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SOME NEW REPLACEABLE TRANSLATION NETS

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1. Introduction. We discuss partial spreads (translation nets) U, V of $\Sigma = PG(3, q)$ where U, V cover the same points of Σ and have no lines in common. Write $t = |U| = |V|$. It has been shown in a previous paper [**4**] that $t \geqq 2(q - 1)$ provided $q \geqq 4$. In this note we analyze further the case $t \leqq 2(q + 1)$. Examples of replaceable translation nets, some of them new, are given for each value of t in the range $2(q - 1) \leqq t \leqq 2(q + 1)$ and for all prime powers q. Moreover, we show that if q is sufficiently large (in particular, if $q > 19$) then, for each value of t in the above range, any pair U, V of replaceable partial spreads that cover the same points, have no lines in common and have cardinality t must be as described in the examples. Our work also complements and generalizes in a number of directions the results of a previous paper by D. A. Foulser and can be modified to yield alternative and combinatorial proofs of a number of Foulser's results. In a later section we discuss some results in $PG(3, 3)$ and $PG(3, 4)$ as well as some general embedding and configurational questions in $PG(3, q)$.

2. The construction. Although this note is more or less self-contained we shall frequently refer to [**4**].

Notation. If A and B are sets then $A - B$ denotes those elements of A not in B. The null set is denoted by $\emptyset$. If R is a regulus of $\Sigma = PG(3, q)$ then R' denotes the opposite regulus so that $(R')' = R$. It is worth noting that R, R' are sets of *lines*. The points lying on the lines of R and R' form the points of a doubly-ruled quadric $Q = Q(R) = Q(R') : Q$ will be regarded as a set of *points* in Σ. Lines of Σ will be denoted by small letters a, b, c, d, etc.

Let R, S denote two distinct reguli of Σ with opposite reguli R', S'. We shall frequently demand that the following condition be satisfied.

Condition 1. $Q(R) \cap Q(S)$ is a union of lines.

THEOREM 1. *Let R, S denote two distinct reguli of Σ satisfying Condition* 1. *Let a, b, c, d denote lines of Σ. Define the line sets U and V as indicated below. Then U and V yield partial spreads of Σ which cover the same points and have no lines in common. Moreover, $t = |U| = |V|$ is as specified.*

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Type 1. $t = 2(q + 1), \quad Q(R) \cap Q(S) = \emptyset.$
$U = R \cup S'.$
$V = R' \cup S.$

Type 2. $t = 2q + 1, \quad R \cap S = \emptyset, \quad R' \cap S' = \{a\}.$
$U = R \cup S' - \{a\}.$
$V = S \cup R' - \{a\}.$

Type 3. $t = 2q, \quad R \cap S = \{a\}, \quad R' \cap S' = \{b\}.$
$U = R - \{a\} \cup S' - \{b\}.$
$V = S - \{a\} \cup R' - \{b\}.$

Type 4. $t = 2q, \quad R \cap S = \emptyset, \quad R' \cap S' = \{a, b\}, a \neq b.$
$U = R \cup S' - \{a, b\}.$
$V = S \cup R' - \{a, b\}.$

Type 5. $t = 2q - 1, \quad R \cap S = \{a\}, \quad R' \cap S' = \{b, c\}, \quad b \neq c.$
$U = R - \{a\} \cup S' - \{b, c\}.$
$V = S - \{a\} \cup R' - \{b, c\}.$

Type 6. $t = 2(q - 1), \quad R \cap S = \{a, b\}, \quad R' \cap S' = \{c, d\}.$
$U = R - \{a, b\} \cup S' - \{c, d\}.$
$V = S - \{a, b\} \cup R' - \{c, d\}.$

Proof. First we claim that $R \cap S' = \emptyset$. Let $x \in R \cap S'$. Suppose $\alpha \in R \cap S$. Then $x \neq \alpha$, since S and S' have no common lines. Since $x \in S'$, x meets each line of S. Thus x meets α, yet $x \in R$ and $\alpha \in R$. This is impossible. Pursuing this we see that $R \cap S' = \emptyset$ unless both $R \cap S = \emptyset$ and $R' \cap S' = \emptyset$. That is, $R \cap S' = \emptyset$ except possibly for Type 1. But here also $R \cap S' = \emptyset$ because $Q(R) \cap Q(S) = \emptyset$. Thus $R \cap S' = \emptyset$. Similarly $R' \cap S = \emptyset$. Therefore the values of t are as indicated. Through each point of $Q(R) \cap Q(S)$ there passes a line of R and a line of S'. Thus since $R \cap S' = \emptyset = R' \cap S$ it follows from Condition 1 that U and V are partial spreads of Σ. It remains to show that U and V cover the same points of Σ. In Type 6, for example, let $P \in u \in U$. If P is covered by $R - \{a, b\}$ then P lies on a line of R'. Thus P is on a line of V unless $P \in c$ or $P \in d$. For example, suppose $P \in c$. Since $c \in S'$, P is on a line of S. Since P lies on a line of (is covered by) $R - \{a, b\}$, P is covered by $S - \{a, b\} \subset V$. Similarly each point of $S' - \{c, d\}$ is also on a line of V. Thus each point P covered by a line of U is also covered by a line of V and, conversely, each point on a line of V is covered by a line of U. Finally note that if $R \cap S = \{a, b\}$, $a \neq b$ and $R' \cap S' = \{c\}$ we obtain an example of Type 5 by a suitable change of notation.

