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# Regularization and generalized double shuffle relations for $p$-adic multiple zeta values

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Abstract

We will introduce a regularization for $p$-adic multiple zeta values and show that the generalized double shuffle relations hold. This settles a question raised by Deligne, given as a project in the Arizona Winter School 2002. Our approach is to use the theory of Coleman functions on the moduli space of genus zero curves with marked points and its compactification. The main ingredients are the analytic continuation of Coleman functions to the normal bundle of divisors at infinity and definition of a special tangential base point on the moduli space.

## Introduction

This paper is a continuation of [BF06]. Let $p$ be a prime number. The first author in [Fur04] introduced $p$-adic multiple zeta values (MZVs) for admissible indices $(n_1, \ldots, n_m) \in \mathbb{Z}^m_{>0}$, that is when $n_m > 1$. In [BF06] the double shuffle relations for these $p$-adic MZVs were proved. In this work we extend their result. By using the moduli space of genus zero curves with marked points and its stable compactification, we introduce a series regularization of $p$-adic MZVs for non-admissible indices and then prove a generalized double shuffle relation. This relation is an extension of the double shuffle relation in [BF06]. This includes a comparison between the integral regularization and the series regularization of $p$-adic MZVs.

Let us first review the story for complex valued MZVs. Recall that for positive integers $n_1, \ldots, n_m$ the (complex) MZV is defined by

$$\zeta(n_1, \ldots, n_m) = \sum_{0<k_1<\cdots<k_m} \frac{1}{k_1^{n_1} \cdots k_m^{n_m}}. \tag{0.1}$$

They were studied first by Euler for $m = 1$ and $m = 2$. It is easy to see that the series is convergent if and only if $n_m > 1$. The *double shuffle relations* consist of *series shuffle relations* and *integral shuffle relations*. Both of them are product formulae between MZVs. The simplest example of the series shuffle relation is

$$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2).$$

It is easily obtained from the expression (0.1) and can be generalized in a similar way to other MZVs. The simplest example of the integral shuffle relation is

$$\zeta(n_1)\zeta(n_2) = \sum_{i=0}^{n_1-1} \binom{n_2 - 1 + i}{i} \zeta(n_1 - i, n_2 + i) + \sum_{j=0}^{n_2-1} \binom{n_1 - j + 1}{j} \zeta(n_2 - j, n_1 + j).$$

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This follows from the iterated integral expression for MZVs. Using these formulae we can get many relations among the MZVs. However, the double shuffle relations are not enough to capture all the relations between MZVs. There are two regularization of the MZVs for non-admissible indices: the *series regularization*, which extends the validity of the series shuffle relation, and the *integral regularization*, which extends the validity of the integral shuffle relation. The several variable multiple polylogarithm (MPL)

$$Li_{n_1,\dots,n_m}(x_1,\dots,x_m)=\sum_{0<k_1<\cdots<k_m}\frac{x_1^{k_1}\cdots x_m^{k_m}}{k_1^{n_1}\cdots k_m^{n_m}} \tag{0.2}$$

with complex variables $x_1,\dots,x_m$ is the device to construct the series shuffle regularization. It is clear that when the indices are admissible the limit of (0.2) when the $x_i$ approach 1 is $\zeta(n_1,\dots,n_m)$. In general one can show that $Li_{n_1,\dots,n_m}(1-\epsilon,\dots,1-\epsilon)=\sum_{i=0}^{N}a_i(\epsilon)\log^i\epsilon$, where $a_i(\epsilon)\in\mathbb{C}[[\epsilon]]$ are analytic functions in a neighborhood of $\epsilon=0$. The *series regularized MZV* $\zeta^S(n_1,\dots,n_m)\in\mathbb{C}[T]$ is by definition the polynomial $\sum_{i=0}^{N}a_i(0)T^i$. The one variable MPL

