1}{4} \times \frac{1}{n} + \frac{3a_0 - 1 - 12a_1}{24} \times \frac{1}{n^2} + \frac{4a_0 - 1 - 24a_1 - 96a_2}{64} \times \frac{1}{n^3}\ldots\right). \qquad (4)$$

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The key idea is that $A(n) - A(n-1) \sim \sqrt{n}$, so that each coefficient of the powers, $\frac{1}{n}, \frac{1}{n^2}$, etc. must be equal to zero. Hence $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{24}$ and $a_2 = 0$. For clarity of exposition, the higher order terms $\left(\frac{1}{n^4}, \frac{1}{n^5}, \ldots\right)$ were not shown in (4). However, when the coefficients of those terms are set equal to zero, we get $a_3 = \frac{-1}{1920}$, $a_4 = 0$, $a_5 = \frac{1}{9216}$. By way of summary, at this stage, we have the following expression for $A(n)$:

$$A(n) \sim \sqrt{n}\left(\frac{2}{3}n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{1920n^3} + \frac{1}{9216n^5} + \ldots\right) + z. \qquad (5)$$

Higher order terms can be calculated, and those calculations are best facilitated by a program like *Mathematica* [2]. (For novelty, this author used it to calculate the coefficients up to a_{19}.) Nevertheless, the higher order terms of $A(n)$ are not of much practical value in approximating $S(n)$ since the error in $A(n)$ after using just the first two terms is of order $1/\sqrt{n}$, and the error is of order $1/n^{5/2}$ when using the first three terms – quite accurate! Of interest is that the result in (5) can be obtained by using the Euler-Maclaurin sum formula, a more advanced mathematical technique outside the scope of this paper [3].

Two side facts are illuminating:

(1) All of the coefficients of the even powers $\left(\frac{1}{n^2}, \frac{1}{n^4}, \frac{1}{n^6}, \ldots\right)$ are zero. It is not immediately obvious why this is the case, but a close look at the pattern in which the higher order terms of the Taylor series combine reveal this to be true. This result can also be seen by using results from the Euler-Maclaurin sum formula.

(2) The first two terms of $A(n)$ in (5) are the terms found when approximating the integral $\int_0^n \sqrt{x}\,dx$ by using the composite trapezoidal rule with a unit step in x; from this $\int_0^n \sqrt{x}\,dx$ is approximately $\sqrt{n}\left(\frac{2}{3}n + \frac{1}{2}\right)$ [4]. In essence, the other terms in (5) are the elusive error terms of the trapezoidal rule for large n.

Finding z

What is interesting at this stage is that we have a procedure for determining the coefficients a_k of $A(n)$, but that procedure cannot find the value of z. This is because the zs subtract out when taking the difference $A(n) - A(n-1)$. So how to find z? The following table shows the first few terms of $A(n)$ (without z) for $n = 10$; it gives us a good sense of what the numerical value of z might be.

Terms of $A(n)$	3	4	5
$A(10)$	22.6761660548	22.6761644078	22.6761644112
$z \approx S(10) - A(10)$	-0.2078878687	-0.2078862216	-0.2078862250

TABLE 2 $n = 10$ $S(10) = 22.4682781862$

Of note is that the value of z in $A(n)$ is the same no matter what n is used; Table 2 indicates that the value of 2 is around -0.207886225.

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The main point of the Article is to use elementary techniques to get accurate approximations for sums like $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}$. So far, that has been the case. However, to get the exact value of z in $A(n)$, we must now take a slight detour and insert results from more advanced mathematical tools. As it turns out,

$$z = \zeta\left(-\tfrac{1}{2}\right),$$

in which ζ is the Riemann zeta function. The details are deferred to the Appendix, in keeping with the elementary flavour of this paper. To twenty decimal places, $\zeta\left(-\tfrac{1}{2}\right) = -0.20788622497735456602$, which is in line with the values in Table 2 above.

By way of summary, we have

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \sim \sqrt{n}\left(\frac{2}{3}n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{1920n^3} + \frac{1}{9216n^5} + \dots\right) + \zeta\left(-\frac{1}{2}\right). \quad (6)$$

Table 3 illustrates how accurate this formula is even when using just the first few terms. The digits in bold are the first digits in the approximations that differ from the exact sum. Interestingly, the formula was derived under the assumption that n was 'large', but we still get good results even for $n = 10$, which is not that large.

n	$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}$	$\sqrt{n}\left(\frac{2}{3}n + \frac{1}{2} + \frac{1}{24n}\right) + z$	$\sqrt{n}\left(\frac{2}{3}n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{1920n^3}\right) + z$
10	22.468278186204100	22.468279**829813398**	22.46827818**2793784**
100	671.462947103147753	671.462947108**355978**	671.462947103147**645**
1000	21097.455887480735355385	21097.455887480**751825**	21097.45588748073535538**1**

