§ 7. Relations among Radii.

*The sum of the radii of the three excircles diminished by the radius of the incircle is double the diameter of the circumcircle.**

FIGURE 64.

Let O be the circumcentre, ID, I_1D_1, I_2D_2, I_3D_3 the radii of the incircle and the three excircles perpendicular to BC.

From O draw OA′ perpendicular to BC, and let OA′ meet the circumcircle below BC at U and above it at U′.

Because OA′ is perpendicular to BC,

therefore A′ is the mid point of BC, and U the mid point of arc BUC.

But since AI_1 bisects ∠ BAC,

therefore it bisects arc BUC, that is, AI_1 passes through U.

Since UU′ is a diameter, and ∠ UAI_3 is right,

therefore I_2I_3 passes through U′.

Now because ID, UA′, I_1D_1 are parallel, and $DA' = D_1A'$,

therefore $$2UA' = I_1D_1 - ID;$$

and because I_2D_2, U′A′, I_3D_3 are parallel, and $D_2A' = D_3A'$,

therefore $$2U'A' = I_2D_2 + I_3D_3.$$

Hence $$2(UA' + U'A') = I_1D_1 + I_2D_2 + I_3D_3 - ID;$$

that is $$4R = r_1 + r_2 + r_3 - r.$$

(1) *The sum of the distances of the circumcentre from the sides of a triangle is equal to the sum of the radii of the incircle and the circumcircle; and the sum of the distances of the orthocentre from the vertices is equal to the sum of the diameters of the incircle and the circumcircle.*

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* Feuerbach in his *Eigenschaften...des...Dreiecks*, § 5 (1822), proves it algebraically. The proof in the text is that of T. S. Davies in the *Ladies' Diary* for 1835, p. 55.

For $$OA' = OU - A'U$$
$$= R - \tfrac{1}{2}(r_1 - r)\ .$$

Similarly $$OB' = R - \tfrac{1}{2}(r_2 - r)\ ;$$
$$OC' = R - \tfrac{1}{2}(r_3 - r)\ ;$$

therefore $$OA' + OB' + OC' = 3R - \tfrac{1}{2}(r_1 + r_2 + r_3 - 3r)$$
$$= 3R - \tfrac{1}{2}(4R + r - 3r)$$
$$= 3R - (2R - r)$$
$$= R + r.$$

Again $$HA + HB + HC = 2OA' + 2OB' + 2OC'\ ,$$
$$= 2(R + r).$$

If one of the angles of the triangle be obtuse, the circumcentre will fall outside the triangle, and its distance from the side opposite the obtuse angle must then be considered negative. Also if the circumcentre fall outside the triangle, so will the orthocentre. In that case the distance of the orthocentre from the vertex of the obtuse angle must be considered negative.

These two properties, as well as the remarks at the end of the proof, are given by Carnot in his *Géométrie de Position*, § 137 (1803).

The following is Carnot's mode of proving the first property.

FIGURE 65.

The quadrilaterals AB′OC′, BA′OC′, CB′OA′ are inscriptible in circles; therefore

$$AO \cdot B'C' = AB' \cdot C'O + AC' \cdot B'O$$
$$BO \cdot C'A' = BC' \cdot A'O + BA' \cdot C'O$$
$$CO \cdot A'B' = CA' \cdot B'O + CB' \cdot A'O\ .$$

Adding these equations, and noting that $AO = BO = CO = R$, that $B'C' + C'A' + A'B' = s$, and that $AB' = CB'$, $AC' = BC'$, $BA = CA'$, we have

$$sR = s(A'O + B'O + C'O) - \tfrac{1}{2}(A'O \cdot BC + B'O \cdot CA + C'O \cdot AB)$$
$$= s(A'O + B'O + C'O) - \Delta$$
$$= s(A'O + B'O + C'O) - sr\ ;$$

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therefore $$R = A'O + B'O + C'O - r,$$

or $$R + r = A'O + B'O + C'O.$$

(2) *The relation* $4R = r_1 + r_2 + r_3 - r$ *has been employed to establish* $$R + r = OA' + OB' + OC' ;$$ *but the method of procedure may be reversed.*

FIGURE 64.