Comments. Type 1 is well-known. It occurs for example, in connection with the Desarguesian planes and certain André planes of order q^2 which yield spreads containing partial spreads of Type 1. The group-net examples of Foulser [**5**] where q is postulated to be odd are of Type 6. Our construction places no restriction on q and so in this case we are extending the work in [**5**]. The fact that Type 6 exists for q even (as is shown below) shows that the existence of the appropriate dihedral subgroup of order $2(q-1)$ described in [**5**] is somewhat irrelevant to the geometry of the situation. However Foulser's work does show that, for appropriate odd values of q, the partial spreads of Type 6 are embedded in a spread, namely, the spread corresponding to the irregular nearfield planes. As is pointed out later on, Types 2 and 4 are well-known. Types 3 and 5 appear to be new.

We proceed to construct examples of all six types. As described in [**1**, Theorem 4.5] and [**2**, p. 536] there is an isomorphism between a regular spread W of Σ with its lines and reguli and the inversive plane $IP(q)$ over $GF(q)$ with its points and circles. This makes it easy to see that in a regular spread W it is possible to find pairs A, B of distinct reguli having 0, 1, or 2 lines in common. We shall also make use of the following result.

LEMMA 2. *Let A, B be distinct reguli of Σ contained in a spread W of $\Sigma = PG(3, q)$. Then $A' \cap B' = \emptyset$. If the point $P \in Q(A) \cap Q(B)$ then P lies on a line of $A \cap B$.*

Proof. Let $P \in Q(A) \cap Q(B)$. There is a unique line x of A through P and a unique line $y \in B$. Since W is a spread containing A and B we have $x = y$ and P lies on a line of $A \cap B$. Let $t \in A' \cap B'$. Through any point P of t there passes a line x of A and a line y of B. Also through P passes a unique line of the spread W. Thus, as above, $x = y$. Each line of A and B meets t in such a point P. This then implies that $A = B$, a contradiction. Therefore $A' \cap B' = \emptyset$. This proves Lemma 2.

As above, let A, B denote distinct reguli of any spread W of Σ, for example a regular spread. Let $A \cap B = \emptyset$. Then, by Lemma 2, $Q(A) \cap Q(B) = \emptyset$; we then obtain an example of Type 1 by putting $A = R$, $B = S'$. If $A \cap B$ is a single line we obtain an example of Type 2 by using Lemma 2 and putting $A' = R$, $B' = S$. Similarly, if $A \cap B$ is two distinct lines we obtain Type 4 with $R = A'$, $S = B'$.

For Type 6, we proceed as follows. Let the line-pairs $\{a, b\}$ and $\{c, d\}$ form the opposite sides of a skew quadrangle of Σ. Let $a \cap c = X_1$, $b \cap c = X_2$, $a \cap d = X_3$, $b \cap d = X_4$. Pick any point P on c with $P \neq X_1, X_2$ and let Q be any point of d with $Q \neq X_3, X_4$. Then the line $t = PQ$ is skew to a and b. Thus a, b, t determine a unique regulus R of Σ. For the fixed point P a different choice of Q will determine a different regulus S. By construction, $\{a, b\} \subset R$ and $\{a, b\} \subset S$. Moreover $\{c, d\} \subset R'$ and $\{c, d\} \subset S'$. Now let

$$X \in Q(R) \cap Q(S).$$

Suppose that X is not on c or d. Then the unique transversal x from X to $\{c, d\}$ is the unique line of R through X and the unique line of S through X. Thus if X

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is not on a or b the reguli R, S have 3 lines in common, namely a, b, x. Then $R = S$, a contradiction. We conclude that if $X \in Q(R) \cap Q(S)$ then either X is covered by $\{a, b\}$ or X is covered by $\{c, d\}$. Therefore we have constructed an example of Type 6. Moreover it is immediate that all examples of Type 6 are constructed in this fashion.