$$Li_{n_1,\dots,n_m}(z)=Li_{n_1,\dots,n_m}(1,\dots,1,z)=\sum_{0<k_1<\cdots<k_m}\frac{z^{k_m}}{k_1^{n_1}\cdots k_m^{n_m}} \tag{0.3}$$

with one complex variable $z$ is the device to construct the integral regularization. It is clear that when the indices are admissible the limit of (0.3) when $z$ approaches 1 is $\zeta(n_1,\dots,n_m)$. It can be shown that $Li_{n_1,\dots,n_m}(1-\epsilon)=\sum_{i=0}^{M}b_i(\epsilon)\log^i\epsilon$, where $b_i(\epsilon)\in\mathbb{C}[[\epsilon]]$ are analytic functions in a neighborhood of $\epsilon=0$. The *integral regularized MZV* $\zeta^I(n_1,\dots,n_m)\in\mathbb{C}[T]$ is by definition the polynomial $\sum_{i=0}^{M}b_i(0)T^i$. We note that $\zeta^I(1)=-T$. There is a comparison relation between these two regularizations that we now describe. Let $\mathbb{L}$ be the $\mathbb{C}$ linear map from the polynomials $\mathbb{C}[T]$ to itself defined via the generating function[1]

$$\sum_{n=0}^{\infty}\mathbb{L}(T^n)\frac{u^n}{n!}=\exp\Big(-\sum_{n=1}^{\infty}\frac{\zeta^I(n)}{n}u^n\Big). \tag{0.4}$$

The *regularization relation* [IKZ06] asserts that

$$\zeta^S(n_1,\dots,n_m)=\mathbb{L}(\zeta^I(n_1,\dots,n_m)). \tag{0.5}$$

The *generalized double shuffle relation* is in fact three types of relations: the series shuffle relation for series regularized MZV, the integral shuffle relation for integral regularized MZV, and the relation (0.5) that gives the comparison between these two regularizations. It is conjectured that the generalized double shuffle relations capture all the possible relations between the MZVs (cf. [Rac02]).

Now we explain the story for $p$-adic MZVs. The one variable MPL in (0.3) has a $p$-adic analogue as a Coleman function on $\mathbb{P}^1(\mathbb{C}_p)-\{0,1,\infty\}$. This function depends on a choice of the branch of $p$-adic logarithm. We denote this function by $Li^a_{n_1,\dots,n_m}(z)$, where $a\in\mathbb{Q}_p$ is the value of the chosen $p$-adic log at $p$. In [Fur04] the $p$-adic MZV $\zeta_p(n_1,\dots,n_m)\in\mathbb{Q}_p$ for an admissible index is defined as a certain limit of this Coleman function as $z$ approaches 1. It is shown in therein that this limit exists and is independent of the choice of the branch of $p$-adic logarithm. Using the language of the tangential base points, it can be explained as follows. The function $Li^a_{n_1,\dots,n_m}(z)$ is a Coleman function on $\mathbb{P}^1-\{0,1,\infty\}$. First analytically continue it to the tangent plane at the point $z=1$, punctured at the origin. Let $z$ denote the canonical parameter of $\mathbb{P}^1-\{0,1,\infty\}$. Then $t=1-z$ is regarded as a local parameter of this tangent plane and it can be shown that the function obtained via this analytic continuation is a polynomial in $T:=\log^a t$. We want to stress

[1] There is a sign error in [Gon02] for this formula.



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that this polynomial is independent of the choice of $a$ and we define the *integral regularized $p$-adic MZV* ${\zeta_p}^I(n_1,\ldots,n_m)\in\mathbb{Q}_p[T]$ to be this polynomial. When $n_m>1$, ${\zeta_p}^I(n_1,\ldots,n_m)$ is a constant polynomial (cf. [Fur04]).