For $OA' = OU - A'U = R - \frac{1}{2}(r_1 - r).$

Similarly $OB' = R - \frac{1}{2}(r_2 - r)$

$OC' = R - \frac{1}{2}(r_3 - r) ;$

therefore $OA' + OB' + OC' = 3R - \frac{1}{2}(r_1 + r_2 + r_3 - 3r)$

that is $R + r = 3R - \frac{1}{2}(r_1 + r_2 + r_3 - 3r) ;$

whence $4R = r_1 + r_2 + r_3 - r.$

(3) $$A'U + B'V + C'W = 2R - r$$

*and** $$A'U' + B'V' + C'W' = 4R + r.$$

These results follow from subtracting $A'O + B'O + C'O$ from $3R$, and adding $A'O + B'O + C'O$ to $3R$.

For another proof of the first of them see *Mathematical Questions from the Educational Times*, XVII. 47 (1872).

(4) *The following relations subsist between the distances of the circumcentre from the sides of a triangle and the radii of the circumcircle and the excircles :* †

$$-OA' + OB' + OC' = -R + r_1$$
$$OA' - OB' + OC' = -R + r_2$$
$$OA' + OB' - OC' = -R + r_3.$$

FIGURE 66.

From U draw UT perpendicular to AB,

and from O draw OR perpendicular to UT.

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* Hind's *Trigonometry*, 4th ed., p. 309 (1841).

† Mr Bernh. Möllmann in Grunert's *Archiv*, XVII. 379 (1851).

It may be proved that $AT = \frac{1}{2}(AB + AC)$

and $BT = \frac{1}{2}(AB - AC)$;

therefore $AB' = \frac{1}{2}AC = \frac{1}{2}(AT - BT)$

$= C'T = OR.$

Hence the right-angled triangles AB′O, ORU are congruent

and $OB' = UR$;

therefore $OB' + OC' = UT$

$= \frac{1}{2}(IF + I_1F_1)$;

therefore $2(OB' + OC') = r + r_1.$

Now $OA' + OB' + OC' = R + r$;

therefore $-OA' + OB' + OC' = -R + r_1.$

(5) *The following is another proof* of the relation*

$$OA' + OB' + OC' = R + r.$$

FIGURE 67.

Through I the incentre draw a parallel to AC meeting OB′ in K; through K draw a parallel to IC meeting OA′ in L and BC in N. From N draw NM perpendicular to AC and meeting OA′ in M.

Since KN is parallel to the bisector of ∠ ACB,

therefore $CN = IK = EB' = CB' - CE$

$$= \frac{AC}{2} - \frac{BC + CA - AB}{2}$$

$$= \frac{AB - BC}{2};$$

therefore $A'N = A'C + CN = \frac{AB}{2} = C'B.$

Again, since MN is perpendicular to AC,

therefore $\angle MNA' = 90° - \angle ACB$

$= 90° - \angle C'OB = \angle OBC'$;

* Mr Lemoine in *Journal de Mathématiques Élémentaires*, 2nd series, IV. 217-8 (1885). (6) also is his.

therefore the right-angled triangles MNA′, OBC′ are congruent,

and $$A'M = OC', \quad MN = OB = R.$$

Since
$$\begin{aligned}\angle MNL &= \angle MNA' + \angle A'NL \\ &= 90^\circ - C + \frac{C}{2} \\ &= 90^\circ - \frac{C}{2} \\ &= \angle MLN;\end{aligned}$$

therefore triangle MLN is isosceles;

therefore „ OLK „ „ .