TABLE 3

Generalisations

The techniques in the prior sections can be readily mimicked to find asymptotic expressions for sums of other fractional powers. For example,

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \dots + \sqrt[3]{n} \sim \sqrt[3]{n}\left(\frac{3n}{4} + \frac{1}{2} + \frac{1}{36n} - \frac{1}{1944n^3}\dots\right) + \zeta\left(-\frac{1}{3}\right). \quad (7)$$

As in the approach above, the first term, $\frac{3}{4}n^{4/3}$ is $\int_0^n \sqrt[3]{x}\, dx$. As in the square root case, this technique cannot find the last term, $\zeta\left(-\tfrac{1}{3}\right)$, but the results shown in the Appendix produce this term. (For the interested reader, $\zeta\left(-\tfrac{1}{3}\right) = -0.2773400478\dots$.)

This approach can be generalised to find the sums of other fractional powers rather than just the square root and cube root cases analysed so far. To outline the method, we first expand the sum:

$$1^s + 2^s + \dots + n^s \sim n^s\left(\frac{n}{s+1} + \frac{1}{2} + \frac{a_1}{n} + \frac{a_3}{n^3} + \dots\right) + z.$$

Similarly,

$$1^s + \dots + (n-1)^s \sim (n-1)^s\left(\frac{n-1}{s+1} + \frac{1}{2} + \frac{a_1}{n-1} + \frac{a_3}{(n-1)^3} + \dots\right) + z.$$

After this latter series is expanded in terms of powers of $1/n$, the difference of the two series yields

$$n^s \sim n^s\left(1 - \frac{(s-1)(s-12a_1)}{12n^2} + \dots\right).$$

From this we see that $a_1 = \frac{s}{12}$. Determining the higher order terms is algebraically more involved, but the net result is

$$1^s + 2^s + \dots + n^s \sim$$

$$n^s\left(\frac{n}{s+1} + \frac{1}{2} + \frac{s}{12n} - \frac{s(s-1)(s-2)}{720n^3} + \frac{s(s-1)(s-2)(s-3)(s-4)}{30240n^5}\right) + z. \quad (8)$$

Again, $z = \zeta(-s)$ follows from the results found in the Appendix. Interestingly, the terms up to $1/n$ were found by G. H. Hardy by using analytic number theory techniques [5].

Conclusion

From the earliest days of school, finding sums has been an important activity for mathematics students. Naturally, the types of sums a student is able to compute become more advanced with time. Arithmetic sums, geometric sums, and sums of powers are often a stopping point for students who do not get as far as calculus.

By using calculus, the types of sums that can be found is considerably broadened. This paper shows how Taylor series can be used to find sums of natural numbers raised to fractional powers. The technique is widely accessible to a large mathematical audience and illustrates a novel way of using common tools to extend results to sums that are, on the face of it, difficult to evaluate.

Appendix

The Riemann zeta function, $\zeta(s)$ is defined by,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and it is convergent for $s > 1$. The definition can be extended analytically to complex values of s. There are two relevant results that pertain directly to the work of this Article. By using the Euler-Maclaurin sum formula [3] and techniques from analytic number theory, G .H. Hardy [5] showed that

$$\zeta(s) = \lim_{n\to\infty}\left\{\sum_{m=1}^{\infty} \frac{1}{m^s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}n^{-s} + \frac{1}{12}sn^{-s-1}\right\}.$$

This expression is valid when $\mathrm{Re}(s) > -3$, except for $s = 1$. where $\mathrm{Re}(s)$

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denotes the real part of s. A related result by Srivastava and Choi is

$$\zeta(s) = \lim_{n\to\infty}\left\{\sum_{m=1}^{\infty}\frac{1}{m^s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}n^{-s} + \frac{1}{12}sn^{-s-1} - \frac{1}{720}s(s+1)(s+2)n^{-s-3}\right\}.$$

This result is valid for $\text{Re}(s) > -5$ also, except for $s = 1$, [6].

The sums of roots formula, developed in this Article are naturally of the form $1^s + 2^s + \dots + n^s$, not in the form $\frac{1}{1^s} + \frac{1}{2^s} + \dots + \frac{1}{n^s}$ as specified in the results of Hardy and Srivastava and Choi. However, if $-s$ is substituted for s into either of their results, then we get

$$1^s + 2^s + \dots + n^s \sim n^s\left(\frac{n}{s+1} + \frac{1}{2} + \frac{s}{12n} - \frac{s(s-1)(s-2)}{720n^3} + \dots\right) + \zeta(-s),$$

in accord with (8).

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