We come to Types 3 and 5. Let us introduce homogeneous coordinates in Σ over the field $F = GF(q)$. Thus we think of Σ as the lattice of non-zero subspaces of the 4-dimensional vector space $V_4(F)$ over F. Let $\{e_1, e_2, e_3, e_4\}$ be a basis for $V_4(F)$ over F. Relative to this basis, each point of Σ has homogeneous coordinates (y_1, y_2, y_3, y_4). We denote by $\langle u, v\rangle$ the line of Σ joining the 2 points of Σ corresponding to the 2 linearly independent vectors u, v. Consider the set A of $q + 1$ lines consisting of the line $\langle e_2, e_4\rangle$ together with the lines $\langle e_1 + \lambda e_2, e_3 + \lambda e_4\rangle$ where λ is any element of F. The different values of λ yield q pairwise skew lines each of which is skew to $\langle e_2, e_4\rangle$. In fact, the lines of A form a regulus, and the quadric $Q(A)$ consists of all points of Σ satisfying $y_1y_4 = y_2y_3$. Let f be the collineation of Σ induced by the linear transformation of $V_4(F)$ given by $f(e_1) = e_1$, $f(e_2) = e_2$, $f(e_3) = e_3$, $f(e_4) = e_1 + e_4$. Then $f(A)$ is another regulus B. The points of $Q(B)$ satisfy $(y_1 - y_4)y_4 = y_2y_3$. From these equations we can easily find $Q(A) \cap Q(B)$. In fact, if $P \in Q(A) \cap Q(B)$ then either $P \in a = \langle e_1, e_3\rangle$ or $P \in b = \langle e_1, e_2\rangle$. Note that $b \in A' \cap B'$ since b meets 3 lines of A and 3 lines of B. In fact $A \cap B = a$, $A' \cap B' = b$. Therefore the reguli A, B satisfy Condition 1. By putting $A = R$, $B = S$ we obtain an example of Type 3.

For Type 5, let A be as above, and let f be the collineation of Σ induced by $f(e_1) = e_1$, $f(e_2) = e_2$, $f(e_3) = e_3$, $f(e_4) = e_3 + e_4$. Then the points of $Q(B)$ where $B = f(A)$ satisfy $y_2(y_3 - y_4) = y_1y_4$. We see that $A \cap B = a = \langle e_1, e_3\rangle$ and $A' \cap B' = \{b, c\}$ with $b = \langle e_1, e_2\rangle$, $c = \langle e_3, e_4\rangle$. Moreover if $P \in Q(A) \cap Q(B)$ then P lies either on a or b or c. Putting $A = R$, $B = S$ we obtain an example of Type 5. Thus we have constructed examples of all six types.

3. Characterization. As before, U and V are partial spreads covering the same points of $\Sigma = PG(3, q)$ and having no lines in common. We also assume that $q + 1 < t \leqq 2(q + 1)$ where $t = |U| = |V|$.

LEMMA 3. *Let $q > 16$. Then some 3 lines of U have at least 5 transversals in V.*

Proof. For each 3-element subset E of the lines of U we denote by $n(E)$ the number of lines of V which are transversals to E. Assume $n(E) \leqq 4$. Then $\Sigma_E\, n(E) \leqq 4\binom{t}{3}$. On the other hand, each line of V meets exactly $q + 1$ lines of U so that $\Sigma_E\, n(E) = \binom{q+1}{3}t$. Thus $4(t - 1)(t - 2) \geqq q(q^2 - 1)$.

If we assume that $t \leqq q(q+1)$ the above implies that $q \leqq 16$. Thus if $q > 16$, $n(E) > 4$ for some E.

The main result of this section now follows.

THEOREM 4. *Let U and V be partial spreads of $\Sigma = PG(3, q)$ which cover the same points and have no lines in common. Assume $q + 1 < t \leqq 2(q+1)$ where $t = |U| = |V|$. Then*

(1) $2(q-1) \leqq t \leqq 2(q+1)$; *and*

(2) *if $q > 19$, then U and V are one the types described in Theorem* 1.

Proof. Part 1 follows from Theorem 3 in [**4**]. For Part 2 we argue as follows. By Lemma 3 some 3 lines of U, say $\{u_1, u_2, u_3\}$, have at least 5 transversals v_1, v_2, v_3, v_4, v_5 in V. As in [**4**, p. 178] let there be exactly β lines $v_1, v_2, v_3, \ldots, v_\beta$ in V which are transversals to $\{u_1, u_2, u_3\}$. Since a regulus contains exactly $q+1$ lines we have $5 \leqq \beta \leqq q+1$. Suppose there are exactly α transversals $u_1, u_2, u_3, \ldots, u_\alpha$ in U to the set $\{v_1, v_2, v_3\}$. Then as in [**4**] we have