We will use a method inspired by [Gon02] to define the series regularized $p$-adic MZV. We use a $p$-adic analogue of the several variable MPL in (0.2). It is denoted by $Li^a_{n_1,\ldots,n_m}(x_1,\ldots,x_m)$ where $a$ is the value of the branch of $p$-adic logarithm at $p$. It is a Coleman function on $\mathcal{M}_{0,m+3}$, the moduli space of genus zero curves with $m+3$ marked points. We identify this space with

$$\mathbb{A}^m-\{x_1\cdots x_m=0 \text{ or } 1-x_i\cdots x_j=0, i\leqslant j\}.$$

Let $\overline{\mathcal{M}}_{0,m+3}$ denote the stable compactification of $\mathcal{M}_{0,m+3}$. The line $(1-t,1-t,\ldots,1-t)$ in $\mathcal{M}_{0,m+3}$ when $t$ approaches 0 intersects a unique divisor $D_0$ (denoted later by $D'_m$) of $\overline{\mathcal{M}}_{0,m+3}$. Let $R$ denote this intersection point and $L$ be the tangent line above $R$ in the normal bundle of $D_0$ minus the zero section. The *series regularized $p$-adic MZV* is defined by

$$\zeta_p^S(n_1,\ldots,n_m):=Li^{a,(D_0)}_{n_1,\ldots,n_m}(\bar{x}_1,\ldots,\bar{x}_m)|_L,$$

where $Li^{a,(D_0)}_{n_1,\ldots,n_m}(\bar{x}_1,\ldots,\bar{x}_m)$ is the analytic continuation of MPL to the part of the normal bundle of $D_0$ minus the zero section that lies over the open part of $D_0$ outside all the other divisors. The series regularized $p$-adic MZV $\zeta_p^S(n_1,\ldots,n_m)$ is a polynomial in $\mathbb{Q}_p[T]$ where $T=\log^a t$. It *a priori* depends on $a$.

The main result of this paper is the following.

Theorem 0.1.

(i) *The series regularized MZV $\zeta_p^S(n_1,\ldots,n_m)\in\mathbb{Q}_p[T]$ is well defined, that is, it is independent of the choice of the branch parameter $a\in\mathbb{Q}_p$.*

(ii) *If $n_m>1$ then $\zeta_p^S(n_1,\ldots,n_m)$ is constant and is equal to the $p$-adic MZV $\zeta_p(n_1,\ldots,n_m)$ in [Fur04].*

(iii) *The generalized double shuffle relation holds, i.e. $\zeta_p^S(n_1,\ldots,n_m)$ satisfies the series shuffle relation and $\zeta_p^I(n_1,\ldots,n_m)$ satisfies the integral shuffle relation. Furthermore, the regularization relation*

$$\zeta_p^S(n_1,\ldots,n_m)=\mathbb{L}_p(\zeta_p^I(n_1,\ldots,n_m))$$

*holds, where $\mathbb{L}_p$ is defined by analogous generating series in (0.4) where we replace $\zeta^I(n)$ by $\zeta_p^I(n)$.*

This theorem is used together with some results of Racinet and the first author to prove the following result.

Theorem 0.2. *Deligne's $p$-adic MZVs satisfy the generalized double shuffle relations.*

The definition of these $p$-adic MZVs is recalled in § 7. This definition was suggested in the 2002 Arizona Winter School and the theorem above solves the project proposed by Deligne.

## 1. Review of Coleman functions

We recall some definitions and properties of Coleman functions and tangential base points as developed in [Bes02], [Bes05] and [BF06]. We fix a branch of $p$-adic logarithm $\log:\mathbb{Q}_p^\times\to\mathbb{Q}_p$ with value $a\in\mathbb{Q}_p$ at $p$ for the rest of this paper.

