

Correlation of the renormalized Hilbert length for convex projective surfaces

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Abstract. In this paper, we focus on dynamical properties of (real) convex projective surfaces. Our main theorem provides an asymptotic formula for the number of free homotopy classes with roughly the same renormalized Hilbert length for two distinct convex real projective structures. The *correlation number* in this asymptotic formula is characterized in terms of their Manhattan curve. We show that the correlation number is not uniformly bounded away from zero on the space of pairs of hyperbolic surfaces, answering a question of Schwartz and Sharp. In contrast, we provide examples of diverging sequences, defined via *cubic rays*, along which the correlation number stays larger than a uniform strictly positive constant. In the last section, we extend the correlation theorem to Hitchin representations.

Key words: low-dimensional dynamics, higher Teichmüller theory, thermodynamic formalism

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1. Introduction

In this paper, we study the *correlation* of length spectra of pairs of convex projective surfaces. We describe how length spectra of two convex projective structures ρ_1 and ρ_2 correlate on a closed connected orientable surface S of genus $g \geq 2$. More precisely, we investigate the asymptotic behavior of the number of free homotopy classes of closed curves whose ρ_1 -Hilbert length and ρ_2 -Hilbert length are roughly the same. This question was first considered by Schwartz and Sharp in the context of hyperbolic surfaces in [56] (see also [27]), and more generally for negatively curved metrics in [23, 49]. The holonomy of a real convex projective structure is an example of a *Hitchin representation* and our correlation theorem holds in this more general setting.

We now discuss our results in greater detail. A (*marked real*) *convex projective structure* on the surface S is described by a strictly convex set Ω_ρ in the real projective plane which admits a cocompact action by a discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ isomorphic to $\Gamma = \pi_1(S)$. We denote by $\rho: \Gamma \rightarrow \mathrm{SL}(3, \mathbb{R})$ the corresponding representation and by X_ρ the surface S equipped with the convex projective structure. Goldman [30] proved that the space of convex projective surfaces $\mathfrak{C}(S)$ is an open cell of dimension $-8\chi(S)$. Hyperbolic structures on S define, via the Klein model of the hyperbolic plane, a $-3\chi(S)$ -dimensional subspace of $\mathfrak{C}(S)$, called the *Fuchsian locus*, which is naturally identified with the Teichmüller space $\mathcal{T}(S)$ of S . As customary, we will blur the distinction between $\mathcal{T}(S)$ and the Fuchsian locus.

Every convex projective structure ρ induces a *Hilbert length* ℓ_ρ^H for non-trivial conjugacy classes of group elements in Γ . Given $[\gamma] \in [\Gamma]$ a non-trivial conjugacy class, we define $\ell_\rho^H([\gamma])$ as the length of the unique closed geodesic with respect to the Hilbert metric on X_ρ that corresponds to the free homotopy class of $[\gamma]$ on S . Concretely, $\ell_\rho^H([\gamma])$ is the logarithm of the spectral diameter of $\rho(\gamma)$, see §2.1 for details. The (*marked*) *Hilbert length spectrum* of ρ is the function $\ell_\rho^H: [\Gamma] \rightarrow \mathbb{R}_{>0}$. We investigate a slightly different notion of length. Denoting the *topological entropy* of (the Hilbert geodesic flow of) ρ as

$$h^H(\rho) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{[\gamma] \in [\Gamma] \mid \ell_\rho^H([\gamma]) \leq T\},$$

we define the *renormalized Hilbert length spectrum* of ρ as $L_\rho^H = h^H(\rho)\ell_\rho^H$. It follows from [7, 10] that $h^H(\rho)$ is positive and finite, as it is the topological entropy of an Anosov flow.

Our first theorem concerns the correlation of the renormalized Hilbert length spectra of two different convex projective structures.

THEOREM 1.1. (Correlation theorem for convex real projective structures) *Fix a precision $\varepsilon > 0$. Consider two convex projective structures ρ_1 and ρ_2 on a surface S with distinct renormalized Hilbert length spectra $L_{\rho_1}^H \neq L_{\rho_2}^H$. There exist constants $C = C(\varepsilon, \rho_1, \rho_2) > 0$ and $M = M(\rho_1, \rho_2) \in (0, 1)$ such that*

$$\#\{[\gamma] \in [\Gamma] \mid L_{\rho_1}^H([\gamma]) \in (x, x + h^H(\rho_1)\varepsilon), L_{\rho_2}^H([\gamma]) \in (x, x + h^H(\rho_2)\varepsilon)\} \sim C \frac{e^{Mx}}{x^{3/2}},$$

where $f(x) \sim g(x)$ means $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$.

Remark 1.2

- (1) There is an involution on the space of convex projective structures, called the *contragredient involution*, which sends ρ to $\rho^* = (\rho^{-1})^t$. Cooper and Delp [21] and Kim [36] show that $L_{\rho_1}^H = L_{\rho_2}^H$ if and only if ρ_2 is ρ_1 or ρ_1^* .
- (2) It follows from the generalized prime geodesic theorem [44, 47] that for any fixed ε , if ρ is a convex projective structure, then

$$\#\{[\gamma] \in [\Gamma] \mid L_{\rho}^H([\gamma]) \in (x, x + h^H(\rho)\varepsilon)\} \sim (e^{h^H(\rho)\varepsilon} - 1) \frac{e^x}{x}.$$

This implies that if ρ_1 and ρ_2 converge to convex projective structures with the same renormalized Hilbert length spectrum, then $M(\rho_1, \rho_2)$ converges to one and $C(\varepsilon, \rho_1, \rho_2)$ diverges.

The topological entropy of any hyperbolic structure is equal to one [34] and its renormalized Hilbert length coincides with its Hilbert length. The entropy varies continuously on $\mathcal{C}(S)$ [52, Proposition 3.8] and Crampon [22] shows that it is strictly less than one away from the Fuchsian locus. Nie and Zhang [46, 60] prove that the entropy can be arbitrarily close to zero. In Example 5.2, we show that there exist a Fuchsian representation ρ_1 and a representation $\rho_2 \in \mathcal{C}(S)$ with topological entropy different from one such that

$$\lim_{x \rightarrow \infty} \#\{[\gamma] \in [\Gamma] \mid \ell_{\rho_1}^H([\gamma]) \in (x, x + \varepsilon), \ell_{\rho_2}^H([\gamma]) \in (x, x + \varepsilon)\} = 0.$$

In particular, for these representations, the size of the set $\{[\gamma] \in [\Gamma] \mid \ell_{\rho_i}^H([\gamma]) \in (x, x + \varepsilon), i = 1, 2\}$ does not grow exponentially in contrast with what happens for the renormalized Hilbert length spectra by Theorem 1.1. However, the renormalized length spectra arise naturally from a dynamical point of view and they play a key role in our discussion.

We refer to the exponent $M(\rho_1, \rho_2)$ from Theorem 1.1 as the *correlation number* of ρ_1 and ρ_2 . An important goal of this paper is to study the correlation number as we vary ρ_1 and ρ_2 in $\mathcal{C}(S)$. One interesting question asked in [56] for the case of the Teichmüller space is whether the correlation number $M(\rho_1, \rho_2)$ is uniformly bounded away from zero as its arguments range over all hyperbolic structures. We answer this question in the negative in §4. We prove the following.

THEOREM 1.3. (Decay of correlation number) *There exist sequences $(\rho_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in the Teichmüller space $\mathcal{T}(S)$ such that the correlation number satisfies*

$$\lim_{n \rightarrow \infty} M(\rho_n, \eta_n) = 0.$$

The sequences in Theorem 1.3 are given by pinching a hyperbolic structure along two different pants decompositions which are filling. Intuitively, these are two families of hyperbolic structures diverging from each other in the Teichmüller space thus suggesting a small correlation number when going to infinity. A key to prove Theorem 1.3 is a characterization of Sharp [58] of the correlation number in terms of the Manhattan curve [17]. In Theorem 4.2, we extend Sharp’s characterization of the correlation number to the case of two convex projective structures.

In §5, we study explicit examples of correlation numbers in the space of convex projective structures. In contrast to Theorem 1.3, we provide pairs of diverging sequences for which the correlation numbers are uniformly bounded below away from zero. These sequences $(\rho_t)_{t \geq 0}$, called *cubic rays*, are defined using holomorphic cubic differentials via Labourie–Loftin’s parameterization of $\mathcal{C}(S)$ [38, 43]. More precisely, Labourie and Loftin describe a mapping class group equivariant homeomorphism between $\mathcal{C}(S)$ and the vector bundle of holomorphic cubic differentials over $\mathcal{T}(S)$. The sequences $(\rho_t)_{t \geq 0}$ lie in fibers of $\mathcal{C}(S)$ with base point $\rho_0 \in \mathcal{T}(S)$ and correspond to rays $(tq)_{t \geq 0}$ for a fixed ρ_0 -holomorphic cubic differential q . We will recall Labourie–Loftin’s parameterization of $\mathcal{C}(S)$ and the definition of cubic rays more precisely in §5.

Using work of Tholozan [59], we show in Lemma 5.1 that for a cubic ray $(\rho_t)_{t \geq 0}$, the renormalized Hilbert length of ρ_t is bi-Lipschitz to the one of ρ_0 with Lipschitz constants independent of t . We deduce that the correlation number of any two convex projective structures in different fibers is uniformly bounded from below by the correlation number of its base hyperbolic structures.

THEOREM 1.4. *Let $(\rho_t)_{t \geq 0}$ and $(\eta_r)_{r \geq 0}$ be two cubic rays associated to two different hyperbolic structures $\rho_0 \neq \eta_0$. Then, there exists a constant $C > 0$ such that for all $t \geq 0$ and $r \geq 0$,*

$$M(\rho_t, \eta_r) \geq CM(\rho_0, \eta_0).$$

A similar statement holds for most pairs of cubic rays with the same base hyperbolic structure.

THEOREM 1.5. *Let $(\rho_t)_{t \geq 0}$ and $(\eta_r)_{r \geq 0}$ be two cubic rays associated to two different holomorphic cubic differentials q_1 and q_2 on a hyperbolic structure X_0 such that q_1, q_2 have unit L^2 -norm with respect to X_0 and $q_1 \neq -q_2$. Then there exists a constant $C > 0$ such that for all $t > 0$ and $r > 0$,*

$$M(\rho_t, \eta_r) \geq C.$$

In §5, we discuss how the assumptions in Theorems 1.4 and 1.5 allows application of the correlation theorem.

In §5.3, which is for the most part independent of the rest of the paper, we show how Lemma 5.1 can be used to study renormalized Hilbert geodesic currents ν_{ρ_t} along a cubic ray $(\rho_t)_{t > 0}$. Geodesic currents are measures on the space of complete geodesics on the universal cover of S and each geodesic current ν has a corresponding length spectrum $\ell_\nu: [\Gamma] \rightarrow \mathbb{R}_{\geq 0}$ with *systole* $\text{Sys}(\nu) = \inf_{c \in [\Gamma]} \ell_\nu(c)$. See §2.5 for a detailed discussion. It follows from [9, 15, 45] that for every convex projective structure ρ , there exists a unique geodesic current ν_ρ whose length spectrum coincides with the renormalized Hilbert length spectrum of ρ . We call ν_ρ the *renormalized Hilbert geodesic current* (also known as the *renormalized Liouville current*) of ρ . We prove the following.

THEOREM 1.6. *As t goes to infinity, the renormalized Hilbert geodesic current $(\nu_{\rho_t})_{t \geq 0}$ along a cubic ray $(\rho_t)_{t \geq 0}$ converges, up to subsequences, to a geodesic current ν with $\text{Sys}(\nu) > 0$.*

This should be compared with what happens for hyperbolic structures: given a sequence of hyperbolic structures which leaves every compact subset of $\mathcal{T}(S)$, up to rescaling and passing to subsequences, the corresponding sequence of geodesic currents converges to a geodesic current with vanishing systole. Burger *et al* [18] show that this fact no longer holds for general sequences of Hilbert geodesic currents. Theorem 1.6 provides new examples of diverging sequences of convex projective structures whose associated geodesic currents converge (projectively and up to subsequences) to a geodesic current with positive systole. In light of [2, Theorem 1.13], Theorem 1.6 extends a theorem of Burger *et al* [18, Theorem 1.12] who prove that the Hilbert geodesic currents of a diverging sequence of convex projective structures of a triangle group converge (projectively and up to subsequences) to a geodesic current with positive systole.

Finally, in §6, we generalize the correlation theorem (Theorem 1.1) to Hitchin components for a large class of length functions. We will replace the Hilbert geodesic flow of a convex projective structure, which is Anosov, with more general metric Anosov translation flows.

Given a hyperbolic structure $\rho \in \mathcal{T}(S)$, seen as a (conjugacy) class of representation $\rho: \Gamma \rightarrow \text{PSL}(2, \mathbb{R})$, we obtain a representation $i \circ \rho: \Gamma \rightarrow \text{PSL}(d, \mathbb{R})$ by post-composing ρ with the unique (up to conjugation) irreducible representation $i: \text{PSL}(2, \mathbb{R}) \rightarrow \text{PSL}(d, \mathbb{R})$. The Teichmüller space $\mathcal{T}(S)$ embeds in this way in the character variety of S and $\text{PSL}(d, \mathbb{R})$. The connected component $\mathcal{H}_d(S)$ of the character variety containing this image is known as the *Hitchin component*. Hitchin [33] showed that $\mathcal{H}_d(S)$ is homeomorphic to an open cell of dimension $-(\dim \text{PSL}(d, \mathbb{R}))\chi(S)$. Choi and Goldman [20] identify $\mathcal{H}_3(S)$ with the space $\mathcal{C}(S)$ which will be our main focus from §2 to §5. Let us remark that, although the spaces $\mathcal{H}_d(S)$ consist of conjugacy classes of representations, for this paper, the distinction between a representation and its conjugacy class is inconsequential. Because of this, when clear from context, we will not distinguish between a representation and its conjugacy class.

To state the general correlation theorem for Hitchin representations (Theorem 1.7), we need to introduce some Lie theoretical notation.

Let

$$\mathfrak{a} = \{\vec{x} \in \mathbb{R}^d \mid x_1 + \dots + x_d = 0\} \quad \text{and} \quad \mathfrak{a}^+ = \{\vec{x} \in \mathfrak{a} \mid x_1 \geq \dots \geq x_d\}$$

denote the (standard) Cartan subspace for $\text{PSL}(d, \mathbb{R})$ and the (standard) positive Weyl chamber, respectively. Let $\lambda: \text{PSL}(d, \mathbb{R}) \rightarrow \mathfrak{a}^+$ be the *Jordan projection* given by $\lambda(g) = (\log \lambda_1(g), \dots, \log \lambda_d(g))$ consisting of the logarithms of the moduli of the eigenvalues of g in non-increasing order. We consider linear functionals in

$$\Delta = \left\{ c_1\alpha_1 + \dots + c_{d-1}\alpha_{d-1} \mid c_i \geq 0, \sum_i c_i > 0 \right\},$$

where $\alpha_i: \mathfrak{a} \rightarrow \mathbb{R}$ are the *simple roots* defined by $\alpha_i(\vec{x}) = x_i - x_{i+1}$ with $i = 1, \dots, d - 1$. Observe that if $\phi \in \Delta$, then $\phi(\vec{x}) > 0$ for all \vec{x} in the interior of \mathfrak{a}^+ . The *length function* ℓ_ρ^ϕ for $\phi \in \Delta$ and $\rho \in \mathcal{H}_d(S)$ is defined by $\ell_\rho^\phi([\gamma]) = \phi(\lambda(\rho(\gamma)))$. The length function is strictly positive because $\lambda(\rho(\gamma))$ is in the interior of \mathfrak{a}^+ for all $[\gamma] \in [\Gamma]$ (see [25, 37]). The *topological entropy* $h^\phi(\rho)$ and *renormalized ϕ -length* $L_\rho^\phi([\gamma]) =$

$h^\phi(\rho)\ell_\rho^\phi([\gamma])$ are defined in similar manner as for convex projective structures. Given a Hitchin representation ρ , we denote by ρ^* its contragredient given by $\rho^* = (\rho^{-1})^t$.