Hence
$$\begin{aligned}OA' + OB' + OC' &= OA' + OK + KB' + A'M \\ &= OA' + OL + IE + A'M \\ &= LM + IE \\ &= MN + IE \\ &= R + r.\end{aligned}$$

(6) If r_1 denote the radius of the first excircle, it may be shown by an analogous proof that

$$OB' + OC' - OA' = r_1 - R.$$

Hence the theorem:

If in a triangle the radius of an excircle be equal to the radius of the circumcircle, one of the three distances of the orthocentre from the vertices is equal to the sum of the other two, and conversely.

(7) *If D, D' be the projections of A' on OB, OC,*
E, E' „ „ „ „ B' „ OC, OA,
F, F' „ „ „ „ C' „ OA, OB

*then**
$$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{R+r}{R}.$$

FIGURE 68.

Since triangle OBC is isosceles, DD′ is parallel to BC,

and $$\frac{DD'}{a} = \frac{OD}{R}.$$

* Mr J. Someritis of Chalcis in Vuibert's *Journal de Mathématiques Élémentaires*, XVI. 128 (1892). The solution in the text is that given on p. 141.

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From the right-angled triangles OA′B, ODA′

$$OD = \frac{OA'^2}{R};$$

therefore $\frac{DD'}{a} = \frac{OA'^2}{R^2}$ and $\sqrt{\frac{DD'}{a}} = \frac{OA'}{R}$.

Similarly $\sqrt{\frac{EE'}{b}} = \frac{OB'}{R}$ and $\sqrt{\frac{FF'}{c}} = \frac{OC'}{R}$;

therefore $$\sqrt{\frac{DD'}{a}} + \sqrt{\frac{EE'}{b}} + \sqrt{\frac{FF'}{c}} = \frac{OA' + OB' + OC'}{R}$$

$$= \frac{R + r}{R}.$$

(8) *The potency (or power) of the incentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the incircle and the circumcircle.**

Figure 69.

In ABC let O be the circumcentre, I the incentre.

Join OI.

Through O draw U′U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I.

Join CU, CU′, CI, AO, and from I draw IE the radius of the incircle perpendicular to AC.

Then $\angle UIC = \angle IAC + \angle ICA,$

$= \frac{1}{2}(A + C);$

and $\angle UCI = \angle BCI + \angle BCU,$

$= \angle BCI + \angle BAU,$

$= \frac{1}{2}(A + C);$

therefore $CU = IU.$

* William Chapple in *Miscellanea Curiosa Mathematica*, I. 123 (1746). Euler gave the property in an inconvenient form about twenty years later. A tolerably full history of Chapple's theorem and its developments during the 18th century will be found in the *Proceedings of the Edinburgh Mathematical Society*, V. 62-78 (1887).

Again, the right-angled triangles AEI, U'CU are similar;

therefore $AI : IE = U'U : UC$;

therefore $AI \cdot UC = U'U \cdot IE$,

that is $AI \cdot IU = 2Rr$.

Lastly from the isosceles triangle OAU

$$OA^2 - OI^2 = AI \cdot IU,$$

that is $R^2 - OI^2 = 2Rr$.

(9) If OI be denoted by d, then

$$R^2 - d^2 = 2Rr,$$

or
$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

(10) *The potency (or power) of an excentre of a triangle with respect to the circumcircle is equal to twice the rectangle under the radii of the excircle and the circumcircle.**

FIGURE 70.

In ABC let O be the circumcentre, I_1 an excentre.

Join OI_1.

Through O draw U'U the diameter of the circumcircle perpendicular to BC. Then U is the mid point of the arc BUC, and AU will pass through I_1.

Join CU, CU', CI_1, AO, and from I_1 draw I_1E_1 the radius of the excircle perpendicular to AC.

Then $\angle UI_1C = 180° - (\angle I_1AC + \angle I_1CA)$,

$= 90° - \frac{1}{2}(A+C)$;

and $\angle UCI_1 = \angle BCI_1 - \angle BCU$,

$= \angle BCI_1 - \angle BAU$,

$= 90° - \frac{1}{2}(A+C)$;

therefore $CU = I_1U$.