$$|U| \geqq \alpha + \tfrac{1}{2}\beta(q + 1 - \alpha),$$

$$|V| \geqq \beta + \tfrac{1}{2}\alpha(q + 1 - \beta).$$

Suppose $\alpha \leqq \beta$. Arguing as in [**4**, p. 178] we obtain the fact that $|U| \geqq \beta + \frac{1}{2}\beta(q+1-\beta)$. Examining this quadratic we obtain $\beta \geqq q - 1$ since $t \leqq 2(q+1)$ and $\beta \geqq 5$. Since, as before, $|U| \geqq \alpha + \frac{1}{2}\beta(q+1-\alpha)$ we obtain $\alpha \geqq q - 1$. Similarly if $\beta \leqq \alpha$ we also obtain $\alpha \geqq q - 1$ and $\beta \geqq q - 1$. Thus in either case the lines of $A = \{u_1, u_2, u_3, \ldots, u_\alpha\}$ are contained in a regulus R and the lines of $B = \{v_1, v_2, \ldots, v_\beta\}$ are contained in the opposite regulus R'. Thus $A = R - \{a, b\}$ and $B = R' - \{c, d\}$ say, with the understanding that either of these sets ($\{a, b\}$ or $\{c, d\}$) may be void, or consist of one line or consist of 2 lines of Σ. Let G denote the remaining lines of U, that is, $G = U - A$, and put $H = V - B$. Notice that $G = U - R$, $H = V - R'$. By Part 1, $t \geqq 2(q-1)$. By hypothesis $t \leqq 2(q+1)$. Since $q - 1 \leqq \alpha$, $\beta \leqq q + 1$, we have $q - 3 \leqq |G|, |H| \leqq q + 3$. Let w be any line of G. Then w meets $Q(R)$ in at most 2 points. So w contains at least $q - 1$ points which must be convered by lines of H. Therefore, at most 4 lines of H fail to meet w. Let w_1, w_2, w_3 be 3 distinct fixed lines of G, and let w be any other line of G. From the above there are at most 16 lines of H that can miss one or other of the 4 lines w_1, w_2, w_3, w. All the remaining lines of H are transversals to $\{w_1, w_2, w_3, w\}$. Now $|H| \geqq q - 3$. Also $(q-3) - 16 \geqq 3$ if $q \geqq 22$. Since q is a prime power and $q > 19$ this is so. Therefore the set $\{w_1, w_2, w_3, w\}$ having 3 or more transversals in Σ (actually in H) is contained in a regulus which we denote by S'. Since w is arbitrary, all lines of G lie in the unique regulus S' of Σ determined by $\{w_1, w_2, w_3\}$. Thus any line of Σ that meets 3 lines of G meets

all of them. In particular, let l be any line of H. Then l meets $Q(R')$ in at most 2 points, so that l contains at least $q - 1$ points that must be covered by lines of G. Since $q - 1 > 2$, we have from the above that l meets all lines of G. In summary, each line of H meets each line of G. Thus $H \subset (S')' = S$. Then, as before, we may write $G = S' - \{z, w\}$, $H = S - \{x, y\}$.
Thus

$$U = R - \{a, b\} \cup S' - \{z, w\};$$

$$V = S - \{x, y\} \cup R' - \{c, d\}.$$

Each point on all lines of $R' - \{c, d\}$ must be covered by a line of U. There are at least $q - 1$ points on a and b which line on lines of $R' - \{c, d\}$ since $\{a, b\} \subset R$. Also $a \notin U$, $b \notin U$. Thus a (and b) meets at least $q - 1$ lines of $G = S' - \{z, w\}$. Thus $a \in (S')' = S$. Similarly $b \in S$. Since V is a partial spread, no line of $S - \{x, y\}$ can meet any line of $R' - \{c, d\}$. In particular, $a \in S$ and since a meets each line of $R' - \{c, d\}$ $a \notin S - \{x, y\}$. Therefore $\{a\} \subset \{x, y\}$. Similarly $\{b\} \subset \{x, y\}$, so that $\{a, b\} \subset \{x, y\}$. By starting out with S and S' rather than R and R' we obtain $\{x, y\} \subset \{a, b\}$. Thus $\{a, b\} = \{x, y\}$. By symmetry, $\{c, d\} = \{z, w\}$. Since U and V are assumed to have no common lines, $R - \{a, b\}$ cannot have any lines in common with $S - \{a, b\}$. Thus $\{a, b\} = R \cap S$. Similarly $\{c, d\} = R' \cap S'$. Now suppose $R \cap S' = \emptyset$. Through each point P of $Q(R) \cap Q(S)$ there passes a line of R' and a line of S. Thus since for example U forms a partial spread it follows that if $P \in Q(R) \cap Q(S)$ then either P lies on a line of $R \cap S$ or on a line of $R' \cap S'$. That is, if $R \cap S' = \emptyset$, the reguli R, S satisfy Condition 1. As in the proof of Theorem 1 we can argue that $R \cap S' = \emptyset$ unless both $\{a, b\}$ and $\{c, d\}$ are empty. In this case $U \supset R$ and $G = S'$. Since U is a partial spread, no line of R meets any line of S'. This yields that $Q(R) \cap Q(S) = \emptyset$ and Condition 1 is again satisfied. This completes the proof of Theorem 4.

4. Combinatorial generalizations. It was pointed out in [**4**, Section 5] that many of the arguments there can be carried out in the more general context of a regulus matrix. Recall that a *regulus matrix* is a $t \times t$ matrix M of zeros and ones containing no 4×4 submatrix having exactly 15 ones. In [**4**] it was shown that if each row and column of M contains exactly k ones then, provided $t \neq k$, we have $t \geqq 2(k - 2)$ for $k \geqq 5$. We remarked that the bound is sharp for $k = q + 1$ with q an odd prime power corresponding to the replaceable group nets in [**5**], but stated that other examples to show the bound is sharp are available for $k \neq q + 1$. In fact, the proof of Theorem 4 indicates immediately how one obtains these examples for any positive integer k by putting two $(k - 2) \times (k - 2)$ blocks of ones consecutively along the diagonal and placing exactly two more ones in each of the remaining rows and columns. We can illustrate this method by using a symmetric matrix with

$k = 3$ as follows.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In constructing this type of regulus matrix we need only ensure that there is no 2×2 submatrix in the bottom left quadrant or the top right quadrant of M with all 4 of its entries being 1.