We are now ready to state the correlation theorem for Hitchin representations.

THEOREM 1.7. *Given a linear functional $\phi \in \Delta$ and a fixed precision $\varepsilon > 0$, for any two different Hitchin representations $\rho_1, \rho_2 : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ such that $\rho_2 \neq \rho_1^*$, there exist constants $C = C(\varepsilon, \rho_1, \rho_2, \phi) > 0$ and $M = M(\rho_1, \rho_2, \phi) \in (0, 1)$ such that*

$$\#\{[\gamma] \in [\Gamma] \mid L_{\rho_1}^\phi([\gamma]) \in (x, x + h^\phi(\rho_1)\varepsilon), L_{\rho_2}^\phi([\gamma]) \in (x, x + h^\phi(\rho_2)\varepsilon)\} \sim C \frac{e^{Mx}}{x^{3/2}}.$$

Remark 1.8

- (1) For $\rho \in \mathcal{H}_d(S)$, when $\phi = \alpha_i$ and $i = 1, \dots, d-1$, we know that $h^{\alpha_i}(\rho) = 1$ thanks to [50, Theorem B]. Thus, $L_\rho^{\alpha_i} = \ell_\rho^{\alpha_i}$ is the simple root length without any renormalization. The correlation theorem in this case can be seen as a natural generalization of Schwartz and Sharp's correlation theorem for hyperbolic surfaces.
- (2) Note that Theorem 1.1 is a corollary of Theorem 1.7 when we set $d = 3$ and consider the positive root $\phi(\vec{x}) = (\alpha_1 + \alpha_2)(\vec{x}) = x_1 - x_3$.

It would be interesting to extend the results on the correlation number proved in §§4 and 5 to the context Hitchin representations. In this spirit, we end the paper by raising Question 6.15 and Conjecture 6.16 which are motivated by Theorems 1.3 and 1.5, respectively.

1.1. Structure of the paper. In §§2–5, we focus on convex projective structures. In this case, the length function can be defined geometrically via the Hilbert distance, and Benoist proved in [7] that the associated geodesic flow is Anosov. These two facts will simplify the exposition. The main results (Theorems 1.3, 1.4, and 1.5) in §§4 and 5 concern the behavior of the correlation number along geometrically defined sequences of convex projective structures. We establish the correlation theorem in full generality in §6, after recalling the precise definitions of Hitchin representations and their length functions and the theory of metric Anosov translation flows.

2. Preliminaries for convex real projective structures

Consider a connected, closed, oriented surface S with genus $g \geq 2$ and denote by Γ its fundamental group. In this preliminary section, we focus on convex projective structures on S . We will discuss in §6 how parts of the material presented here hold for general Hitchin components.

The structure of §2 is as follows. In §2.1, we briefly recall the relevant geometric aspects of the theory of convex projective structures on surfaces. We refer to [7, 30] for further details and background. In §2.2, we collect dynamical properties of convex projective surfaces which will play an important role in §§3 and 4. In §2.4, we prove the independence lemma (Lemma 2.11) which plays a key role in the proof of Theorem 1.1. Finally, we briefly survey the theory of *geodesic currents* in §2.5 which will be used in the proof of Theorem 1.3 and in §5.3.

2.1. *Convex projective surfaces.* A properly convex set Ω in \mathbb{RP}^2 is a bounded open convex subset of an affine chart. A properly convex set whose boundary does not contain open line segments is *strictly convex*. We will exclusively focus on strictly convex sets in this paper. We equip a strictly convex set Ω with its *Hilbert metric* d_Ω . More precisely, if $x, y \in \Omega$, the projective line \overline{xy} passing through x and y intersects the boundary of Ω in two points a, b , where a, x, y, b appear in this order along \overline{xy} . The Hilbert distance between x and y is

$$d_\Omega(x, y) = \frac{1}{2} \log[a, x, y, b],$$

where $[a, x, y, b]$ denotes the crossratio of four points on a projective line. With the Hilbert metric, geodesics are segments of a projective line intersecting Ω . Typically, the Hilbert metric is not Riemannian, but it derives from a Finsler norm. Thus, one can study the *unit tangent bundle* $T^1\Omega$ of a strictly convex set Ω .

The main objects of interest are representations $\rho: \Gamma \rightarrow \text{SL}(3, \mathbb{R})$ such that $\rho(\Gamma)$ preserves a properly convex set Ω_ρ on which it acts properly discontinuously with quotient homeomorphic to the closed surface S . In this case, we say that ρ is a *(marked real) convex projective structure* which *divides* Ω_ρ and denote by X_ρ the surface S equipped with the convex projective structure ρ . Since S is a closed surface of negative Euler characteristic, Ω_ρ is strictly convex and if $\gamma \in \Gamma$ is non-trivial, then the moduli $\lambda_1(\rho(\gamma)) > \lambda_2(\rho(\gamma)) > \lambda_3(\rho(\gamma)) > 0$ of the eigenvalues of $\rho(\gamma)$ are distinct. (See for example [30, Theorem 3.2] and references therein).

The Hilbert distance on Ω_ρ induces the *Hilbert length spectrum* ℓ_ρ^H for non-trivial conjugacy classes of group elements in Γ . Algebraically, if $[\Gamma]$ is the set of conjugacy classes of non-identity elements in Γ , namely $[\gamma] \in [\Gamma]$ is the conjugacy class of $\gamma \neq \text{id}$, then

$$\ell_\rho^H([\gamma]) = \frac{1}{2} \log \frac{\lambda_1(\rho(\gamma))}{\lambda_3(\rho(\gamma))}.$$

Benoist [7] proved that if ρ is a convex projective structure on S , then $(\Omega_\rho, d_{\Omega_\rho})$ is Gromov hyperbolic, the Gromov boundary and the topological boundary of Ω_ρ coincide, and $\partial\Omega_\rho$ is of class $C^{1+\alpha}$ for some $\alpha \in (0, 1]$. For a(ny) point $o \in \Omega_\rho$, the orbit map $\gamma \mapsto \rho(\gamma).o$ is a quasi-isometric embedding. It follows (see e.g. [26, Ch. 7, Proposition 14]) that the orbit map induces a *limit map* $\xi_\rho: \partial\Gamma \rightarrow \partial\Omega_\rho$ between Gromov boundaries which is a ρ -equivariant bi-Hölder homeomorphism.

If $\rho(\Gamma)$ divides a strictly convex set Ω_ρ in \mathbb{RP}^2 , then the representation ρ^* divides a (typically different) strictly convex set Ω_{ρ^*} . Here we recall that the *contragredient* representation is defined as $\rho^*(\gamma) = (\rho(\gamma)^{-1})^t$ for all $\gamma \in \Gamma$. We refer to this operation on the space of convex projective structures as the *contragredient involution*. The contragredient involution preserves the Hilbert length because for all $[\gamma] \in [\Gamma]$,

$$\ell_\rho^H([\gamma]) = \frac{1}{2} \log \frac{\lambda_1(\rho(\gamma))}{\lambda_3(\rho(\gamma))} = \frac{1}{2} \log \frac{1/\lambda_3(\rho(\gamma))}{1/\lambda_1(\rho(\gamma))} = \ell_{\rho^*}^H([\gamma]).$$

A standard computation using the irreducible representation $\mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(3, \mathbb{R})$ shows that if ρ is a hyperbolic structure, then $\rho = \rho^*$. The converse holds by [5, Theorem 1.3]: if ρ is a convex projective structure such that $\rho = \rho^*$, then ρ is a hyperbolic structure.

2.2. The Hilbert geodesic flow and reparametrization function. Suppose that ρ is a convex projective surface. The *Hilbert geodesic flow* Φ^ρ is defined on the unit tangent bundle of the surface T^1X_ρ . The image $\Phi_t^\rho(w)$ of a point $w = (x, v)$ is obtained by following the unit speed geodesic for time t leaving x in the direction v . When it is clear from context, we simply write Φ^ρ as Φ .

The Hilbert geodesic flow Φ on T^1X_ρ is an example of a topologically mixing Anosov flow by [7, Propositions 3.3 and 5.6]. A standard reference for the theory of Anosov flows is [35, §6]. A key property for our discussion is that topologically mixing Anosov flows can be modeled by Markov partitions and symbolic dynamics in the sense of Bowen [11].

Given a positive Hölder continuous function $f: T^1X_\rho \rightarrow \mathbb{R}$, one can define a *Hölder reparameterization* of the flow Φ by time change. We construct the flow Φ^f following [53, §2]. First, we define $\kappa: T^1X_\rho \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\kappa(x, t) = \int_0^t f(\Phi_s(x)) ds.$$

Given the fact that f is positive and T^1X_ρ is compact, the function $\kappa(x, \cdot)$ is an increasing homeomorphism of \mathbb{R} . We therefore have an inverse $\alpha: T^1X_\rho \times \mathbb{R} \rightarrow \mathbb{R}$ that verifies

$$\alpha(x, \kappa(x, t)) = \kappa(x, \alpha(x, t)) = t$$

for every $(x, t) \in T^1X_\rho \times \mathbb{R}$. The Hölder reparameterization of Φ by the Hölder continuous function f is given by $\Phi_t^f(x) = \Phi_{\alpha(x,t)}(x)$. We say that f is a *reparameterization function* for Φ^f . The new flow $\Phi^f = \{\Phi_t^f\}_{t \in \mathbb{R}}$ shares the same set of periodic orbits of Φ . For any periodic orbit τ of Φ with period $\lambda(\tau)$, its period as a Φ^f periodic orbit is

$$\lambda(f, \tau) = \int_0^{\lambda(\tau)} f(\Phi_s(x)) ds. \quad (2.1)$$

Property (2.1) is a simple application of the definitions of $\alpha(x, t)$ and $\kappa(x, t)$.

Remark 2.1. Each oriented closed geodesic γ on a convex projective structure ρ is associated with a periodic orbit τ of Φ . However, an oriented closed geodesic γ corresponds to a free homotopy class $[\gamma] \in [\Gamma]$. We adopt different perspectives depending on necessity in this paper while keeping in mind that they are the same object described from different points of view.

Remark 2.2

- (1) One can check from the definition that Φ is a Hölder reparameterization of Φ^f with the Hölder reparameterization function given by $1/f$.
- (2) Set $\Psi = \Phi^f$ and consider g a positive Hölder reparameterization of Ψ , then Ψ^g is a Hölder reparameterization of Φ by the Hölder function $g \cdot f$.

Let ρ_1 and ρ_2 be two convex projective structures on a surface S . The next lemma, which is essentially due to Sambarino [53, §5], states that there exists a positive Hölder continuous reparameterization function $f_{\rho_1}^{\rho_2} : T^1X_{\rho_1} \rightarrow \mathbb{R}$ encoding the Hilbert length spectrum of ρ_2 . We include a proof for completeness.

LEMMA 2.3. (Sambarino [53]) *Let ρ_1 and ρ_2 be convex projective structures on a surface S . There exists a positive Hölder continuous function $f_{\rho_1}^{\rho_2} : T^1X_{\rho_1} \rightarrow \mathbb{R}$ such that for every periodic orbit τ corresponding to $[\gamma] \in [\Gamma]$, one has*

$$\lambda(f_{\rho_1}^{\rho_2}, \tau) = \ell_{\rho_2}^H([\gamma]).$$

Proof. Let us lift the picture to the universal cover. By [7, equation (20)], there exists a Hölder continuous ρ_1 -equivariant homeomorphism $T^1\Omega_{\rho_1} \rightarrow \partial^2\Omega_{\rho_1} \times \mathbb{R}$, where $\partial^k\Omega_{\rho_1}$ is the set of ordered k -tuples of distinct points in $\partial\Omega_{\rho_1}$. Since the limit map of ρ_1 is Hölder continuous, we have a bi-Hölder homeomorphism $\partial^3\Omega_{\rho_1} \cong \partial^2\Omega_{\rho_1} \times \mathbb{R}$. We thus obtain a Hölder continuous ρ_1 -equivariant homeomorphism $\chi : T^1\Omega_{\rho_1} \rightarrow \partial^3\Omega_{\rho_1}$. Since the Hilbert geodesic flow is Anosov [7, Proposition 3.3], it follows from [53, Theorem 3.2] (see also [16, Proposition 5.21]) that for any choice of an auxiliary hyperbolic surface ρ_0 , there exists a Hölder continuous positive reparameterization function $g_{\rho_0}^{\rho_2} : T^1\Omega_{\rho_0} \cong \partial^3\mathbb{H}^2 \rightarrow \mathbb{R}$ of the geodesic flow of ρ_0 with periods $\ell_{\rho_2}^H([\gamma])$. Considering the Hölder continuous function $\xi^{(3)} : \partial^3\Omega_{\rho_1} \rightarrow \partial^3\mathbb{H}^2$ induced by the inverse of the limit map ξ_{ρ_1} and the limit map $\xi_{\rho_0} : \partial\Gamma \rightarrow \partial\mathbb{H}^2$, we obtain the composition $g_{\rho_0}^{\rho_2} \circ \xi^{(3)} \circ \chi : T^1\Omega_{\rho_1} \rightarrow \mathbb{R}$ which is the lift of the desired reparameterization function $f_{\rho_1}^{\rho_2}$. The equality $\lambda(f_{\rho_1}^{\rho_2}, \tau) = \ell_{\rho_2}^H([\gamma])$ follows from equivariance of the limit maps. \square

2.3. *Thermodynamic formalism.* In this subsection, we will introduce several important concepts from thermodynamic formalism in our context that will be needed later. Standard references for thermodynamic formalism and Markov codings are [12, 47].

For a continuous function $f : T^1X_{\rho} \rightarrow \mathbb{R}$, we define its *pressure* with respect to Φ as

$$P(\Phi, f) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(\sum_{\tau \in R_T} e^{\lambda(f, \tau)} \right),$$

where $R_T := \{\tau \text{ periodic orbit of } \Phi \mid \lambda(\tau) \in [T - 1, T]\}$. One can check that the topological entropy $h^H(\rho)$ is $P(\Phi, 0)$. For simplicity, we omit the geodesic flow Φ from the notation and write $P(\cdot)$ for $P(\Phi, \cdot)$. The pressure can be characterized as follows.

PROPOSITION 2.4. (Variational principle) *The pressure of a continuous function $f : T^1X_{\rho} \rightarrow \mathbb{R}$ satisfies*

$$P(f) = \sup_{\mu \in \mathcal{M}^{\Phi}} \left(h(\mu) + \int f \, d\mu \right),$$

where \mathcal{M}^{Φ} is the space of Φ -invariant probability measures on T^1X_{ρ} and $h(\mu) = h(\Phi, \mu)$ denotes the measure-theoretic entropy of Φ with respect to $\mu \in \mathcal{M}^{\Phi}$.