Again, the right-angled triangles AE_1I_1, U'CU are similar;

therefore $AI_1 : I_1E_1 = U'U : UC$;

therefore $AI_1 \cdot UC = U'U \cdot I_1E_1$,

that is $AI_1 \cdot I_1U = 2Rr_1$.

* John Landen in *Lucubrationes Mathematicae*, pp. 1-6 (1755).

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Lastly from the isosceles triangle OAU,

$$OI_1^2 - OA^2 = AI_1 \cdot I_1U,$$

that is $\quad OI_1^2 - R^2 = 2Rr_1.$

Hence also $\quad OI_2^2 - R^2 = 2Rr_2,$

and $\quad OI_3^2 - R^2 = 2Rr_3.$

(11) If OI_1, OI_2, OI_3 be denoted by d_1, d_2, d_3 then

$$d_1^2 - R^2 = 2Rr_1, \quad d_2^2 - R^2 = 2Rr_2, \quad d_3^2 - R^2 = 2Rr_3;$$

or

$$\frac{1}{R+d_1} + \frac{1}{R-d_1} = -\frac{1}{r_1}$$

$$\frac{1}{R+d_2} + \frac{1}{R-d_2} = -\frac{1}{r_2}$$

$$\frac{1}{R+d_3} + \frac{1}{R-d_3} = -\frac{1}{r_3}.$$

(12) The potency of I with respect to the circumcircle is *

$$\frac{abc}{a+b+c}.$$

For

$$2Rr = \frac{abc}{2\Delta} \cdot \frac{2\Delta}{a+b+c}.$$

(13) The potency of I_1 with respect to the circumcircle is

$$\frac{abc}{-a+b+c}.$$

For

$$2Rr_1 = \frac{abc}{2\Delta} \cdot \frac{2\Delta}{-a+b+c}.$$

(14) If the first excircle cut the circumcircle at S, and I_1S be produced to intersect the circumcircle at T, then $I_1T = 2R$.

For $I_1S \cdot I_1T$ = potency of I_1 with respect to circumcircle,

$$= OI_1^2 - R^2 = 2Rr_1;$$

and $\quad I_1S = r_1.$

---

* C. J. Matthes, *Commentatio de Proprietatibus Quinque Circulorum*, p. 41 (1831).

(15) If IO be produced to meet the circumcircle in M, N, and the incircle in P, Q (the order of the letters is MPOIQN), then *

$$MP \cdot NQ = r^2$$
$$MQ \cdot NP = 4Rr + r^2.$$

For $MP = (R - r) + OI,\ NQ = (R - r) - OI$

and $MQ = (R + r) + OI,\ NP = (R + r) - OI.$

(16) If I_1O be produced to meet the circumcircle in M, N, and the first excircle in P, Q (the order of the letters is $MOQNI_1P$), then

$$MP \cdot NQ = r_1^2$$
$$MQ \cdot NP = 4Rr_1 - r_1^2.$$

(17) $$IM \cdot IN = 2Rr$$
$$OP \cdot OQ = -R^2 + 2Rr + r^2.$$

(18) $$I_1M \cdot I_1N = 2Rr_1$$
$$OP \cdot OQ = R^2 + 2Rr_1 - r_1^2.$$

(19) The product of the potencies † of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r^2(4R + r).$$

The product of the potencies of M and N with respect to the incircle

$$MP \cdot MQ \times NP \cdot NQ = r^3(4R + r).$$

(20) The product of the potencies of P and Q with respect to the circumcircle

$$MP \cdot NP \times MQ \cdot NQ = r_1^2(4R - r_1).$$

The product of the potencies of M and N with respect to the first excircle

$$MP \cdot MQ \times NP \cdot NQ = r_1^3(4R - r_1).$$

---

* The first part is given by Mr Néorouzian in the *Nouvelles Annales*, IX. 216-7 (1850); the second part occurs in *Exercices de Géométrie*, by F.I.C., 2nd ed., p. 506 (1882).