5. Structure and embedding. In general, two partial spreads of the same type in Theorem 1 need not be equivalent under a collineation of Σ. This is discussed later. We proceed to show that *any net* U (or V) of Type 2 or Type 4 is embeddable in a spread (in fact, in many spreads). I am indebted to Professor D. A. Foulser for pointing this out to me. First we need a lemma on the structure of $R' \cup S'$, due to Foulser. The lemma can be established using the methods of indicator sets [**2**]. However, an elegant proof of this lemma, an outline of which we now present, has been constructed by Professor Foulser, as follows.

LEMMA 5. *In Types* 2 *and* 4, $R' \cup S'$ *is contained in a unique regular spread* $W = W(F)$ *corresponding to a field* $F = GF(q)$.

Proof. In Type 4, we can represent R' and S' as follows, in

$$V_4 = \{(x, y) : x = (x_1, x_2), y = (y_1, y_2)\}.$$

R' consists of the lines $x = 0$ and $y = \lambda x$, $\lambda \in GF(q)$. S' consists of $x = 0$ and $y = \lambda Tx$, $\lambda \in GF(q)$, where T is a 2×2 matrix over $GF(q)$. T has no eigenvalues in $GF(q)$, so T is irreducible. Hence $F = \{\lambda T + \mu I : \lambda, \mu \in GF(q)\}$ is a field isomorphic to $GF(q)$, and $R' \cup S' \subseteq W(F)$. In Type 2, we can represent R' as above, and S' by $x = 0$ and $y = (\lambda T + Z)x$, $\lambda \in GF(q)$, where T and Z are 2×2 matrices. As before, Z is irreducible, and det $(\lambda T + Z + \mu I) \neq 0$ for λ, $\mu \in GF(q)$. Let each 2×2 matrix $M = (m_{ij})$ represent the point $m = (m_{11}, m_{12}, m_{21}, m_{22})$ in $\Sigma = PG(3, q)$. Det $M \neq 0$ if and only if $m \notin H$, where H is the hyperbolic quadric $x_1x_4 - x_2x_3 = 0$ in Σ. Hence T, Z and I determine three points of Σ which span a subspace which misses H. This subspace cannot be a plane, so it must be a line. That is, T, Z and I are linearly dependent and hence generate a field F isomorphic to $GF(q)$. As before, $R' \cup S' \subseteq W(F)$.

Let U be any partial spread of Type 2 or Type 4. By Lemma 5, $R' \cup S'$ is contained in a regular spread W. By replacing R' by R we get a new spread W'

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which contains U. Thus the Hall spread contains examples of Types 2 and 4. There are many other spreads containing partial spreads U (or V) of Type 2 or Type 4. For example, the regular near field spread N of order q^2 with q odd is a union of $q + 1$ reguli sharing 2 lines. Therefore, by reversing an appropriate regulus in N, we obtain again examples of Type 4 embedded in spreads.

Two partial spreads U_1, U_2 of type 2 are isomorphic. For let $U_i = R_i \cup S_i' - \{a_i\}$, $i = 1, 2$. By Lemma 5, $R_i' \cup S_i' \subset W_i$, where W_i is a regular spread, $i = 1, 2$. By a collineation of Σ [**1**, Theorem 4.4] we may take $W_1 = W_2 = W$. Also, by [**1**, Theorem 4.5] we can assume that

$$U_i = R \cup S_i' - \{a\},$$

$i = 1, 2$. As mentioned earlier, there is an isomorphism between W and the inversive plane $M = IP(q)$. Let $H = PGL(2, q)_{(\infty)}$ be the subgroup of automorphisms of M leaving a circle C_0 invariant and fixing a point (∞) of M. Then $|H| = q(q - 1) = |M - C_0|$. Moreover, if $\sigma \in H$, $\sigma \neq 1$ then σ fixes no point of $M - C_0$. If follows that H is regular, and hence, transitive on the circles of L, where L is the linear pencil of circles tangent to C_0 at (∞). This yields the desired result.