A Φ -invariant probability measure μ on T^1X_{ρ} is called an *equilibrium state* for f if the supremum is attained at μ .

Remark 2.5

- (1) For a Hölder continuous function $f : T^1 X_\rho \rightarrow \mathbb{R}$, there exists a unique equilibrium state μ_f by [13, Theorem 3.3].
- (2) The equilibrium state μ_0 for $f = 0$ is called a *probability measure of maximal entropy* or a *Bowen–Margulis measure*, denoted as μ_Φ . The Hilbert geodesic flow Φ is topologically mixing and Anosov, thus it admits a unique measure of maximal entropy on $T^1 X_\rho$. The entropy of the measure of maximal entropy coincides with the topological entropy. See for instance [35, §20].

The following lemma, derived from Abramov’s formula [1], allows us to rescale a reparameterization function to be pressure zero.

LEMMA 2.6. (Sambarino [53, Lemma 2.4], Bowen and Ruelle [13, Proposition 3.1]) *For a positive Hölder reparameterization function f on $T^1 X_\rho$ and $h \in \mathbb{R}$, the pressure function satisfies*

$$P(-hf) = 0$$

if and only if $h = h(\Phi^f)$, where

$$h(\Phi^f) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{\tau \text{ periodic orbit} \mid \lambda(f, \tau) \leq T\}.$$

By definition, $h(\Phi^f)$ is the topological entropy of the reparameterized flow Φ^f .

We will use Lemma 2.6 in the proofs of Theorems 1.1, 1.3, and 1.5.

Remark 2.7. By construction, if $f = f_{\rho_1}^{\rho_2}$ is the reparameterization function defined in Lemma 2.3, then the topological entropy of the flow Φ^f is equal to the topological entropy $h^H(\rho_2)$ of ρ_2 as defined in the introduction.

Finally, we introduce (*Livšić*) *cohomology*. We say two Hölder continuous functions f and g are (*Livšić*) *cohomologous* if there exists a Hölder continuous function $V : T^1 X_\rho \rightarrow \mathbb{R}$ that is differentiable in the flow’s direction such that for all $x \in T^1 X_\rho$,

$$f(x) - g(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\Phi_t(x)).$$

Remark 2.8. (Livšić’s theorem, [41])

- (1) Two Hölder continuous functions f and g are cohomologous on $T^1 X_\rho$ if and only if $\lambda(f, \tau) = \lambda(g, \tau)$ for any periodic orbit τ of Φ . It follows that the pressure of a Hölder continuous function depends only on its cohomology class.
- (2) Two Hölder continuous functions f and g have the same equilibrium state on $T^1 X_\rho$ if and only if $f - g$ is cohomologous to a constant C . In this case, we have $P(f) = P(g) + C$ [35, §20].

2.4. *Independence of convex projective surfaces.* We start by recalling the notion of (topologically) weakly mixing flows which motivates the concept of independence of representations. A flow φ on $T^1 X_\rho$ is *weakly mixing* if its periods do not generate a discrete

subgroup of \mathbb{R} . In particular, we can ask whether the Hilbert geodesic flow Φ is weakly mixing. Equivalently, we ask whether there exists some non-zero real number $a \in \mathbb{R}$ such that $a\ell_\rho^H(\gamma) \in \mathbb{Z}$ for all $[\gamma] \in \Gamma$. In the proof of Theorem 1.1, we will need a strengthening of this property for a pair of representations.

Two convex projective structures ρ_1 and ρ_2 are *dependent* if there exist $a_1, a_2 \in \mathbb{R}$, not both equal to zero, such that $a_1\ell_{\rho_1}^H([\gamma]) + a_2\ell_{\rho_2}^H([\gamma]) \in \mathbb{Z}$ for all $[\gamma] \in \Gamma$. Otherwise, ρ_1 and ρ_2 are *independent* over \mathbb{Z} . The next definition clarifies that this notion of independence is of a dynamical nature.

Definition 2.9. Two positive Hölder continuous functions $f_1, f_2: T^1X_\rho \rightarrow \mathbb{R}$ are *dependent* if there exists $a_1, a_2 \in \mathbb{R}$ not both equal to zero and a complex valued C^1 function $u: T^1X_\rho \rightarrow S^1$ such that $a_1 f_1 + a_2 f_2 = (1/2\pi i)(u'/u)$. Here, u' denotes the derivative of u along the flow, that is, $\partial/\partial t|_{t=0}u \circ \Phi_t$. Otherwise, f_1 and f_2 are *independent*. In particular, f is said to be *(in)dependent* if f and the constant function $g \equiv 1$ are (in)dependent.

Remark 2.10. The integral over a closed orbit of u'/u is an integer multiple of $2\pi i$. Thus, by integrating along closed orbits and using equation (2.1) and Remark 2.2, we see that if ρ_1 and ρ_2 are independent, then the reparameterization function $f_{\rho_1}^{\rho_2}$ on $T^1X_{\rho_1}$ is independent.

We now prove that convex projective surfaces with distinct Hilbert length spectra are independent.

LEMMA 2.11. (Independence lemma) *Let ρ_1 and ρ_2 be convex projective structures with distinct Hilbert length spectra. If there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1\ell_{\rho_1}^H([\gamma]) + a_2\ell_{\rho_2}^H([\gamma]) \in \mathbb{Z}$ for all $[\gamma] \in \Gamma$, then $a_1 = a_2 = 0$.*

Proof. Our proof follows from combining the results of Benoist [5, 6] and an argument of Glorieux [28].

We prove this statement by contradiction. Consider the product representation $\eta = \rho_1 \times \rho_2: \Gamma \rightarrow G_1 \times G_2$, where G_i denotes the Zariski closure of ρ_i . Benoist [5, Theorem 1.3] proved that G_i is $\text{PSL}(3, \mathbb{R})$ or isomorphic to $\text{PSO}(1, 2)$. Either way, this Zariski closure is connected, center-free, and simple, so $G_1 \times G_2$ is semi-simple. Choose a Cartan subspace of $G_1 \times G_2$ such that

$$\mathfrak{a} \subseteq \{(\vec{x}, \vec{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 = y_1 + y_2 + y_3\}$$

and a positive Weyl chamber \mathfrak{a}^+ contained in $\{(\vec{x}, \vec{y}) \in \mathfrak{a} \mid x_1 \geq x_2 \geq x_3 \text{ and } y_1 \geq y_2 \geq y_3\}$. Denote by $\lambda: G_1 \times G_2 \rightarrow \mathfrak{a}^+$ the corresponding Jordan projection and by ϕ_{a_1, a_2}^H the non-zero linear functional $\phi_{a_1, a_2}^H(\vec{x}, \vec{y}) = a_1(x_1 - x_3) + a_2(y_1 - y_3)$ so that $\phi_{a_1, a_2}^H(\lambda(\eta(\gamma))) = a_1\ell_{\rho_1}^H([\gamma]) + a_2\ell_{\rho_2}^H([\gamma])$.

Denote by $H \subseteq G_1 \times G_2$ the Zariski closure of $\eta(\Gamma)$. As a first step, observe that $H \neq G_1 \times G_2$. Otherwise, Benoist [6, Proposition on p. 2] shows that the additive group generated by $\lambda(\eta(\Gamma))$ is dense (in the standard topology) in \mathfrak{a} . We obtain a contradiction as we assumed $\phi_{a_1, a_2}^H(\lambda(\eta(\gamma))) \in \mathbb{Z}$ for all $\gamma \in \Gamma$ and ϕ_{a_1, a_2}^H is continuous.

Let $\pi_i: H \rightarrow G_i$ for $i = 1, 2$ denote the projection maps. It follows from [24, Lemma 3.1] that π_i is surjective for $i = 1, 2$. Denote by $N_1 = \pi_2^{-1}(\text{id})$ (respectively

$N_2 = \pi_1^{-1}(\text{id})$ the kernel of π_2 (respectively π_1) which is naturally identified with a normal subgroup of G_1 (respectively G_2) (note the indices in the definition of N_i). Then, Goursat’s lemma [32, Theorem 5.5.1] states that the image of H in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \cong G_2/N_2$. Since G_1 is simple, connected, and center-free, then $N_1 = \{e\}$ or G_1 .

Case 1: Suppose $N_1 = G_1$. Then $N_2 = G_2$ and H is the direct product $G_1 \times G_2$, which is a contradiction.

Case 2: Suppose $N_1 = \{e\}$. Since G_2 is simple, $N_2 = \{e\}$ and $G_1 \cong G_2$. In other words, H is the graph of an automorphism $\iota: G_1 \rightarrow G_2$. This induces an automorphism of the corresponding Lie algebras and, by the classification of their outer automorphisms given in [31] (see also [14, Theorem 11.9]), we deduce that ρ_2 is conjugated to either ρ_1 or ρ_1^* , which contradicts our hypothesis. \square

Observe that Lemma 2.11 readily implies that the geodesic flow of a convex projective structure is weakly mixing. Otherwise there exist $a \in \mathbb{R}, a \neq 0$ such that $a\ell_\rho^H([\gamma]) \in \mathbb{Z}$ which directly contradicts Lemma 2.11 with $a_1 = a, a_2 = 0, \rho_1 = \rho,$ and $\rho_2 \in \mathcal{C}(S)$ different from ρ_1 and ρ_1^* .

2.5. *Geodesic currents for convex projective surfaces.* Fix an auxiliary hyperbolic structure m on S . A *geodesic current* is a Borel, locally finite, $\pi_1(S)$ -invariant measure on the set of complete geodesics of the universal cover \tilde{S} . An important example is the geodesic current δ_γ given by Dirac measures on the axes of the lifts of a closed geodesic γ in S .

The space $\mathcal{C}(S)$ of geodesic currents is a convex cone in an infinite dimensional vector space. Bonahon [8] extended the intersection pairing on closed curves to the space of geodesic currents, that is, there exists a continuous, positive, symmetric, bilinear pairing

$$i: \mathcal{C}(S) \times \mathcal{C}(S) \rightarrow \mathbb{R}_{\geq 0}$$

such that $i(\delta_c, \delta_d)$ equals the intersection number of the closed geodesics c and d .

Extending the work of Bonahon [9], in [15, 45], it was shown that for each convex projective surface ρ , there exists a *Hilbert geodesic current* ν_ρ such that for every $[\gamma] \in [\Gamma]$,

$$i(\nu_\rho, \delta_\gamma) = \ell_\rho^H([\gamma]),$$

where γ denotes the unique closed geodesic in its free homotopy class $[\gamma]$. Bonahon [9, Proposition 15] proves that the geodesic current ν_ρ of a hyperbolic structure ρ has self-intersection $i(\nu_\rho, \nu_\rho) = -\pi^2\chi(S)$. However, if $\rho \in \mathcal{C}(S)$ is not in the Fuchsian locus, then $i(\nu_\rho, \nu_\rho) > -\pi^2\chi(S)$ by Corollary 5.3 in [15].

In general, given a geodesic current ν , we can use the intersection number to define its length spectrum $\ell_\nu: [\Gamma] \rightarrow \mathbb{R}^+$ as $\ell_\nu([\gamma]) = i(\nu, \delta_\gamma)$. The *sysole* of ν is then $\text{Sys}(\nu) := \inf_{[\gamma] \in [\Gamma]} \ell_\nu([\gamma])$. Corollary 1.5 in [18] shows that $\text{Sys}: \mathcal{C}(S) \rightarrow [0, \infty)$ is a continuous function.

A geodesic current is *period minimizing* if for all $T > 0$, the set $\#\{[\gamma] \in [\Gamma] \mid \ell_\nu([\gamma]) < T\}$ is finite. We define the *exponential growth rate* of a period minimizing

geodesic current ν by

$$h(\nu) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \#\{\gamma \in [\Gamma] \mid \ell_\nu([\gamma]) < T\}.$$

The notation is motivated by the fact that if ρ is a convex projective structure and ν_ρ is the corresponding Hilbert geodesic current, then $h(\nu_\rho)$ is equal to the topological entropy $h^H(\rho)$ of ρ .

The systole and the exponential growth rate of a geodesic current are related by the following inequality, which will play an important role in the proof of Theorem 1.3.

THEOREM 2.12. (Corollary 7.6 in [45]) *Let S be a closed, connected, oriented surface of genus $g \geq 2$. There exists a constant $C > 0$ depending only on g such that for every period minimizing geodesic current $\nu \in \mathcal{C}(S)$,*

$$\text{Sys}(\nu)h(\nu) \leq C.$$

3. The correlation theorem

In this section, we study the length spectra of two convex real projective structures simultaneously.

This idea appeared first in [56] for studying correlation of hyperbolic structures. We adapt their argument to the context of convex real projective structures. Theorem 3.1, which was proved independently by Lalley and Sharp with slightly different conditions, gives the asymptotic formula for the number of closed orbits of an Axiom A flow under constraints. Anosov flows are important examples of Axiom A flows and Theorem 3.1 will be a crucial ingredient for our proof of Theorem 1.1.

Fix a convex projective surface ρ . Let $f : T^1X_\rho \rightarrow \mathbb{R}$ be a Hölder continuous function and consider the function $t \rightarrow P(tf)$ for $t \in \mathbb{R}$, where P denotes the pressure. This function is real analytic and strictly convex in t when f is not cohomologous to a constant [47, Proposition 4.12]. Its derivative satisfies

$$P'(tf) := \frac{d}{dt}P(tf) = \int f d\mu_{tf}, \tag{3.1}$$

where μ_{tf} is the equilibrium state for tf . We denote by $J(f)$ the open interval of values $P'(tf)$. If $a \in J(f)$, we let $t_a \in \mathbb{R}$ be the unique real number for which $P'(t_a f) = \int f d\mu_{t_a f} = a$. We ease notation and set $\mu_a = \mu_{t_a f}$.

The following is the key result needed to establish our Theorem 1.1 (and Theorem 1.7).

THEOREM 3.1. (Lalley [39, Theorem I], Sharp [57, Theorem 1]) *Let $f : T^1X_\rho \rightarrow \mathbb{R}$ be an independent Hölder continuous function and let $a \in J(f)$. Then, for fixed $\varepsilon > 0$, there is a constant $C = C(f, \varepsilon)$ such that*

$$\#\{\tau : \lambda(\tau) \in (x, x + \varepsilon), \lambda(f, \tau) \in (ax, ax + \varepsilon)\} \sim C \frac{\exp(h(\mu_a)x)}{x^{3/2}}.$$

Proof. Recall that the Hilbert geodesic flow on T^1X_ρ is Anosov and, as pointed out at the end of §2, weakly mixing. Thus, we can apply Lalley and Sharp’s results which hold for all weakly mixing Axiom A flows. □

Remark 3.2. The constant $C = C(f, \varepsilon) > 0$ has the same order of magnitude as ε^2 and is related to $P''(t_a f)$. See [39, Theorem 5] and [56].

We introduce the concepts of *pressure intersection* and *renormalized pressure intersection* which we will need for the proof of Theorem 1.1.