† The first part is given in *Nouvelles Annales*, XVII. 358, 447-8 (1858), and attributed to Grunert.

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(21) The radius of the circumcircle is never less than the diameter of the incircle.*

For OI^2 is positive ;

therefore $R - 2r$ cannot be negative.

(22) When the radius of the circumcircle is equal to the diameter of the incircle, the circumcentre and the incentre coincide, and the triangle is equilateral.

(23) When the straight line joining the incentre and the circumcentre passes through one of the vertices, the triangle is isosceles.

(24) Since the value of OI^2 is independent of the sides of the triangle ABC, if two circles whose radii are R and r be so situated that the square of the distance between their centres equals $R(R - 2r)$, then any number of triangles may be drawn, each of which shall be inscribed in the larger circle, and circumscribed about the smaller†; and if the two circles be so situated that the square of the distance between their centres is not equal to $R(R - 2r)$, then no triangle can be inscribed and circumscribed.

(25) Since the value of $OI_1{}^2$ is independent of the sides of the triangle, a corresponding statement may be made regarding two circles whose radii are R and r_1.

(26) If one side of a triangle inscribed in and circumscribed about two given circles be given, the other two sides may be found.

(27) Of the innumerable triangles that may be inscribed in and circumscribed about two given circles, two will be isosceles; and the common diameter of the two circles will pass through their vertices and cut their bases at right angles. That isosceles triangle which has the least base and the greatest altitude will be the greatest, and the other isosceles triangle will be the least of all the triangles that can be inscribed and circumscribed.

* Theorems (21)-(24), (26), (27) are given by Chapple; (28) part of which is given by Chapple, is due to Dr Otto Böklen. See Grunert's *Archiv*, XXXVIII. 143 (1862).

† A detailed proof of this statement, if such should be considered necessary, is given by Dr W. H. Besant in the *Quarterly Journal of Mathematics*, XII. 276 (1873).

(28) In connection with these innumerable triangles a large number of constant magnitudes may be found. A few are here enumerated.

(*a*) The sum of the perpendiculars from the circumcentre to the sides is constant.

(*b*) The sum of the distances of the orthocentre from the vertices is constant.

(*c*) The sum of the radii of the excircles is constant.

(*d*) The sum of the reciprocals of the radii of the excircles is constant.

(*e*) The ratio of the product of the sides to the sum of the sides is constant.

(*f*) The ratio of the area to the perimeter is constant.

The proofs of these statements are

(*a*) $$OA' + OB' + OC' = R + r$$

(*b*) $$\tfrac{1}{2}(HA + HB + HC) = R + r$$

(*c*) $$r_1 + r_2 + r_3 = R + r$$

(*d*) $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

(*e*) $$\frac{abc}{a+b+c} = 2Rr$$

(*f*) $$\frac{\Delta}{s} = r$$

(29) The sum of the squares of the distances of the circumcentre from the incentre and the excentres is equal to three times the square of the diameter of the circumcircle.*

For
$$\begin{aligned}\Sigma(OI^2) &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\ &= 4R^2 + 2R \cdot 4R \\ &= 12R^2.\end{aligned}$$

* Feuerbach, *Eigenschaften...des...Dreiecks*, § 50 (1822).

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(30) The preceding theorem may be derived from the following*:

The sum of the squares of the tangents drawn from the centres of the four circles of contact of a triangle to any circle which passes through the circumcentre is equal to three times the square of the diameter of the circumcircle.

FIGURE 71.

Let Q be the centre of a circle passing through O the circumcentre.

Draw the diameter of the circumcircle UU′ perpendicular to BC and bisecting II_1 at U and I_2I_3 at U′.

Join Q with O, I, I_1, I_2, I_3, U, U′, and draw CU, CU′.