If $q \geqq 4$, two partial spreads U_1, U_2 of type 4 need not be isomorphic. For, let $U_1 = R_1 \cup S_1' - \{c_1, d_1\}$ and $U_2 = R_2 \cup S_2' - \{c_2, d_2\}$. Let $T \in PL(\Sigma)$ with $T(U_1) = U_2$. Since R_1 is a regulus containing $q + 1$ lines and since $q \geqq 4$ we must have $T(R_1) = R_2$. Thus, $T(R_1') = R_2'$. Also $T(S_1') = S_2'$. Thus T maps $R_1' \cup S_1'$ onto $R_2' \cup S_2'$. By Lemma 5, $R_i' \cup S_i'$ is contained in a regular spread W_i, $i = 1, 2$. In particular, let $W_1 = W_2 = W$. If T maps $R_1' \cup S_1'$ to $R_2' \cup S_2'$ then T maps the regular spread W containing $R_1' \cup S_1'$ onto a regular spread containing $R_2' \cup S_2'$. By Theorem 4.3 in [**1**] there is only one such regular spread, namely W. Therefore T fixes W. As mentioned earlier, there is an isomorphism α between a regular spread W of $\Sigma = PG(3, q)$ with its lines and reguli, and the inversive plane $IP(q)$ with its points and circles. Under α, the subgroup of $G = PL(\Sigma)$ fixing W corresponds to an automorphism group $\bar{G}$ of $IP(q)$. In $IP(q)$ we cannot, for example, find an element of $\bar{G}$ that maps a pair of intersecting but non-orthogonal circles into a pair of intersecting but orthogonal circles. It therefore follows that there may be no element T of G mapping U_1 to U_2. Similarly, two partial spreads U_1, U_2 of Type 2 need not be equivalent under G. For a more detailed discussion of the action of $G = PL(\Sigma)$ on pairs of reguli in a regular spread W we refer to Bruck [**1**, Theorem 7.5].

For certain values of q, partial spreads of Type 6 are embedded in the spread corresponding to the irregular nearfield planes as pointed out in Foulser [**5**]. It can be shown that under the *linear isomorphisms* of Σ there are $q - 1$ non-isomorphic examples of Type 6.

In Section 2 it was pointed out that certain partial spreads U, V which are of Type 1 in Theorem 1 are embedded in spreads. In general, two partial spreads U_1, U_2 of Type 1 need not be isomorphic under $G = PL(\Sigma)$. For example, let W be a regular spread of $\Sigma = PG(3, 3)$. Choose any two lines l, m of W. The remaining 8 lines of W are partitioned into two reguli A and B with $A \cap B = \emptyset$. From Lemma 2, $Q(A) \cap Q(B) = \emptyset$. By putting $A = R$ and $B = S'$ we obtain an example U_1 of a partial spread of Type 1 where $U_1 = R \cup S' = A \cup B$. Similarly we can obtain another example of a partial spread U_2 of Type 1 where $U_2 = A \cup B'$. Let T be a collineation of Σ mapping U_1 to U_2. Now U_1 is contained in a regular spread W. Thus $U_2 = T(U_1)$ must be contained in a regular spread. But (see [**3**]) there is only one spread containing U_2, namely, $W_2 = A \cup B' \cup \{l, m\}$. Moreover, W_2 is not regular: it is a Hall spread which is subregular of index 1. Thus, T does not exist, and in general, two partial spreads of Type 1 in Theorem 6 need not be equivalent under a collineation of Σ.

We do not know if the partial spreads of Types 3 and 5 are embeddable in spreads. However, we can show that none of the partial spreads U of Type 1–6 is embeddable in a *regular spread* of Σ. As a preliminary we have

LEMMA 6. *Let U be a partial spread of Σ such that U is contained in a regular spread W of Σ. Suppose that V is another partial spread such that U and V cover the same points and have no lines in common. Then $|U| = |V| \geqq 2q$. Moreover the case $|U| = |V| = 2q$ can occur.*

Proof. Suppose U is replaceable by V. Let v be a line of $V - U$. Through the $q + 1$ points of v there passes $q + 1$ lines of U forming a regulus R since S is regular. Not all lines of V are lines of R' for otherwise U would just be a regulus R. Thus let w be a line of $V - R'$. Then $w \notin U$, and w can meet $Q(R)$ in at most two points. The remaining points of w that are not on $Q(R)$ must be covered by lines of U. Thus $|U| \geqq (q + 1) + (q - 1) = 2q$. Suppose $q = 3$, and let T be any regulus of $\Sigma = PG(3, 3)$. Let us take two different regular spreads W_1, W_2 of Σ containing R. If $W_1 - R \cap W_2 - R \neq \emptyset$ we have $W_1 = W_2$. Thus $W_1 - R \cap W_2 - R = \emptyset$. We can now put $U = W_1 - R$, $V = W_2 - R$. Then $|U| = |V| = 10 - 4 = 6 = 2 \cdot 3$.

Lemma 6 immediately shows that the partial spreads in Theorem 1 corresponding to Types 5, 6 are not embeddable in a regular spread. In fact none of Types 2–6 are contained in a regular spread. We shall only prove this for Type 2 as the proof is easily modified to cover the remaining Types 3, 4, 5, and 6. Recall that, in Type 2, $U = R \cup S' - \{a\}$ where $R' \cap S' = \{a\}$. Let W be a regular spread containing, say, U. By hypothesis W contains R. Since W is regular W contains all lines of the regulus formed by any 3 of its lines. Thus $S' \subset W$ so that $a \in W$. Then if P is any point of the line a, P is covered by two lines of W, namely the line of R through P and the line a. But this is impossible. Similarly V is not contained in any regular spread of Σ.