Definition 3.3. Let ρ_1 and ρ_2 be two convex projective structures and let $f : T^1X_{\rho_1} \rightarrow \mathbb{R}$ be a Hölder continuous reparameterization function. The *pressure intersection* of ρ_1 and ρ_2 is

$$\mathbf{I}(\rho_1, \rho_2) := \int f \, d\mu_{\Phi^{\rho_1}},$$

where $\mu_{\Phi^{\rho_1}}$ is the measure of maximal entropy for Φ^{ρ_1} . The *renormalized pressure intersection* of ρ_1 and ρ_2 is

$$\mathbf{J}(\rho_1, \rho_2) := \frac{h^H(\rho_2)}{h^H(\rho_1)} \mathbf{I}(\rho_1, \rho_2).$$

By Livšić’s theorem, the definitions of pressure intersection and renormalized pressure intersection do not depend on the choice of reparameterization function (see Remark 2.8).

PROPOSITION 3.4. [14, Proposition 3.8] *For every $\rho_1, \rho_2 \in \mathfrak{C}(S)$, the renormalized pressure intersection is such that*

$$\mathbf{J}(\rho_1, \rho_2) \geq 1$$

with equality if and only if $L_{\rho_1}^H = L_{\rho_2}^H$.

Proof. Notice that by Livšić’s theorem, $L_{\rho_1}^H = L_{\rho_2}^H$ if and only if the function $(h^H(\rho_2)/h^H(\rho_1))f$ is Livšić cohomologous to the constant function 1. Then the proposition is a direct application of [14, Proposition 3.8]. □

Now we are ready to prove our main theorem stated in the introduction and repeated below for the convenience of the reader.

THEOREM 1.1. *Fix a precision $\varepsilon > 0$. Consider two convex projective structures ρ_1 and ρ_2 on a surface S with distinct renormalized Hilbert length spectra $L_{\rho_1}^H \neq L_{\rho_2}^H$. There exist constants $C = C(\varepsilon, \rho_1, \rho_2) > 0$ and $M = M(\rho_1, \rho_2) \in (0, 1)$ such that*

$$\#\{[\gamma] \in [\Gamma] \mid L_{\rho_1}^H([\gamma]) \in (x, x + h^H(\rho_1)\varepsilon), L_{\rho_2}^H([\gamma]) \in (x, x + h^H(\rho_2)\varepsilon)\} \sim C \frac{e^{Mx}}{x^{3/2}},$$

where $f(x) \sim g(x)$ means $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$.

Proof. This proof is inspired by the proof for hyperbolic surfaces in [56]. Our first goal is to show that for the reparameterization function $f_{\rho_1}^{\rho_2}$ described in Lemma 2.3, the value a in Theorem 3.1 can be chosen to be $h^H(\rho_1)/h^H(\rho_2)$. To ease notation, we set $f = f_{\rho_1}^{\rho_2}$ and we write Φ for the Hilbert geodesic flow Φ^{ρ_1} on $T^1X_{\rho_1}$.

By Lemma 2.6, we have that

$$0 = P(-h^H(\rho_2)f) = h(\mu_{-h^H(\rho_2)f}) - \int h^H(\rho_2)f \, d\mu_{-h^H(\rho_2)f},$$

where $\mu_{-h^H(\rho_2)f}$ is the equilibrium state of $-h^H(\rho_2)f$. Hence,

$$h^H(\rho_2) \int f d\mu_{-h^H(\rho_2)f} = h(\mu_{-h^H(\rho_2)f})$$

and

$$\int f d\mu_{-h^H(\rho_2)f} = \frac{h(\mu_{-h^H(\rho_2)f})}{h^H(\rho_2)} \leq \frac{h^H(\rho_1)}{h^H(\rho_2)}. \tag{3.2}$$

Notice the equality can be attained only when $\mu_{-h^H(\rho_2)f} = \mu_\Phi$, where μ_Φ is the (Bowen–Margulis) measure of maximal entropy for the geodesic flow Φ . This happens only if $-h^H(\rho_2)f$ is cohomologous to a constant which, by integrating, implies the length spectrum $\ell_{\rho_1}^H$ is a multiple of $\ell_{\rho_2}^H$. This is impossible by Lemma 2.11 and therefore the inequality is strict.

However, by Proposition 3.4, we have that

$$1 \leq \mathbf{J}(\rho_1, \rho_2) = \frac{h^H(\rho_2)}{h^H(\rho_1)} \mathbf{I}(\rho_1, \rho_2) = \frac{h^H(\rho_2)}{h^H(\rho_1)} \int f d\mu_\Phi.$$

This yields

$$\int f d\mu_\Phi \geq \frac{h^H(\rho_1)}{h^H(\rho_2)}. \tag{3.3}$$

Moreover, the equality is attained only when $h^H(\rho_2)f$ is cohomologous to $h^H(\rho_1)$ which is impossible given our hypotheses.

Combining the inequalities (3.2) and (3.3), together with the fact that $J(f)$ is an open interval, we conclude that $(h^H(\rho_1)/h^H(\rho_2) - \delta, h^H(\rho_1)/h^H(\rho_2) + \delta) \subset J(f)$ for small $\delta > 0$. In particular, because $P'(tf)$ is a strictly increasing continuous function, there exists some $t_{a_0} \in (-h^H(\rho_2), 0)$ such that $a_0 = P'(t_{a_0}f) = h^H(\rho_1)/h^H(\rho_2)$, as desired.

We now show that $0 < h(\mu_{a_0}) < h^H(\rho_1)$. Because the Hilbert geodesic flow is Anosov, it has positive entropy with respect to any equilibrium state of a Hölder reparameterization, so $0 < h(\mu_{a_0}) \leq h(\mu_\Phi) = h^H(\rho_1)$ and the equality occurs if and only if $\mu_a = \mu_\Phi$ is the measure of maximal entropy. However, as shown above,

$$\int f d\mu_\Phi > \frac{h^H(\rho_1)}{h^H(\rho_2)} = \int f d\mu_{a_0}.$$

This shows that μ_{a_0} can not be the Bowen–Margulis measure and hence $h(\mu_{a_0}) < h^H(\rho_1)$.

Thanks to Lemma 2.11 and Theorem 3.1, we can conclude that for ρ_1, ρ_2 convex projective structures such that $\rho_2 \neq \rho_1, \rho_1^*$, there exists $\tilde{C} = \tilde{C}(\rho_1, \rho_2, \varepsilon)$ such that

$$\begin{aligned} & \#\left\{[\gamma] \in [\Gamma]: \ell_{\rho_1}^H([\gamma]) \in (y, y + \varepsilon), \ell_{\rho_2}^H([\gamma]) \in \left(\frac{h^H(\rho_1)}{h^H(\rho_2)}y, \frac{h^H(\rho_1)}{h^H(\rho_2)}y + \varepsilon\right)\right\} \\ & \sim \tilde{C} \frac{\exp(h(\mu_{a_0})y)}{y^{3/2}}. \end{aligned}$$

Setting $x = h^H(\rho_1)y$ and clearing denominators, we have the desired statement with $C = h^H(\rho_1)^{3/2}\tilde{C}$ and $M = (h(\mu_{a_0})/h^H(\rho_1)) \in (0, 1)$. □

4. Correlation number, Manhattan curve, and decay of correlation number

In this section, we focus on the correlation number $M(\rho_1, \rho_2)$ from Theorem 1.1. In Theorem 4.2, we express the correlation number in terms of Burger’s *Manhattan curve* [17] generalizing the main result in [58]. In §4.2, we prove that the correlation number is *not* uniformly bounded away from zero in $\mathfrak{C}(S)$. Specifically, we provide two sequences of *hyperbolic structures* along which the correlation number goes to zero, thus answering a question from [56].

4.1. *Manhattan curve and correlation number.* Let ρ_1 and ρ_2 be two convex projective structures. The *Manhattan curve* of ρ_1, ρ_2 is the curve $\mathcal{C}(\rho_1, \rho_2)$ that bounds the convex set

$$\left\{ (a, b) \in \mathbb{R}^2 : \sum_{[\gamma] \in [\Gamma]} e^{-(a\ell_{\rho_1}^H([\gamma]) + b\ell_{\rho_2}^H([\gamma]))} < +\infty \right\}.$$

Equivalently (see [58]), the Manhattan curve can be defined in terms of the pressure function and the reparameterization function $f = f_{\rho_1}^{\rho_2} : T^1 X_{\rho_1} \rightarrow \mathbb{R}$ from Lemma 2.3 as

$$\mathcal{C}(\rho_1, \rho_2) = \{(a, b) \in \mathbb{R}^2 : P(-a - bf) = 0\} = \{(a, b) \in \mathbb{R}^2 : P(-bf) = a\}.$$

The next theorem collects the properties of the Manhattan curve we will need which were first discussed in the setting of representations into isometry groups of rank one symmetric spaces (see [17, Theorem 1]). We recall that a convex curve is a simple curve in the Euclidean plane which lies completely on one side of any of its tangent lines. The boundary of a convex set is always a convex curve. A strictly convex curve is a convex curve that does not contain any line segments.

THEOREM 4.1. *Let $\rho_1, \rho_2 \in \mathfrak{C}(S)$ denote two convex projective surfaces.*

- (1) *The Manhattan curve is a real analytic convex curve passing through the points $(h^H(\rho_1), 0)$ and $(0, h^H(\rho_2))$.*
- (2) *The normals to the Manhattan curve at the points $(h^H(\rho_1), 0)$ and $(0, h^H(\rho_2))$ have slopes $\mathbf{I}(\rho_1, \rho_2)$ and $1/\mathbf{I}(\rho_2, \rho_1)$, respectively.*
- (3) *The Manhattan curve is strictly convex if and only if $\rho_2 \neq \rho_1, \rho_1^*$.*

Proof. For item (1), convexity of the Manhattan curve follows from its definition since it bounds a convex set. However, because the Manhattan curve is the graph of a function q defined implicitly as $P(-q(s)f) = s$, the implicit function theorem and real analyticity of the pressure function yield real analyticity of the Manhattan curve.

For item (2), we notice $P(0) = h^H(\rho_1)$ and $P(-h^H(\rho_2)f) = 0$ (recall Lemma 2.6). Therefore, $\mathcal{C}(\rho_1, \rho_2)$ passes through $(h^H(\rho_1), 0)$ and $(0, h^H(\rho_2))$. To see that the normal to the Manhattan curve at $(h^H(\rho_1), 0)$ has slope $\mathbf{I}(\rho_1, \rho_2)$, we differentiate the equality $P(-q(s)f) = s$ using equation (3.1) at $(h^H(\rho_1), 0)$ to obtain

$$1 = \frac{d}{ds} \Big|_{s=h^H(\rho_1)} P(-q(s)f) = \left(- \int f d\mu_{\Phi^{\rho_1}} \right) q'(h^H(\rho_1)).$$

Therefore, the normal to the Manhattan curve at $(h^H(\rho_1), 0)$ has slope $\mathbf{I}(\rho_1, \rho_2) = \int f d\mu_{\Phi^{\rho_1}}$. The rest of item (2) follows by symmetry.

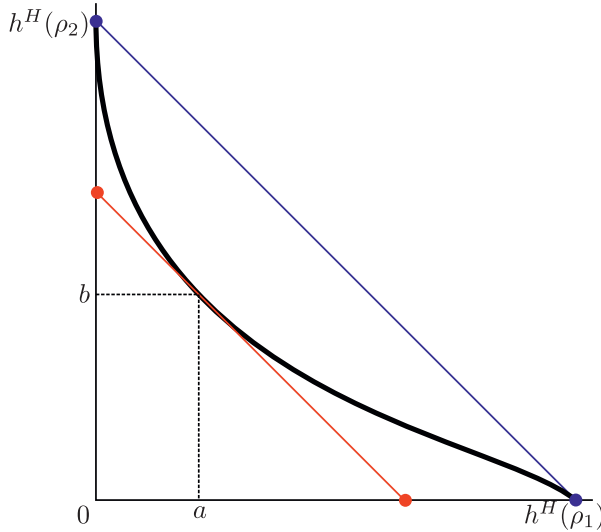


FIGURE 1. Manhattan curve and the point (a, b) described in Theorem 4.2.

For item (3), if $\mathcal{C}(\rho_1, \rho_2)$ is not strictly convex, then by real analyticity, it must be a line segment. From item (2), we have $\mathbf{I}(\rho_1, \rho_2) = 1/\mathbf{I}(\rho_2, \rho_1)$. Therefore, we conclude $\mathbf{J}(\rho_1, \rho_2) = 1/\mathbf{J}(\rho_2, \rho_1)$. By Proposition 3.4, we conclude $\mathbf{J}(\rho_1, \rho_2) = \mathbf{J}(\rho_2, \rho_1) = 1$ and therefore $L_{\rho_1}^H = L_{\rho_2}^H$. Then Remark 1.2 gives the results. \square

Sharp [58] expressed the correlation number $M(\rho_1, \rho_2)$ in terms of the Manhattan curve for hyperbolic structures. We establish an analogous result for convex real projective structures.

THEOREM 4.2. *Let ρ_1 and ρ_2 be convex projective structures with distinct renormalized Hilbert length spectra. Then, their correlation number can be written as*

$$M(\rho_1, \rho_2) = \frac{a}{h^H(\rho_1)} + \frac{b}{h^H(\rho_2)},$$

where $(a, b) \in \mathcal{C}(\rho_1, \rho_2)$ is the unique point on the Manhattan curve at which the tangent line is parallel to the line passing through $(h^H(\rho_1), 0)$ and $(0, h^H(\rho_2))$. See Figure 1.

Proof. By the proof of Theorem 1.1, the correlation number is such that $h^H(\rho_1)M(\rho_1, \rho_2) = h(\mu_{a_0})$, where $a_0 = \int f d\mu_{a_0} = (h^H(\rho_1)/h^H(\rho_2))$. By definition,

$$h(\mu_{a_0}) = P(t_{a_0}f) - t_{a_0} \frac{h^H(\rho_1)}{h^H(\rho_2)}.$$

Note that $\mathcal{C}(\rho_1, \rho_2)$ is the graph of a real analytic function q defined implicitly as $P(-q(s)f) = s$. Setting $q(s_0) = -t_{a_0}$, it follows that

$$M(\rho_1, \rho_2) = \frac{h(\mu_{a_0})}{h^H(\rho_1)} = \frac{P(-q(s_0)f)}{h^H(\rho_1)} + \frac{q(s_0)}{h^H(\rho_2)} = \frac{s_0}{h^H(\rho_1)} + \frac{q(s_0)}{h^H(\rho_2)}.$$

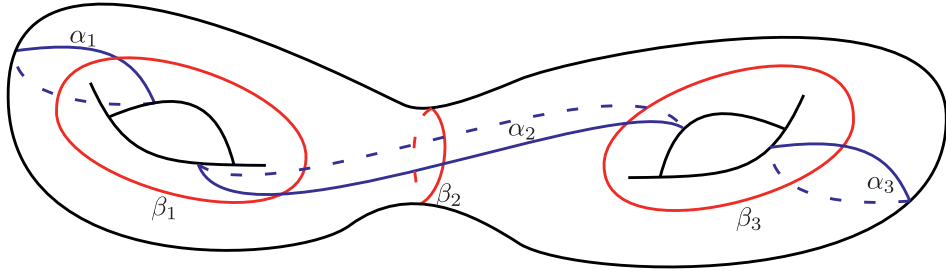


FIGURE 2. Closed curves $\alpha_1, \alpha_2,$ and α_3 and $\beta_1, \beta_2,$ and β_3 give two pair-of-pants decompositions whose union fills the surface of genus two.