If the four tangents be denoted by t, t_1, t_2, t_3,

$$\begin{aligned} \text{then } t^2+t_1^2+t_2^2+t_3^2 &= (QI^2-QO^2)+(QI_1^2-QO^2) \\ &\quad+(QI_2^2-QO^2)+(QI_3^2-QO^2) \\ &=(QI^2+QI_1^2)+(QI_2^2+QI_3^2)-4QO^2 \\ &=2(QU^2+UI^2)+2(QU'^2+U'I_2^2)-4QO^2 \\ &=2(QU^2+UC^2)+2(QU'^2+U'C^2)-4QO^2 \\ &=2(QU^2+QU'^2)+2(UC^2+U'C^2)-4QO^2 \\ &=4(QO^2+OU^2)+\quad 2U'U^2 \quad -4QO^2 \\ &=4OU^2+8OU^2 \\ &=12R^2. \end{aligned}$$

When QO becomes zero, or the circle with centre Q vanishes to a point,

$$t^2+t_1^2+t_2^2+t_3^2=OI^2+OI_1^2+OI_2^2+OI_3^2.$$

(31) Since $-\dfrac{1}{r}+\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=0,$

therefore $-\dfrac{1}{2Rr}+\dfrac{1}{2Rr_1}+\dfrac{1}{2Rr_2}+\dfrac{1}{2Rr_3}=0;$

therefore $\dfrac{1}{d^2-R^2}+\dfrac{1}{d_1^2-R^2}+\dfrac{1}{d_2^2-R^2}+\dfrac{1}{d_3^2-R^2}=0;$

therefore $\dfrac{1}{12(R^2-d^2)}+\dfrac{1}{12(R^2-d_1^2)}+\dfrac{1}{12(R^2-d_2^2)}+\dfrac{1}{12(R^2-d_3^2)}=0.$

* Philip Beecroft in the *Lady's and Gentleman's Diary* for 1845, p. 63.

But
$$12(R^2-d^2)=d_1^2+d_2^2+d_3^2-11d^2$$
$$12(R^2-d_1^2)=d^2+d_2^2+d_3^2-11d_1^2$$
$$12(R^2-d_2^2)=d_1^2+d^2+d_3^2-11d_2^2$$
$$12(R^2-d_3^2)=d_1^2+d_2^2+d^2-11d_3^2\,;$$

hence
$$\frac{1}{d_1^2+d_2^2+d_3^2-11d^2}+\frac{1}{d^2+d_2^2+d_3^2-11d_1^2}+$$
$$\frac{1}{d_1^2+d^2+d_3^2-11d_2^2}+\frac{1}{d_1^2+d_2^2+d^2-11d_3^2}=0,$$

the equation* by which the four distances d, d_1, d_2, d_3 are connected together.

(32)
$$\left.\begin{aligned}
-\frac{1}{d^2-R^2}+\frac{1}{d_1^2-R^2}+\frac{1}{d_2^2-R^2}+\frac{1}{d_3^2-R^2}&=\frac{1}{Rr}\\
-\frac{1}{d^2-R^2}+\frac{1}{d_1^2-R^2}-\frac{1}{d_2^2-R^2}-\frac{1}{d_3^2-R^2}&=\frac{1}{Rr_1}\\
-\frac{1}{d^2-R^2}-\frac{1}{d_1^2-R^2}+\frac{1}{d_2^2-R^2}-\frac{1}{d_3^2-R^2}&=\frac{1}{Rr_2}\\
-\frac{1}{d^2-R^2}-\frac{1}{d_1^2-R^2}-\frac{1}{d_2^2-R^2}+\frac{1}{d_3^2-R^2}&=\frac{1}{Rr_3}
\end{aligned}\right\}$$

A large number of formulae expressive of the relations between r, r_1, r_2, r_3, R, and a, b, c, h_1, h_2, h_3, α, β, γ, etc., will be found in subsequent Sections.

* Mr Franz Unferdinger in Grunert's *Archiv*, XXXIII. 428 (1859).