6. The cases q = 3, 4. We wish to discuss briefly partial spreads in $PG(3, 3)$ and $PG(3, 4)$ in relation to Section 2. Along the way we come across some

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questions which relate to $PG(3, q)$ for any q. First we sketch some miscellaneous results on the embedding of partial spreads in $PG(3, F)$ where $F = GF(3) = \{0, 1, -1\}$. The following is Theorem 3.3 in [**3**].

THEOREM 7. *Let W be a maximal partial spread of $PG(3, 3)$. Then either $|W| = 7$ or $|W| = 10$. In this last case W is a spread.*

Using Theorem 7 and some combinatorial arguments we obtain

THEOREM 8. *Let L be any partial spread of $\Sigma = PG(3, 3)$ with $|L| \leqq 5$. Then there exists a spread S of Σ containing L.*

Recall that the classical double-six theorem (see [**6**]) is equivalent to the following statement. Given 6 skew lines u_i $(1 \leqq i \leqq 6)$ in a 3-dimensional projective space $\Sigma = PG(3, F)$ over a commutative field F and given another set v_j $(1 \leqq j \leqq 5)$ of 5 skew lines such that

(i) v_k is skew to u_k, $1 \leqq k \leqq 5$;

(ii) v_j intersects u_i if $i \neq j$, $1 \leqq j \leqq 5$, $1 \leqq i \leqq 6$.

Then there exists a line v_6 skew to each of the lines v_1, v_2, v_3, v_4, v_5 such that v_6 intersects u_i (if $1 \leqq i \leqq 5$) but is skew to u_6. The resulting collection of 12 lines above is referred to as a *double-six configuration*. In [**6**] it is shown that double-six configurations exist in every $PG(3, F)$ except in the case that $F = GF(q)$ and $q = 2, 3, 5$. In a manner analagous to the above, a *double-five theorem* would say that given 5 skew lines u_i, $1 \leqq i \leqq 5$ and a set of 4 skew lines v_j, $1 \leqq j \leqq 4$ such that v_j intersects u_i if $i \neq j$ and such that u_k is skew to v_k, $1 \leqq k \leqq 4$, then there exists a fifth line v_5 skew to v_1, v_2, v_3, v_4 and meeting u_1, u_2, u_3, u_4 but skew also to u_5. The resulting set of 10 lines consisting of two sets of 5 skew lines would then be called a *double-five configuration*.

The following question may be of interest.

Problem. For what values of q does a double-five theorem hold in $PG(3, q)$?

In the work of G. Pellegrino [**9**] this next result is implicit (see Lemma 1 in [**9**]).

LEMMA 9. *Let A be a partial spread of $PG(3, 3)$ with $|A| = 4$. Then either A is a regulus or A has at most one transversal in Σ.*

Using the previous results of this section it is then possible to prove a double-five theorem, actually a stronger version of it, in $PG(3, 3)$ to the effect that given *just three* of the lines v_j we can always find the remaining *two*. A key to this last result is Theorem 8. Theorem 8 suggests a generalization but before stating it, we mentioned theorem 3.1 in [**3**], as follows.

THEOREM 10. *Let W be a maximal partial spread in $\Sigma = PG(3, q)$ such that W is not a spread. Then $q + \sqrt{q} + 1 \leqq |W| \leqq q^2 - \sqrt{q}$.*

Theorem 8 then might suggest something along the following lines.

CONJECTURE 11. *There exists a maximal integer $k = k(q)$ such that in $\Sigma =$*

$PG(3, q)$ the following statement is valid: if W is a partial spread of Σ with $|W| < k$, then W is embeddable in a spread of Σ. Also, $k \geqq 2q$.

In view of Theorem 10 we see that Conjecture 11 implies Conjecture 12 below.

CONJECTURE 12. *Let U be a partial spread of Σ with $|U| < k$. Suppose that V_1 is a partial spread of Σ with $|V_1| > |U| - \sqrt{q} - 1$ and such that every point on each line of V_1 is covered by a line of U. Then $V_1 \subset V$ where V is a partial spread of Σ with $|V| = |U|$ such that U and V cover the same points of Σ.*

Let us use these ideas for the case $q = 4$. Suppose that U is a partial spread of $\Sigma = PG(3, 4)$ with $|U| = 6$. Using the notation of Conjecture 12 let $|V_1| = 4$. Assume $k(4) = 8$. Then, by Conjecture 12, $V_1 \subset V$ with $|V| = 6$, and by doing a little more work one can then show that in this case U and V form the two halves of a double-six Ω in Σ. In other words, *a stronger double-six theorem would hold in $PG(3, 4)$ if Conjecture 11 holds with $k(4) = 8$*, analogous to the strong double-five theorem in $PG(3, 3)$ previously described.