Moreover, observe that

$$1 = \frac{d}{ds} P(-q(s)f) = \left(- \int f d\mu_{-q(s)f} \right) q'(s).$$

We conclude by recalling that $\int f d\mu_{\iota_{a_0} f} = (h^H(\rho_1)/h^H(\rho_2))$ and that the line passing through $(h^H(\rho_1), 0)$ and $(0, h^H(\rho_2))$ has slope $-(h^H(\rho_2)/h^H(\rho_1))$. \square

Remark 4.3. It follows from strict convexity of the Manhattan curve and Theorem 4.2 that $M(\rho_1, \rho_2) \in (0, 1)$. This fact is independently proved in Theorem 1.1.

4.2. *Decay of correlation number.* This section is dedicated to the proof of Theorem 1.3 from the introduction, which we restate here for the convenience of the reader.

THEOREM 1.3. *There exist sequences $(\rho_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in the Teichmüller space $\mathcal{T}(S)$ such that the correlation numbers $M(\rho_n, \eta_n)$ satisfy*

$$\lim_{n \rightarrow \infty} M(\rho_n, \eta_n) = 0.$$

Proof. We construct two special families of hyperbolic structures ρ_n, η_n and consider their corresponding geodesic currents $\nu_{\rho_n}, \nu_{\eta_n}$ as in §2.5. Our proof proceeds in two steps. First, we take geodesic currents $\nu_{\rho_n} + \nu_{\eta_n}$ given by the sum of the currents of ρ_n and η_n and show that their exponential growth rates satisfy $\lim_{n \rightarrow \infty} h(\nu_{\rho_n} + \nu_{\eta_n}) = 0$. Then we show that this condition implies that the correlation number $M(\rho_n, \eta_n)$ goes to zero as well.

We consider two pair-of-pants decompositions (α_i) and (β_i) whose union is filling on a hyperbolic structure ρ_0 . See Figure 2 for an example. A family of simple closed curves is filling if the complement of their union consists of topological discs. We take $(\rho_n)_{n=1}^\infty$ to be a sequence of hyperbolic structures obtained by pinching all α_i on ρ_0 so that the hyperbolic length $\ell_{\rho_n}(\alpha_i) = \epsilon_n$ with $\epsilon_n \rightarrow 0$ when $n \rightarrow \infty$. Similarly, we take (η_n) to be another sequence of hyperbolic structures obtained by pinching all β_i on ρ_0 so that the hyperbolic length $\ell_{\eta_n}(\beta_i) = \epsilon_n$. Note in such cases, we have that $\ell_{\rho_n}^H = \ell_{\rho_n}$ and $\ell_{\eta_n}^H = \ell_{\eta_n}$ and that the topological entropy $h^H(\rho_n) = h^H(\eta_n) = 1$ for all n . We now proceed to prove $\lim_{n \rightarrow \infty} h(\nu_{\rho_n} + \nu_{\eta_n}) = 0$.

By definition, the systole of $\nu_{\rho_n} + \nu_{\eta_n}$ is equal to $L_n = \inf_{[\gamma] \in [\Gamma]} \{ \ell_{\rho_n}^H([\gamma]) + \ell_{\eta_n}^H([\gamma]) \}$. Note that for a fixed n , $\#\{[\gamma] \in [\Gamma] \mid \ell_{\rho_n}^H([\gamma]) + \ell_{\eta_n}^H([\gamma]) < T\} < \infty$ for all $T > 0$. In

other words, the geodesic current $\nu_{\rho_n} + \nu_{\eta_n}$ is period minimizing and so we can apply Theorem 2.12 to find a constant C depending only on the topology of S such that

$$h(\nu_{\rho_n} + \nu_{\eta_n}) \leq \frac{C}{L_n}.$$

Therefore, to show $\lim_{n \rightarrow \infty} h(\nu_{\rho_n} + \nu_{\eta_n}) = 0$, it suffices to show that $\lim_{n \rightarrow \infty} L_n = \infty$. Because of the filling condition, the geodesic representative of any $[\gamma] \in [\Gamma]$ must intersect transversally either curves in (α_i) or curves in (β_i) . By the collar lemma (see [19, Ch. 4]), each geodesic representative of α_i (respectively β_i) for ρ_n (respectively η_n) is enclosed in a standard collar neighborhood of width approximately $\log(1/\epsilon_n)$. In particular, every closed curve traverses a collar neighborhood of width approximately $\log(1/\epsilon_n)$ for the hyperbolic metric ρ_n or the hyperbolic metric η_n which implies that $\lim_{n \rightarrow \infty} L_n = \infty$.

We now show that $\lim_{n \rightarrow \infty} h(\nu_{\rho_n} + \nu_{\eta_n}) = 0$ implies that the correlation number goes to zero as well. Consider the reparameterization functions $f_{\rho_n}^{\eta_n}$ as in Lemma 2.3 and notice that $\lambda(1 + f_{\rho_n}^{\eta_n}, \tau) = \ell_{\rho_n}^H([\gamma]) + \ell_{\eta_n}^H([\gamma])$ for every periodic orbit corresponding to $[\gamma] \in [\Gamma]$. By Lemma 2.6, for all n ,

$$P(-h(\nu_{\rho_n} + \nu_{\eta_n}) - h(\nu_{\rho_n} + \nu_{\eta_n})f_{\rho_n}^{\eta_n}) = P(-h(\nu_{\rho_n} + \nu_{\eta_n})(1 + f_{\rho_n}^{\eta_n})) = 0.$$

We have that $(h(\nu_{\rho_n} + \nu_{\eta_n}), h(\nu_{\rho_n} + \nu_{\eta_n})) \in \mathcal{C}(\rho_n, \eta_n)$ thanks to the characterization of the Manhattan curve in terms of the pressure function. If the line $y + x = 2h(\nu_{\rho_n} + \nu_{\eta_n})$ is tangent to the Manhattan curve $\mathcal{C}(\rho_n, \eta_n)$, then by Theorem 4.2, we obtain $M(\rho_n, \eta_n) = 2h(\nu_{\rho_n} + \nu_{\eta_n})$. If not, then the line $y + x = 2h(\nu_{\rho_n} + \nu_{\eta_n})$ must intersect transversally $\mathcal{C}(\rho_n, \eta_n)$ at two points. See Figure 3. Now, by the mean value theorem and the convexity of the Manhattan curve, we have from Theorem 4.2 that there exists $0 < a_n, b_n < 2h(\nu_{\rho_n} + \nu_{\eta_n})$ such that

$$0 < M(\rho_n, \eta_n) = a_n + b_n < 2h(\nu_{\rho_n} + \nu_{\eta_n}).$$

This immediately implies that $M(\rho_n, \eta_n)$ goes to zero as n goes to infinity. □

5. The renormalized Hilbert length along cubic rays

5.1. *Cubic rays and affine spheres.* Building on work of Hitchin [33], Labourie [38], and Loftin [43] have independently parameterized $\mathcal{C}(S)$ as the vector bundle of holomorphic cubic differentials over $\mathcal{T}(S)$. This is called the Labourie–Loftin’s parameterization. We briefly recall this parameterization since we will use it in this subsection to study the correlation number. The reader can refer to [38, 43] for a detailed construction. This parameterization shows that the space of pairs formed by a complex structure J on S and a J -holomorphic cubic differential q is in one-to-one correspondence with the space of convex real projective structures on S .

Because of the classical correspondence between hyperbolic structures and complex structures, we will sometimes blur the difference between a Riemann surface X and a hyperbolic structure X . We recall that a holomorphic cubic differential q is a holomorphic section of $K \otimes K \otimes K$, where K is the canonical line bundle of a Riemann surface X . Locally, a holomorphic cubic differential can be written in complex coordinate charts

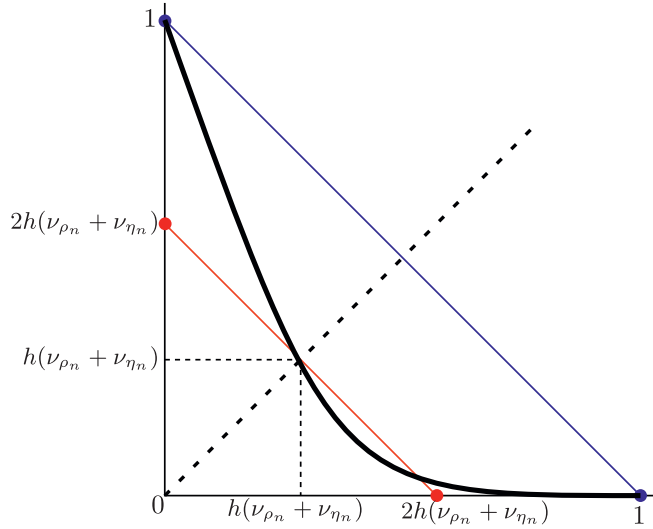


FIGURE 3. Manhattan curve and the point $(h(\nu_{\rho_n} + \nu_{\eta_n}), h(\nu_{\rho_n} + \nu_{\eta_n}))$ described in the proof of Theorem 1.3.

as $q = q(z)dz \otimes dz \otimes dz$ (often written as $q(z)dz^3$) with $q(z)$ holomorphic. The cubic differential q is invariant under change of coordinates, meaning that if we pick a different coordinate chart w , then $q(w)dw^3 = q(z)dz^3$. The hyperbolic metric σ that corresponds to the complex structure of X induces a pointwise Hermitian metric $\langle \cdot, \cdot \rangle$ for holomorphic cubic differentials on X defined as follows:

$$\langle q_1, q_2 \rangle = \frac{q_1 \overline{q_2}}{\sigma^3}.$$

The L^2 -norm of a cubic differential q is defined as

$$\|q\|_X^2 = \int_X \langle q, q \rangle d\sigma.$$

The correspondence between the space of pairs of complex structures on S and holomorphic cubic differentials on associated Riemann surfaces with the set of convex real projective structures on S goes through a geometrical object invariant under affine transformations, called an *affine sphere*. We briefly describe the relation. Consider a pair (J, q) , where J is a complex structure and q is a holomorphic cubic differential on the Riemann surface X_J associated to J . Then one can obtain an affine sphere, which is a hypersurface in \mathbb{R}^3 that is invariant under affine transformations, by solving a second-order elliptic partial differential equation (PDE) called Wang’s equation (see [43, §4]). The affine sphere can be projected to \mathbb{RP}^2 so as to obtain the developing image Ω_ρ of some convex projective structure ρ . Conversely, given a convex projective structure ρ and its developing image Ω_ρ , we can construct an affine sphere by solving a Monge–Ampère equation (see [43, §7]). The affine sphere will provide a complex structure J and a holomorphic cubic differential q on X_J .

Since hyperbolic structures correspond to complex structures, the above identification provides a way to parameterize the space of convex projective structures $\mathfrak{C}(S)$ as the vector bundle of holomorphic cubic differentials over $\mathcal{T}(S)$. In particular, we can fix a non-zero cubic differential q and consider the associated family of representations $(\rho_t)_{t \in \mathbb{R}}$ in $\mathfrak{C}(S)$ parameterized by $(tq)_{t \in \mathbb{R}}$. When $t \geq 0$, we call such a family a *cubic ray*. In this section, we study the properties of the renormalized Hilbert length spectrum along cubic rays in $\mathfrak{C}(S)$.

Note that we can use the holomorphic data introduced above to describe when two convex projective structures ρ_1 and ρ_2 have the same renormalized Hilbert length spectra (see Remark 1.2). Fixing a Riemann surface X_J , the case $\rho_1 = \rho_2$ happens when their corresponding holomorphic cubic differentials q_1 and q_2 with respect to X_J are equal in the Labourie–Loftin’s parameterization. However, if ρ_1 is the contragredient of ρ_2 , then their corresponding holomorphic objects must be (X_J, q_1) and $(X_J, q_2) = (X_J, -q_1)$, respectively. For a detailed explanation of this fact, we refer the reader to [42, §8].

For the remainder of this section, when there is no ambiguity, we ease notation by writing $h_t = h^H(\rho_t)$ for the topological entropy, and $L_t^H = L_{\rho_t}^H$ for the renormalized Hilbert length spectrum of a cubic ray $(\rho_t)_{t \geq 0}$. With these notation, ρ_0 is the hyperbolic structure associated to X_0 . The entropy h_0 equals 1 and the renormalized Hilbert length spectrum L_0^H is the X_0 -length spectrum.

The following observation is important and follows immediately from the work of Tholozan [59].

LEMMA 5.1. *Let $(\rho_t)_{t > 0}$ be a cubic ray. There exists $D > 1$ such that for all $t \geq 0$,*

$$\frac{1}{D}L_0^H \leq L_t^H \leq DL_0^H.$$

Proof. Tholozan [59, Theorem 3.9] proves that there exists $B > 1$ such that for all $t > 0$,

$$\frac{1}{B}t^{1/3}\ell_0^H \leq \ell_t^H \leq Bt^{1/3}\ell_0^H. \tag{5.1}$$

In turn, this implies that for all $T \geq 0$ and $t > 0$,

$$\#\left\{[\gamma] \mid \ell_0^H([\gamma]) \leq \frac{1}{Bt^{1/3}}T\right\} \leq \#\{[\gamma] \mid \ell_t^H([\gamma]) \leq T\} \leq \#\left\{[\gamma] \mid \ell_0^H([\gamma]) \leq \frac{B}{t^{1/3}}T\right\}.$$

Since $h_0 = 1$, by definition of topological entropy, the above inequalities imply that for all $t > 0$,

$$\frac{1}{Bt^{1/3}} \leq h_t \leq \frac{B}{t^{1/3}}. \tag{5.2}$$

The result follows by taking $D = B^2$ and combining the inequalities (5.1) and (5.2). The case for $t = 0$ is obvious. □

The following example points out a problem with comparing un-renormalized length spectra and partially motivates our study of the renormalized Hilbert length spectrum. In particular, equation (5.3) below should be compared to the correlation theorem.

Example 5.2. Fix $\epsilon > 0$. Let ρ_0 be a Fuchsian representation in $\mathfrak{C}(S)$ and consider a cubic ray $(\rho_t)_{t \geq 0} \subset \mathfrak{C}(S)$. Consider any $t > 0$. Then,

$$\lim_{x \rightarrow \infty} \#\{[\gamma] \in [\Gamma] \mid \ell_0^H([\gamma]) \in (x, x + \epsilon), \ell_t^H([\gamma]) \in (x, x + \epsilon)\} = 0. \tag{5.3}$$

Proof. For $t > 0$, consider $\delta > 0$ small enough so that $h_t < 1 < D - \delta$, where D is as in Lemma 5.1. Here we recall that $h^H(\rho)$ is strictly less than one when $\rho \in \mathfrak{C}(S)$ is not in the Fuchsian locus (see [22]). There exists M sufficiently large such that $(Dx/x + \epsilon) \geq D - \delta$ for all $x > M$. Therefore, for all $x > M$, if $x < \ell_0^H([\gamma]) < x + \epsilon$, then

$$\ell_t^H([\gamma]) \geq \frac{D}{h_t} \ell_0^H([\gamma]) > \frac{D}{D - \delta} x \geq x + \epsilon.$$

This shows that $\{[\gamma] \in [\Gamma] \mid \ell_0^H([\gamma]) \in (x, x + \epsilon), \ell_t^H([\gamma]) \in (x, x + \epsilon)\} = \emptyset$ for all $x > M$, which implies that equation (5.3) holds. \square

5.2. Cubic rays and correlation numbers. In contrast with Theorem 1.3, we show some instances in which the correlation numbers $M(\rho_t, \eta_t)$ arising from two different cubic rays are uniformly bounded away from zero. We first introduce a convenient lemma that will be used in the proof of Theorems 1.4 and 1.5.

LEMMA 5.3. *Suppose ρ and η are different convex real projective structures such that $\rho^* \neq \eta$. Let ν_ρ and ν_η be the renormalized Hilbert currents associated to ρ and η , respectively. For $s \in [0, 1]$, let u_s denote the geodesic current $s\nu_\rho + (1 - s)\nu_\eta$. Then there exists a unique $s_0 \in (0, 1)$ such that*

$$M(\rho, \eta) = h(u_{s_0}).$$

Proof. Since $\eta \neq \rho, \rho^*$, the Manhattan curve $\mathcal{C}(\rho, \eta)$ is strictly convex. The line $(x/h^H(\rho)) + (y/h^H(\eta)) = 1$ intersects the Manhattan curve in the first quadrant only at the points $(h^H(\rho), 0)$ and $(0, h^H(\eta))$. It follows that for every $s \in [0, 1]$, the straight line connecting $(sh^H(\rho), (1 - s)h^H(\eta))$ and the origin must intersect $\mathcal{C}(\rho, \eta)$. By Lemma 2.6,

$$\begin{aligned} &P(-h(u_s)sh^H(\rho) - h(u_s)(1 - s)h^H(\eta)f_\rho^\eta) \\ &= P(-h(u_s)(sh^H(\rho) + (1 - s)h^H(\eta)f_\rho^\eta)) = 0, \end{aligned}$$

where f_ρ^η is the reparameterization function from Lemma 2.3. It follows from the definition of the Manhattan curve that every point on $\mathcal{C}(\rho, \eta)$ in the first quadrant can be written in the form of $(h(u_s)sh^H(\rho), h(u_s)(1 - s)h^H(\eta))$ for some $s \in [0, 1]$. By Theorem 4.2, there exists $s_0 \in (0, 1)$ such that

$$M(\rho, \eta) = \frac{s_0 h(u_{s_0}) h^H(\rho)}{h^H(\rho)} + \frac{(1 - s_0) h(u_{s_0}) h^H(\eta)}{h^H(\eta)} = h(u_{s_0}). \tag{5.4} \quad \square$$

We first study the case of two cubic rays which lie in two different fibers of the vector bundle $\mathfrak{C}(S) \rightarrow \mathcal{T}(S)$ given by the Labourie–Loftin parameterization of the space of convex projective surfaces.

THEOREM 1.4. *Let $(\rho_t)_{t \geq 0}$, $(\eta_r)_{r \geq 0}$ be two cubic rays associated to two different hyperbolic structures $\rho_0 \neq \eta_0$. Then, there exists a constant $C > 0$ such that for all $t, r \geq 0$,*

$$M(\rho_t, \eta_r) \geq CM(\rho_0, \eta_0).$$

Proof. Fix $t, r \geq 0$ and write $\rho = \rho_t$ and $\eta = \eta_r$, for simplicity. Note that by hypothesis, $L_\rho^H \neq L_\eta^H$ and $L_{\rho_0}^H \neq L_{\eta_0}^H$. This is because $L_\rho^H = L_\eta^H$ if and only if ρ equals η or its contragredient involution η^* . Clearly, $\rho \neq \eta$. Moreover, $\rho = \eta^*$ only happens if their associated pairs of complex structures and holomorphic cubic differentials by the above construction are (X_0, q) and $(X_0, -q)$, respectively (see [42, §8]), which is excluded as well. For $s \in [0, 1]$, let u_s denote the geodesic current given by $s\nu_\rho + (1 - s)\nu_\eta$, where ν_ρ and ν_η are the renormalized Hilbert geodesic currents of ρ and η , respectively. Denote by $h(u_s)$ the exponential growth rate for the geodesic current u_s .

By Lemma 5.1, there exist constants D_1, D_2 depending on ρ_0 and η_0 , respectively, such that

$$sL_\rho^H + (1 - s)L_\eta^H \leq sD_1L_{\rho_0}^H + (1 - s)D_2L_{\eta_0}^H \leq \max\{D_1, D_2\}(sL_{\rho_0}^H + (1 - s)L_{\eta_0}^H).$$

Set $C = 1/\max\{D_1, D_2\}$ and $w_s = s\nu_{\rho_0} + (1 - s)\nu_{\eta_0}$, where ν_{ρ_0} and ν_{η_0} are the renormalized Hilbert geodesic currents of ρ_0 and η_0 , respectively. Then,

$$h(u_s) \geq Ch(w_s).$$

By Lemma 2.6, the intersection between the Manhattan curve $\mathcal{C}(\rho_0, \eta_0)$ and the line passing through the origin and the point $(s, (1 - s))$ has coordinates $(sh(w_s), (1 - s)h(w_s))$ and lies on the line $y + x = h(w_s)$. Then, by Theorem 4.2,

$$h(w_s) \geq M(\rho_0, \eta_0).$$

See Figure 4. Finally, by Lemma 5.3, there exists $s_0 \in (0, 1)$ such that

$$M(\rho, \eta) = h(u_{s_0}) \geq Ch(w_{s_0}) \geq CM(\rho_0, \eta_0),$$

which concludes the proof. □

Finally, we prove Theorem 1.5 from the introduction which can be seen as a ‘limiting case’ of Theorem 1.4 if we avoid pairs of sequences related by the contragredient involution.

THEOREM 1.5. *Let $(\rho_t)_{t \geq 0}$ and $(\eta_r)_{r \geq 0}$ be two cubic rays associated to two different holomorphic cubic differentials q_1 and q_2 on a hyperbolic structure X_0 such that q_1, q_2 have unit L^2 -norm with respect to X_0 and $q_1 \neq -q_2$. Then the correlation number $M(\rho_t, \eta_r)$ is uniformly bounded away from zero for all $t > 0$ and $r > 0$.*

Proof. This proof is similar to the proof of Theorem 1.4. We note that $\rho_0 = \eta_0$ and $L_{\rho_0}^H = L_{\eta_0}^H = L_0^H$. Recall from Lemma 5.1, there exists $D_1, D_2 > 1$ such that for all $t, r \geq 0$,

$$\frac{1}{D_1}L_0^H \leq L_{\rho_t}^H \leq D_1L_0^H \quad \text{and} \quad \frac{1}{D_2}L_0^H \leq L_{\eta_r}^H \leq D_2L_0^H.$$

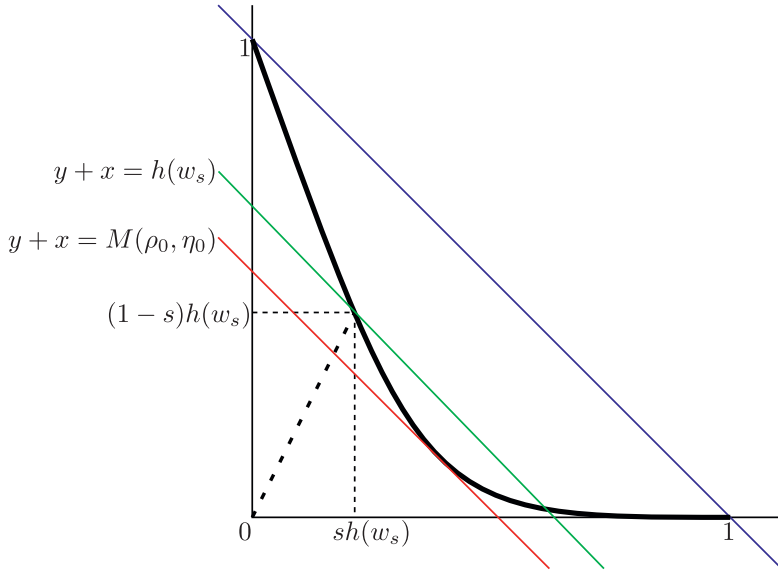


FIGURE 4. The correlation number of ρ_0 and η_0 is less or equal to the exponential growth rate of the geodesic current w_s .

Consider the renormalized Hilbert currents ν_{ρ_t} and ν_{η_r} for ρ_t and η_r , respectively. Let $s \in [0, 1]$ and set the geodesic current $u_{t,r,s} = s\nu_{\rho_t} + (1-s)\nu_{\eta_r}$. As a first step, we show that the entropy of the geodesic current $u_{t,r,s} = s\nu_{\rho_t} + (1-s)\nu_{\eta_r}$ is uniformly bounded away from zero. We have from Lemma 5.1, for all $[\gamma] \in [\Gamma]$,

$$sL_{\rho_t}^H([\gamma]) + (1-s)L_{\eta_r}^H([\gamma]) \leq (sD_1 + (1-s)D_2)L_0^H([\gamma]) \leq \max\{D_1, D_2\}L_0^H([\gamma]).$$

Thus, we obtain $h(u_{t,r,s}) \geq 1/D$ for every $s \in (0, 1)$, where $D = \max\{D_1, D_2\}$.

Next we want to show this implies the correlation number $M(\rho_t, \eta_r)$ is bounded away from zero. Because q_1 and q_2 both have unit L^2 -norm and $q_2 \neq -q_1$, we have $L_{\rho_t}^H \neq L_{\eta_t}^H$ as $\rho_t^* \neq \eta_t$ when $t = r > 0$. Again, here we use the fact that $\rho_t = \eta_t^*$ only if their associated pairs of complex structures and holomorphic cubic differentials (via affine spheres) are (X_0, tq_1) and $(X_0, -tq_1)$, respectively. For each $t > 0$ and $r > 0$, by Lemma 5.3, we can find some $s_0 \in (0, 1)$ such that

$$M(\rho_t, \eta_r) = h(u_{t,r,s_0}).$$

We conclude that $M(\rho_t, \eta_r) \geq 1/D$ for all $t > 0$ and $r > 0$, as desired. □

5.3. *Cubic rays and geodesic currents.* In this section, we observe two properties of renormalized Hilbert currents $\nu_{\rho_t} = h_t\nu_{\rho_t}$ along cubic rays $(\rho_t)_{t \geq 0}$ which readily follow from Lemma 5.1. Let us start by showing that the self-intersection of the renormalized Hilbert geodesic current is uniformly bounded along cubic rays.

PROPOSITION 5.4. *There exists $C > 1$ such that for all $t \geq 0$,*

$$\frac{1}{C} \leq i(v_{\rho_t}, v_{\rho_t}) \leq C,$$

where v_{ρ_t} is the renormalized Hilbert geodesic current of ρ_t .

Proof. Recall from §2.5 that Bonahon proved that $i(v_{\rho_0}, v_{\rho_0}) = -\pi^2 \chi(S)$, where $\chi(S)$ is the Euler characteristic of S . Corollary 5.2 in [15] states that

$$\left(\inf_{[\gamma] \in [\Gamma]} \frac{L_t^H([\gamma])}{L_0^H([\gamma])} \right)^2 \leq \frac{i(v_{\rho_t}, v_{\rho_t})}{i(v_{\rho_0}, v_{\rho_0})} \leq \left(\sup_{[\gamma] \in [\Gamma]} \frac{L_t^H([\gamma])}{L_0^H([\gamma])} \right)^2,$$

so

$$-\pi^2 \chi(S) \left(\inf_{[\gamma] \in [\Gamma]} \frac{L_t^H([\gamma])}{L_0^H([\gamma])} \right)^2 \leq i(v_{\rho_t}, v_{\rho_t}) \leq -\pi^2 \chi(S) \left(\sup_{[\gamma] \in [\Gamma]} \frac{L_t^H([\gamma])}{L_0^H([\gamma])} \right)^2.$$

We conclude by applying Lemma 5.1. □

Bonahon [9] showed that Thurston’s compactification of the Teichmüller space can be understood via geodesic currents. Explicitly, given a diverging sequence of hyperbolic structures m_t , there exists $\lambda_t > 0$ and a geodesic current α with $i(\alpha, \alpha) = 0$, a *measured lamination*, such that $\lambda_t v_{m_t}$ converges up to subsequences to α . In this case, the systole of α vanishes, that is,

$$\text{Sys}(\alpha) = \inf_{[\gamma] \in [\Gamma]} i(\alpha, \delta_\gamma) = 0.$$

Burger *et al* [18] show that there exist diverging sequences of Hilbert geodesic currents which converge projectively to a current α with $\text{Sys}(\alpha) > 0$. We use Lemma 5.1 to show that this happens along cubic rays.

THEOREM 1.6. *As t goes to infinity, the renormalized Hilbert geodesic current v_{ρ_t} along a cubic ray converges, up to passing to a subsequence, to a geodesic current v with $\text{Sys}(v) > 0$.*

Proof. Suppose (α_i) and (β_j) are pair-of-pants decompositions which fill the surface, that is, they are pants decompositions such that the complement of their union is a collection of topological discs. Set $u = \sum_{\gamma \in (\alpha_i) \cup (\beta_j)} \delta_\gamma$ and let $M = \max_{\gamma \in (\alpha_i) \cup (\beta_j)} L_0^H([\gamma])$. For every $t \in \mathbb{R}$, we have

$$i(v_{\rho_t}, u) = \sum_{\gamma \in (\alpha_i) \cup (\beta_j)} L_t^H([\gamma]).$$

By Lemma 5.1, v_t lies in $\{v \in \mathcal{C}(S) \mid i(v, u) \leq (6g - 6)DM\}$ for some $D > 1$. This set is compact by [9, Proposition 4] and by linearity of the intersection number. Thus, v_t converges, up to passing to a subsequence, to $v \in \mathcal{C}(S)$. Applying Lemma 5.1 again, we see that $\text{Sys}(v_{\rho_t})$ is greater or equal to $D^{-1} \text{Sys}(v_{\rho_0}) > 0$. By continuity, the systole $\text{Sys}(v)$ is strictly positive. □

6. Generalization to Hitchin representations

In this section, we illustrate how to generalize the main results of §§3 and 4 to the context of Hitchin representations.

We start with introducing Hitchin components and Hitchin representations. Given $\rho \in \mathcal{T}(S)$, we can postcompose the corresponding holonomy representation $\rho: \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R})$ with the unique (up to conjugation) irreducible representation $i: \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$ given by the action of $A \in \mathrm{PSL}(2, \mathbb{R})$ on the space of degree $d - 1$ homogeneous polynomials in two variables. The Hitchin component $\mathcal{H}_d(S)$ is the connected component of the character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(d, \mathbb{R})) // \mathrm{PSL}(d, \mathbb{R})$ containing $i \circ \rho$. The copy of the Teichmüller space $\mathcal{T}(S)$ embedded in the Hitchin component is its *Fuchsian locus*. Hitchin proves in [33], using Higgs bundles techniques, that $\mathcal{H}_d(S)$ is homeomorphic to an open Euclidean ball of dimension $(2g - 2) \cdot \dim(\mathrm{PSL}(d, \mathbb{R}))$. When $d = 2$, the Hitchin component $\mathcal{H}_2(S)$ coincides with the Teichmüller space $\mathcal{T}(S)$. Choi and Goldman [20] identify the Hitchin component $\mathcal{H}_3(S)$ with the space of convex projective structures on the surface S , which is the main focus of §§2–5.