In connection with Theorem 3 in [**4**] we mention the following result.

THEOREM 13. *A double-five configuration exists in $PG(3, 3)$.*

Proof. Let
$$\begin{aligned}
u_1 &= \langle (1, 0, 0, 0), (1, -1, 0, 0)\rangle;\\
u_2 &= \langle (1, 1, 1, 0), (-1, 1, 1, -1)\rangle;\\
u_3 &= \langle (1, 1, 1, 1), (1, -1, 1, -1)\rangle;\\
u_4 &= \langle (0, 0, 0, 1), (0, 0, 1, -1)\rangle;\\
u_5 &= \langle (1, 1, 1, -1), (1, 0, -1, 0)\rangle;\\
v_1 &= \langle (1, 1, 1, 0), (0, 0, 0, 1)\rangle;\\
v_2 &= \langle (1, 0, 0, 0), (0, 0, 1, 0)\rangle;\\
v_3 &= \langle (0, 1, 0, 0), (0, 0, 1, 1)\rangle;\\
v_4 &= \langle (1, 1, 0, 0), (0, 1, 0, 1)\rangle;\\
v_5 &= \langle (1, -1, 0, 0), (-1, 1, 1, -1)\rangle.
\end{aligned}$$

Then it is easy to check that u_k is skew to v_k, $1 \leqq k \leqq 5$, and that u_i intersects v_j if $i \neq j$, $1 \leqq i, j \leqq 5$.

Definition. A *potential double-five* consists of 2 partial spreads $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $V = \{v_1, v_2\}$, with $|U| = 5$ and $|V| = 2$ such that v_j misses u_j but intersects all remaining lines of U for $j = 1, 2$. We say that a potential double-five $Z = U \cup V$ is an *unrealized double-five* if there does not exist a double-five configuration which contains the lines of Z.

THEOREM 15. *Unrealized double-fives exist in $PG(3, 3)$.*

Proof. With the notation of Theorem 13 set

$$U = \{u_2, u_3, u_4, u_5\} \cup \langle (1, 0, 0, 0), (0, 1, -1, 0)\rangle$$

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and put $V = \{v_1, v_2\}$. Then it can be verified that $Z = U \cup V$ is an unrealized double-five.

LEMMA 15. *Let* $Z = U \cup V$ *be an unrealized double-five with* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *and* $V = \{v_1, v_2\}$. *Then there does not exist any line* x *of* Σ *such that* x *is skew to* v_1 *and* v_2 *and such that* x *intersects* 4 *of the* 5 *lines of* U.

Proof. Using Lemma 9, x would have to meet both u_1 and u_2. But then, by the strong double-five theorem, Z would be contained in a double-five configuration.

We now mention two types of examples of maximal partial spreads W of $PG(3, 3)$ which are not spreads. It will follow from Theorem 7 that $|W| = 7$.

Example 1. Let S' be any non-regular spread of $\Sigma = PG(3, 3)$. Let l be a line of Σ such that l is not a line of S', and such that the 4 lines $A = \{a_1, a_2, a_3, a_4\}$ of S' passing through the 4 points of l do not form a regulus. Set $W = (S - \text{A}) \cup \{l\}$. Then, from Lemma 9, W is a *maximal* partial spread of Σ with $|W| = 7$.

Example 2. Let $Z = U \cup V$ be an unrealized double-five in Σ (see Theorem 14) with $|U| = 5$, $|V| = 2$. By Theorem 8, U is embeddable in a spread S, so that $U \subset S$. Let $S = U \cup L$ say with $|L| = 5$. Now put $W = L \cup V$. Then W is a partial spread of Σ with $|W| = 7$. Furthermore, by Lemma 15, W is maximal.

By making use of Theorem 8 and Lemma 9 the following can be shown.

THEOREM 16. *Let* W *be a maximal partial spread of* $PG(3, 3)$ *such that* W *is not a spread. Then* $|W| = 7$. *Furthermore, either* W *is obtainable as in Example* 1 *above or* W *is as in Example* 2.

We return to the case $q = 4$. It follows from the work of Kleinfeld [**8**] (see also Johnson [**7**]) that, up to a collineation, there are just three types of spreads in $\Sigma = PG(3, 4)$. These three spreads are

(1) the regular spread, corresponding to the Desarguesian plane of order 16;
(2) a subregular spread of index 1 corresponding to the Hall plane; and
(3) the spread corresponding to a semifield of order 16 with kernel isomorphic to $GF(4)$.

As explained in Hirschfeld [**6**] all double-sixes in Σ are projectively equivalent. One can show that neither the Desarguesian spread nor the Hall spread contain the 6 lines E of one half of a double-six. However R. H. F. Denniston has pointed out to me that the spread of Type (3) above does contain half-double sixes E in great profusion. Note that the 5 lines of one half of a double-six in $PG(3, 4)$ and the five lines of one half of a double-five in $PG(3, 3)$ are examples of *replaceable* partial spreads, in $PG(3, 4)$ and $PG(3, 3)$ respectively.

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