In this section, we establish the correlation theorem 1.7 for pairs of Hitchin representations for any $d \geq 3$. In this setting, the length spectra will not be defined geometrically as for the Hilbert length spectrum of a convex projective structure, but they will be interpreted as periods of the *Busemann cocycle*, which was first introduced by Quint [51]. Then, we follow a construction of Sambarino [53, 54] to replace the geodesic flow on the unit tangent bundle of a convex projective surface with Sambarino's *translation flows*. These flows are not necessarily Anosov, but fit in the more general framework of *metric Anosov flows*. Nevertheless, the main results coming from thermodynamic formalism needed for this paper still hold in this setting [48, §3].

6.1. Length functions, Busemann cocycles, and entropy. In this section, we define length functions for Hitchin representations and the Busemann cocycles associated to them. We refer to [51, 53] for a more detailed construction.

We need to recall some Lie theory. Let $G = \mathrm{PSL}(d, \mathbb{R})$ and consider the standard choices of Cartan subspace

$$\mathfrak{a} = \{\vec{x} \in \mathbb{R}^d \mid x_1 + \cdots + x_d = 0\}$$

and positive Weyl chamber $\mathfrak{a}^+ = \{\vec{x} \in \mathfrak{a} \mid x_1 \geq \cdots \geq x_d\}$. Let $\lambda: G \rightarrow \mathfrak{a}^+$ be the *Jordan projection*

$$\lambda(g) = (\log \lambda_1(g), \dots, \log \lambda_d(g))$$

consisting of the logarithms of the moduli of the eigenvalues of g in non-increasing order. We will use the following fundamental property of the Jordan projection of the image of a Hitchin representation.

THEOREM 6.1. (Fock–Goncharov [25], Labourie [37]) *Every Hitchin representation $\rho \in \mathcal{H}_d(S)$ is discrete and faithful and for every $[\gamma] \in [\Gamma]$,*

$$\lambda_1(\rho(\gamma)) > \cdots > \lambda_d(\rho(\gamma)) > 0.$$

The *limit cone* \mathcal{L}_ρ of a Hitchin representation $\rho \in \mathcal{H}_d(S)$, introduced in [6], is the closed cone of \mathfrak{a}^+ generated by $\lambda(\rho(\gamma))$. This cone contains all the rays spanned by positive multiples of $\lambda(\rho(\gamma))$ for all $\gamma \in \Gamma$. The (positive) *dual cone* of \mathcal{L}_ρ is defined as $\mathcal{L}_\rho^* = \{\phi \in \mathfrak{a}^* : \phi|_{\mathcal{L}_\rho} \geq 0\}$. For every $i = 1, \dots, d - 1$, the *i*th *simple root* $\alpha_i : \mathfrak{a} \rightarrow \mathbb{R}$, defined by $\alpha_i(\vec{x}) = x_i - x_{i+1}$. We will focus on linear functionals in the set

$$\Delta = \left\{ c_1\alpha_1 + \dots + c_{d-1}\alpha_{d-1} \mid c_i \geq 0, \sum_i c_i > 0 \right\},$$

which is contained in the interior of \mathcal{L}_ρ^* by a result of Potrie–Sambarino [50, Theorem B and Lemma 2.7].

Given ϕ in the interior of \mathcal{L}_ρ^* , the ϕ -length of $[\gamma] \in [\Gamma]$ is defined as

$$\ell_\rho^\phi([\gamma]) = \phi(\lambda(\rho(\gamma))) > 0,$$

and its *exponential growth rate* is

$$h^\phi(\rho) = \limsup_{T \rightarrow \infty} \frac{1}{T} \# \log\{[\gamma] \in [\Gamma] \mid \ell_\rho^\phi([\gamma]) \leq T\}.$$

Sambarino [54] established the following important property of linear functionals in Δ . We include the proof for completeness.

LEMMA 6.2. (Sambarino [54, Lemma 3.2]) *If $\phi \in \Delta$, then $h^\phi(\rho) \in (0, \infty)$.*

Proof. Since we know $h^{\alpha_i}(\rho) = 1$ from [50, Theorem B], by Lemma 2.7 in [50], a linear functional $\varphi \in \mathcal{L}_\rho^*$ has finite and positive entropy if and only if it belongs to the interior of \mathcal{L}_ρ^* . Since elements in Δ are contained in the interior of \mathcal{L}_ρ^* , the entropy $h^\phi(\rho)$ is positive and finite for every $\phi \in \Delta$. □

The ϕ -length function can be realized via the period of the ϕ -Busemann cocycle. To define ϕ -Busemann cocycles, we need to introduce the *Frenet curve* ξ_ρ for a Hitchin representation ρ . A *complete flag* F in \mathbb{R}^d is a maximal nested sequence of subspaces of \mathbb{R}^d . Explicitly, F is a collection of subspaces $F^{(i)}$ of \mathbb{R}^d , with the properties that $F^{(i)} \subset F^{(i+1)}$ and $\dim F^{(i)} = i$ for all $i = 0, 1, \dots, d$. Denote by F_d the space of complete flags in \mathbb{R}^d . Fock and Goncharov [25] and Labourie [37] show that for every Hitchin representation ρ , there exists a unique (up to $\text{PSL}(d, \mathbb{R})$) ρ -equivariant *Frenet curve* $\xi_\rho : \partial\Gamma \rightarrow F_d$ which is bi-Hölder continuous onto its image, transverse, positive, and it satisfies certain contraction properties. Moreover, the Frenet curve is *dynamics preserving*: if $\gamma^+ \in \partial\Gamma$ is the attracting fixed point of $\gamma \in \Gamma$, then $\xi_\rho(\gamma^+)$ is the attracting eigenflag of $\rho(\gamma)$. We refer to [25, Theorem 1.15] and [37, Theorem 4.1] for precise statements.

We are now ready to define ϕ -Busemann cocycles for $\phi \in \Delta$. Fix $F_0 \in F_d$ and observe that for every $F \in F_d$, there exists $k \in \text{SO}(d)$ such that $F = kF_0$. Every $M \in \text{SL}(d, \mathbb{R})$ can be written as $M = L(\exp \sigma(M))U$, for some $L \in \text{SO}(d)$, $\sigma(M) \in \mathfrak{a}$, and U is unipotent and upper triangular. This is known as the Iwasawa decomposition of M . The *Iwasawa cocycle* $B : \text{SL}(d, \mathbb{R}) \times F_d \rightarrow \mathfrak{a}$ is defined as $B(A, F) = \sigma(Ak)$.

The vector valued Busemann cocycle $B_\rho : \Gamma \times \partial\Gamma \rightarrow \mathfrak{a}$ of a Hitchin representation ρ with Frenet curve ξ_ρ is

$$B_\rho(\gamma, x) = B(\rho(\gamma), \xi_\rho(x)).$$

For every linear functional $\phi \in \Delta$, we set $B_\rho^\phi(\gamma, x) = \phi(B_\rho(\gamma, x))$. Lemma 7.5 in [53] directly shows that the cocycle B_ρ^ϕ encodes the ϕ -length via the equality $B_\rho^\phi(\gamma, \gamma^+) = \ell_\rho^\phi([\gamma])$, for every $\gamma \in \Gamma$.

The ϕ -Busemann cocycle is an example of a Hölder cocycle.

Definition 6.3. A Hölder cocycle is a map $c : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ such that

$$c(\gamma\eta, x) = c(\gamma, \eta x) + c(\eta, x)$$

for any $\gamma, \eta \in \Gamma$ and $x \in \partial\Gamma$, and there exists $\alpha \in (0, 1)$ such that $c(\gamma, \cdot)$ is α -Hölder for all $\gamma \in \Gamma$.

For a general Hölder cocycle, let $\ell_c(\gamma) = c(\gamma, \gamma^+)$ denote the c -length (also known as *period*) of $\gamma \in \Gamma$. Two Hölder cocycles are *cohomologous* if they have the same periods. We can define the *exponential growth rate* h_c of a general Hölder cocycle $c : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ as

$$h_c = \limsup_{T \rightarrow \infty} \frac{1}{T} \# \log\{[\gamma] \in [\Gamma] \mid \ell_c(\gamma) \leq T\}.$$

Note that the exponential growth rate $h^\phi(\rho)$ of the linear functional $\phi \in \Delta$ is the same as the exponential growth rate $h_{B_\rho^\phi}$ of the cocycle B_ρ^ϕ . The number $h^\phi(\rho)$ is also called the *topological entropy* of a Hitchin representation ρ with respect to the ϕ -length. Indeed, $h^\phi(\rho)$ is the topological entropy of a *translation flow*, as we recall in the next section.

6.2. Translation flows and metric Anosov flows. In this subsection, we recall how to equip Hitchin representations with translation flows which will allow us to use thermodynamic formalism tools to study them. We start by recalling the construction of Sambarino's translation flow [53], which is analogous to Hopf's parameterization of the geodesic flow of a negatively curved manifold.

Let $\partial^2\Gamma = \{(x, y) \in \partial\Gamma \times \partial\Gamma : x \neq y\}$. The *translation flow* $\{\varphi_t\}_{t \in \mathbb{R}}$ on $\partial^2\Gamma \times \mathbb{R}$ is

$$\varphi_t(x, y, s) = (x, y, s - t).$$

Given a Hölder cocycle $c : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$, the group Γ acts on $\partial^2\Gamma \times \mathbb{R}$ by

$$\gamma(x, y, t) = (\gamma x, \gamma y, t - c(\gamma, y)).$$

The translation flow $\{\varphi_t\}_{t \in \mathbb{R}}$ then descends to the quotient M_c of $\partial^2\Gamma \times \mathbb{R}$ by the Γ -action via c . Sambarino proves a reparameterization theorem for the translation flow φ_t .

First, recall that two (Hölder) continuous flows ψ_t and ψ'_t on compact metric spaces X and Y , respectively, are (*Hölder*) *conjugated* if there exists a (bi-Hölder) homeomorphism $h : X \rightarrow Y$ such that $\psi'_t = h \circ \psi_t \circ h^{-1}$ for every $t \in \mathbb{R}$.

THEOREM 6.4. (Sambarino [53, Theorem 3.2]) *Given a Hölder cocycle c with non-negative periods and $h_c \in (0, \infty)$, the action defined by the cocycle c on $\partial^2\Gamma \times \mathbb{R}$ is proper and co-compact. Moreover, the translation flow φ_t on the quotient space M_c is Hölder conjugated to a Hölder reparameterization of the geodesic flow on T^1X_0 , where X_0 is a(ny) hyperbolic structure on S . The translation flow on M_c is topologically mixing and its topological entropy equals h_c .*

From now on, we will always assume that the cocycle c is of the form B_ρ^ϕ for some Hitchin representation ρ and $\phi \in \Delta$. In this case, the hypotheses of Theorem 6.4 are satisfied. We will still denote the translation flow associated to B_ρ^ϕ by φ_t and write $M_\rho^\phi = M_{B_\rho^\phi}$.

We want to use thermodynamic formalism methods for the translation flow φ_t and we briefly explain why we can do so in this setting. While it is well known that a Hölder reparameterization of an Anosov flow is an Anosov flow [3, p. 122], Hölder conjugacy does not necessarily preserve the Anosov property [29]. To construct symbolic codings and use results from the thermodynamic formalism for Hitchin representations, we will work in the more general setting of metric Anosov flows. *Metric Anosov flows (or Smale flows)* were introduced by Pollicott in [48] to generalize classical results for Anosov flows and Axiom A flows. In particular, he constructed a symbolic coding for metric Anosov flows [48].

Our definition of metric Anosov flow, which is better suited to our purposes, differs from the original definition in [48]. The main difference is that we allow an α -power of the distance $d(x, y)$ for the exponential diverging and expanding properties in part (1) of Definition 6.5 below. This property guarantees that Hölder conjugacy preserves our notion of metric Anosov flow (see Proposition 6.7) and is useful in our setting, for example, in Proposition 6.8.

Recall for any continuous flow ψ_t on a compact metric space X , the *local stable set* of a point $x \in X$ is defined for $\epsilon > 0$ as

$$W_\epsilon^s(x) = \{y \in X : d(\psi_t x, \psi_t y) \leq \epsilon, \text{ for all } t \geq 0 \text{ and } d(\psi_t x, \psi_t y) \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

The *local unstable set* of a point x for $\epsilon > 0$ is

$$W_\epsilon^u(x) = \{y \in X : d(\psi_{-t} x, \psi_{-t} y) \leq \epsilon, \text{ for all } t \geq 0 \text{ and } d(\psi_{-t} x, \psi_{-t} y) \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

Definition 6.5. A continuous flow ψ on a compact metric space X is *metric Anosov* if the following hold.

- (1) There exist positive C, λ, ϵ and $\alpha \in (0, 1]$ such that

$$d(\psi_t x, \psi_t y) \leq C e^{-\lambda t} d(x, y)^\alpha \quad \text{when } y \in W_\epsilon^s(x) \text{ and } t \geq 0$$

and

$$d(\psi_{-t} x, \psi_{-t} y) \leq C e^{-\lambda t} d(x, y)^\alpha \quad \text{when } y \in W_\epsilon^u(x) \text{ and } t \geq 0.$$

- (2) There exists $\delta > 0$ and a continuous map ν on the set $\{(x, y) \in X \times X : d(x, y) \leq \delta\}$ such that $\nu = \nu(x, y)$ is the unique value for which $W_\epsilon^u(\psi_\nu x) \cap W_\epsilon^s(y)$ is non-empty. The set $W_\epsilon^u(\psi_\nu x) \cap W_\epsilon^s(y)$ consists of a single point, denoted as $\langle x, y \rangle$.

Remark 6.6. Metric Anosov flows in Definition 6.5 have a symbolic coding. Indeed, these flows verify one of the key properties [10, Lemma 1.5] used by Bowen to build symbolic codings for Axiom A flows. From this, and after adapting Bowen's arguments to accommodate for the exponent α in the definition, we observe that the metric Anosov flows in Definition 6.5 satisfy the expansivity, tracing, and specification properties [10, Proposition 1.6 and §2]. These are also the crucial properties used by Pollicott in his construction of symbolic codings for Smale flows [48].

PROPOSITION 6.7. *Let ψ^1 be a Hölder continuous metric Anosov flow on a compact metric space X . If ψ^2 is a flow on a compact metric space Y and ψ^2 is Hölder conjugate to ψ^1 , then ψ^2 is a Hölder continuous metric Anosov flow.*

Proof. We show here that a Hölder conjugacy preserves metric Anosov properties. By hypothesis, there exists a bi-Hölder homeomorphism $h : X \rightarrow Y$ with Hölder exponent $\alpha_0 \in (0, 1]$ (for both h and h^{-1}) such that $h \circ \psi_t^1 = \psi_t^2 \circ h$. We want to show that ψ_t^2 also satisfies the metric Anosov property.

Given $x_2 \in Y$, we want to find suitable parameters so that conditions (1) and (2) in Definition 6.5 hold. We show condition (1) for local stable sets. Suppose $x_2 = h(x_1)$. Because ψ_t^1 is metric Anosov, we can find $\varepsilon_1, C_1, \lambda_1, \delta_1$ positive and $\alpha_1 \in (0, 1]$ so that

$$\begin{aligned} W_{\varepsilon_1}^s(x_1) &= \{y_1 \in X : d_X(\psi_t^1 x_1, \psi_t^1 y_1) \leq \varepsilon_1 \text{ and } d_X(\psi_t^1 x_1, \psi_t^1 y_1) \\ &\leq C_1 e^{-\lambda_1 t} d_X(x_1, y_1)^{\alpha_1} \text{ for all } t \geq 0\}. \end{aligned}$$

Note,

$$\begin{aligned} d_Y(\psi_t^2 h(x_1), \psi_t^2 h(y_1)) &= d_Y(h(\psi_t^1 x_1), h(\psi_t^1 y_1)) \leq C_h d_X(\psi_t^1 x_1, \psi_t^1 y_1)^{\alpha_0} \\ &\leq C_h (C_1 e^{-\lambda_1 t} d_X(x_1, y_1)^{\alpha_1})^{\alpha_0} \leq C_2 e^{-\lambda_1 \alpha_0 t} d_Y(h(x_1), h(y_1))^{\alpha_0^2 \alpha_1}. \end{aligned}$$

Here we have used the Hölder properties of both h and h^{-1} , and C_h is a constant from the Hölder property of h . This implies that we can find $\varepsilon_2 > 0$ so that $W_{\varepsilon_2}^s(x_2)$ satisfies condition (1) in Definition 6.5 for ψ_t^2 with positive $C_2, \lambda_2 = \lambda_1 \alpha_0$ and $\alpha_2 = \alpha_0^2 \alpha_1$. The argument is similar for local unstable sets. Furthermore, given $x_2, y_2 \in Y$ close enough, condition (2) in Definition 6.5 can be verified by taking $\langle x_2, y_2 \rangle = h(\langle h^{-1}(x_2), h^{-1}(y_2) \rangle)$. We conclude that ψ_t^2 is metric Anosov. \square

The translation flow has been introduced and extensively studied by Sambarino in [53]. For completeness, we provide a proof that the translation flow is metric Anosov.

PROPOSITION 6.8. *The translation flow φ_t on M_ρ^ϕ is a metric Anosov flow which admits a symbolic coding with Hölder continuous roof functions.*

Proof. Let X_0 be a base hyperbolic surface. The geodesic flow on $T^1 X_0$ is a (metric) Anosov flow which admits a symbolic coding with Hölder roof function. From Theorem 6.4, we know that the translation flow φ_t associated to the cocycle B_ρ^ϕ defined on the quotient space M_ρ^ϕ is Hölder conjugate to a Hölder reparameterization of the geodesic flow ψ_t on $T^1 X_0$. In other words, there exists a Hölder continuous function $f : T^1 X_0 \rightarrow \mathbb{R}_{>0}$

and a bi-Hölder homeomorphism $h : T^1X_0 \rightarrow M_\rho^\phi$ such that $(h^{-1} \circ \varphi \circ h)_t = \psi_t^f$ for all $t \in \mathbb{R}$. Since ψ_t^f is an Anosov flow (see [3, p. 122]), φ_t is a metric Anosov flow by Proposition 6.7.

The coding is therefore preserved. The roof function remains Hölder by either Hölder conjugacy or Hölder reparameterization as it is given by an opportune composition of Hölder functions. The translation flow is therefore a metric Anosov flow on a compact metric space that admits a symbolic coding with Hölder roof function. \square

6.3. *Reparameterization functions.* After constructing flows for Hitchin representations, the next thing that we want to do is to define the *reparameterization function* with domain M_ρ^ϕ for ρ a Hitchin representation. This construction is well known and it is contained in the work of Sambarino [53]. We include this discussion to make our exposition self-contained.

In §2.2, given a positive Hölder continuous function f on the unit tangent bundle T^1X_ρ of a convex real projective structure, we defined a reparameterization of the flow by time change. This can be done more generally for any Hölder continuous flow on a compact metric space X . In particular, given two Hitchin representations ρ_1 and ρ_2 in $\mathcal{H}_d(S)$, we will show the existence of a positive Hölder continuous *reparameterization function* $f_{\rho_1}^{\rho_2} : M_{\rho_1}^\phi \rightarrow \mathbb{R}_{>0}$ that encodes the ϕ -length spectrum of ρ_2 .

We start with a lemma that relates the positivity of the Hölder reparameterization function to the positivity of its entropy. The *entropy* of a Hölder function f on a compact metric space X equipped with a metric Anosov flow is defined as

$$h_f = \limsup_{T \rightarrow \infty} \frac{1}{T} \# \log \left\{ \tau \text{ periodic} \mid \int_\tau f \leq T \right\}.$$

LEMMA 6.9. (Ledrappier [40, Lemma 1], Sambarino [53, Lemma 3.8]) *Let $f : X \rightarrow \mathbb{R}$ be a Hölder continuous function with non-negative periods on a compact metric space X equipped with a topological transitive metric Anosov flow. The following are equivalent:*

- (1) *the function f is cohomologous to a positive Hölder continuous function;*
- (2) *there exists $\kappa > 0$ such that $\int_\tau f > \kappa p(\tau)$ where $p(\tau)$ is the period of τ ;*
- (3) *the entropy $h_f \in (0, \infty)$.*

We also need the following theorem of Ledrappier [40] which establishes the correspondence between cohomologous Hölder cocycles and cohomologous Hölder continuous functions on T^1X_0 . We will state this theorem for vector valued Hölder cocycles. Their definition is analogous to Definition 6.3.

THEOREM 6.10. (Ledrappier [40, p. 105]) *Let V be a finite-dimensional vector space. For each Hölder cocycle $c : \Gamma \times \partial\Gamma \rightarrow V$, there exists a Hölder continuous map $F_c : T^1X_0 \rightarrow V$, such that for every $\gamma \in \Gamma - \{e\}$,*

$$\ell_c(\gamma) = \int_{[\gamma]} F_c.$$

This map $c \rightarrow F_c$ induces a bijection between the set of cohomology classes of V -valued Hölder cocycles and the set of cohomology classes of Hölder maps from T^1X_0 to V .

For a Hitchin representation ρ , this tells us that we can find a Hölder continuous map $g_\rho : T^1X_0 \rightarrow \mathfrak{a}$ with periods equal to the periods of the vector valued Busemann cocycle B_ρ . Since $h_{\phi \circ g_\rho} = h_{B_\rho^\phi} = h^\phi(\rho)$ is finite and positive, by Lemma 6.9, the reparameterization $\phi \circ g_\rho$ on T^1X_0 is cohomologous to a positive Hölder continuous function.

Now we are ready to state our lemma about the existence of positive reparameterization functions, which should be compared to Lemma 2.3.

LEMMA 6.11. *Let ρ_1 and ρ_2 be two different representations in $\mathcal{H}_d(S)$. There exists a positive Hölder continuous function $f_{\rho_1}^{\rho_2} : M_{\rho_1}^\phi \rightarrow \mathbb{R}_{>0}$ such that for every periodic orbit τ corresponding to $[\gamma] \in [\Gamma]$,*

$$\lambda(f_{\rho_1}^{\rho_2}, \tau) = \ell_{\rho_2}^\phi([\gamma]).$$

Proof. We denote the translation flow for $M_{\rho_i}^\phi$ with respect to the Busemann cocycle $B_{\rho_i}^\phi$ as φ_t^i for $i = 1, 2$. Now since $B_{\rho_i}^\phi$ has positive entropy, by Theorem 6.4, both φ_t^i are Hölder conjugate to Hölder reparameterizations of the geodesic flow on T^1X_0 , where X_0 is an auxiliary hyperbolic surface. The reparameterization functions for φ_t^i on T^1X_0 can be chosen to be positive by the discussion after Theorem 6.10. Generalizing Remark 2.2, we conclude that the flow φ_t^2 is Hölder conjugate to a Hölder reparameterization of φ_t^1 on $M_{\rho_1}^\phi$. Therefore, there exists a Hölder function $f : M_{\rho_1}^\phi \rightarrow \mathbb{R}$ such that the flow φ_t^2 is Hölder conjugate to $(\varphi_t^1)^f$ where the flow $(\varphi_t^1)^f$ is a reparameterization of φ_t^1 by f . Because $h_f = h_{B_\rho^\phi}$ is positive and finite, we can always choose $f = f_{\rho_1}^{\rho_2}$ in its cohomology class to be a positive function by Lemma 6.9. One easily checks that $\lambda(f, \tau) = \ell_{\rho_2}^\phi([\gamma])$. □

Once we have defined reparameterization functions, we see that all definitions, remarks, and results in §2.3 regarding thermodynamic formalism readily generalize to this context by simply changing the domain from T^1X_ρ to M_ρ^ϕ .

6.4. *Correlation and Manhattan curve theorem revisited.* First, we state the independence lemma for Hitchin representations which generalizes Lemma 2.11.

LEMMA 6.12. (Independence lemma) *Consider $\phi \in \Delta$ and Hitchin representations $\rho_1, \rho_2 \in \mathcal{H}_d(S)$ such that $\rho_2 \neq \rho_1$ or ρ_1^* . If there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1 \ell_{\rho_1}^\phi([\gamma]) + a_2 \ell_{\rho_2}^\phi([\gamma]) \in \mathbb{Z}$ for all $[\gamma] \in [\Gamma]$, then $a_1 = a_2 = 0$.*

Proof. The Zariski closure G_i of $\rho_i(\Gamma)$ is simple, center-free, and connected by a result of Guichard (see for example [14, Theorem 11.7]. See also [55, Corollary 1.5]). Then, we argue by contradiction, as in the proof of Lemma 2.11. □

Recall that a flow is weakly mixing if its periods do not generate a discrete subgroup of \mathbb{R} . In particular, the independence lemma implies that the translation flow φ_t is weakly mixing.

We are now ready to state our correlation theorem for Hitchin representations. Recall that we denote the *renormalized ϕ length spectrum* of ρ by $L_\rho^\phi = h^\phi(\rho)\ell_\rho^\phi$.

THEOREM 1.7. *Given a linear functional $\phi \in \Delta$ and a fixed precision $\varepsilon > 0$, for any two different Hitchin representations $\rho_1, \rho_2: \Gamma \rightarrow \text{PSL}(d, \mathbb{R})$ such that $\rho_2 \neq \rho_1^*$, there exist constants $C = C(\varepsilon, \rho_1, \rho_2, \phi) > 0$ and $M = M(\rho_1, \rho_2, \phi) \in (0, 1)$ such that*

$$\#\{\gamma \in [\Gamma] \mid L_{\rho_1}^\phi([\gamma]) \in (x, x + h^\phi(\rho_1)\varepsilon), L_{\rho_2}^\phi([\gamma]) \in (x, x + h^\phi(\rho_2)\varepsilon)\} \sim C \frac{e^{Mx}}{x^{3/2}}.$$

Proof. By Proposition 6.8 and Lemma 6.12, the translation flow associated to the linear functional ϕ is a weakly mixing metric Anosov flow on $M_{\rho_1}^\phi$ that admits a symbolic coding with Hölder continuous roof function. Moreover, ρ_1 and ρ_2 are independent thanks to the independence Lemma 6.12. Thus we can apply Lalley and Sharp’s Theorem 3.1 which holds in the context of flows with a symbolic coding with Hölder roof function.

Recall $J(f_{\rho_1}^{\rho_2})$ is the open interval of values $P'(tf_{\rho_1}^{\rho_2})$ for $t \in \mathbb{R}$. We then want to verify $h^\phi(\rho_1)/h^\phi(\rho_2) \in J(f_{\rho_1}^{\rho_2})$. The proof of this fact follows from the same argument as in the proof for Theorem 1.1. □

Remark 6.13

- (1) Fix $\phi \in \Delta$. Sambarino’s orbit counting theorem [53, Theorem 7.8] implies that the correlation number $M = M(\rho_1, \rho_2, \phi)$ converges to one and $C(\varepsilon, \rho_1, \rho_2, \phi)$ diverges if ρ_1 and ρ_2 converge to Hitchin representations with the same ϕ -length spectrum.
- (2) Note that Theorem 1.7 holds more generally for any functional ϕ which is contained in the interiors of $\mathcal{L}_{\rho_1}^*$ and $\mathcal{L}_{\rho_2}^*$.

For two Hitchin representations ρ_1, ρ_2 with $\rho_2 \neq \rho_1, \rho_1^*$ and $\phi \in \Delta$, consider the *Manhattan curve*

$$C^\phi(\rho_1, \rho_2) = \{(a, b) \in \mathbb{R}^2: P(-a - bf) = 0\},$$

where f is the reparameterization function from Lemma 6.11. We obtain a characterization of the correlation number analogous to Theorem 4.2.

THEOREM 6.14. *Fix $\phi \in \Delta$, and let ρ_1 and ρ_2 be Hitchin representations in $\mathcal{H}_d(S)$ such that $\rho_2 \neq \rho_1, \rho_1^*$. Their correlation number can be written as*

$$M(\rho_1, \rho_2, \phi) = \frac{a}{h^\phi(\rho_1)} + \frac{b}{h^\phi(\rho_2)},$$

where $(a, b) \in C^\phi(\rho_1, \rho_2)$ is the unique point on the Manhattan curve at which the tangent line is parallel to the line passing through $(h^\phi(\rho_1), 0)$ and $(0, h^\phi(\rho_2))$.

We conclude by raising two questions which are motivated by Theorems 1.3 and 1.5.

Recall that Potrie and Sambarino [50, Theorem B] showed that for every Hitchin representation ρ , the simple root lengths are such that $h^{\alpha_i}(\rho) = 1$ for $i = 1, \dots, d - 1$. Moreover, the simple root lengths restrict to the hyperbolic length on the Fuchsian locus $\mathcal{T}(S) \subset \mathcal{H}_d(S)$. Thus, Theorem 1.7 in this case is a particularly natural generalization of the correlation theorem for hyperbolic surfaces [56]. Theorem 1.3 exhibits examples of

sequences for which the α_i -correlation number decays. We ask whether there exist similar examples which lie outside the Fuchsian locus.

Question 6.15. For $d \geq 3$ and $i = 1, \dots, d-1$, do there exist sequences $(\rho_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}_d(S)$ which leave every compact neighborhood of the Fuchsian locus and such that the correlation numbers satisfy $\lim_{n \rightarrow \infty} M(\rho_n, \eta_n, \alpha_i) = 0$?

Finally, we raise a conjecture motivated by Theorem 1.5. Similarly to cubic rays introduced in §5, in general for the Hitchin component $\mathcal{H}_d(S)$, one can consider a d th-order holomorphic differential q over X_0 and its associated family of representations $(\rho_t)_{t \in \mathbb{R}}$ in $\mathcal{H}_d(S)$ parameterized by $(tq)_{t \in \mathbb{R}}$. These representations are given by holonomies of *cyclic Higgs bundles*. We refer to [4] for a detailed definition and discussion. When $t \geq 0$, we say this family of representations $(\rho_t)_{t \geq 0}$ is a *ray associated to q* .

Conjecture 6.16. Let $(\rho_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$ be two rays associated to two different d th-order holomorphic differentials q_1 and q_2 on a hyperbolic structure X_0 such that q_1, q_2 have unit L^2 -norm with respect to X_0 and $q_1 \neq -q_2$. Then the correlation number $M(\rho_t, \eta_t, \sum_{i=1}^{d-1} \alpha_i)$ is uniformly bounded away from zero as t goes to infinity.

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