Let $X$ be a smooth variety over $K$, a finite extension of $\mathbb{Q}_p$. Let $\mathcal{NC}(X)$ denote the category of unipotent flat vector bundles on $X$, i.e. a vector bundle together with a flat connection on it such that it is an iterative extension of trivial vector bundles together with a trivial flat connection. This category is a neutral tannakian category and any point $x \in X(K)$ defines a fiber functor $\omega_x$ from $\mathcal{NC}(X)$ to the category $Vec_K$ of finite dimensional $K$-vector spaces (cf. [Del89]). In [Vol03], Vologodsky has constructed a canonical system (after fixing a branch of $p$-adic logarithm) of isomorphisms $a^X_{x,y} : \omega_x \longrightarrow \omega_y$ for any pair of points in $X(K)$. The properties of these isomorphisms are summarized in [Bes05, § 2]. Following [Bes05], an abstract Coleman function is a triple $(M, s, y)$ where $M \in \mathcal{NC}(X)$, $s \in \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$ and $y$ is a collection of $y_x \in M_x$ for all $x \in X(L)$ for any finite extension $L$ of $K$, where $M_x$ is the fiber of $M$ over $x$ and is an $L$-vector space defined by the fiber functor $\omega_x : \mathcal{N}C(X_L) \to Vec_L$.

This data must satisfy the following:

(a) for any two points $x_1, x_2 \in X(L) = X_L(L)$ we have $a^{X_L}_{x_1,x_2}(y_{x_1}) = y_{x_2}$;

(b) for any field homomorphism $\sigma : L \longrightarrow L'$ that fixes $K$ and $x \in X(L)$ we have $\sigma(y_x) = y_{\sigma(x)}$.

There is a natural notion of morphism between the abstract Coleman functions. The connected component of an abstract Coleman function is called a Coleman function. A Coleman function is also interpreted as a function on $X(\overline{K})$ by assigning to $x$ the value $s(y_x)$. This is indeed a locally analytic function. In this paper, we will use both approaches for Coleman functions, i.e. the interpretation as a triple $(M, s, y)$ as above and the interpretation as a locally analytic function. The set of Coleman functions on $X$ is a ring which we denote by $\mathrm{Col}^a(X)$. Here $a \in \mathbb{Q}_p$ is the value of the chosen branch of the $p$-adic logarithm at $p$.

Let $X$ be a smooth $\mathcal{O}_K$-scheme and $D = \sum_{i\in I} D_i$ be a divisor with relative normal crossings over $\mathcal{O}_K$, with the $D_i$ smooth and irreducible over $\mathcal{O}_K$. Let $J$ be a non-empty subset of $I$. In [BF06] a tangential morphism $Res_{D,J} : \mathcal{NC}((X - D)_K) \longrightarrow \mathcal{NC}(\mathcal{N}^{00}_J)$ was constructed. Here $\mathcal{N}^{00}_J$ is the normal bundle of $D_J = \bigcap_{j\in J} D_j$ minus the normal bundles of $D_{J-\{j\}}$ for all $j \in J$ (the normal bundle $\mathcal{N}_\emptyset$ is considered as the zero section of $\mathcal{N}_{D_J}$), and then restricted to $D_J - \bigcup_{j\notin J}(D_j \cap D_J)$. The construction is given as follows (cf. [BF06, § 3]): For each $j \in J$ consider the valuation $v_j$ on $K(X)$ associated with the divisor $D_j$. Let $\mathcal{O}_X(D^{-1})$ be the localization of $\mathcal{O}_X$ at $D$. There exists a multi-filtration $F_J$ on $\mathcal{O}_X(D^{-1})$, indexed by tuples $\chi = (\chi_j \in \mathbb{Z})_{j\in J}$, such that $F^\chi_J$ is the $\mathcal{O}_X$-module generated by $\{f \in \mathcal{O}_X(D^{-1}),\ v_j(f) \geqslant \chi_j$ for all $j \in J\}$. It is easy to see that $Spec(Gr_J\mathcal{O}_X(D^{-1}))$ is precisely $\mathcal{N}^{00}_J$. Suppose we have a connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$ with logarithmic singularities along $D$. We give $\Omega^1_X(D^{-1}) = \Omega^1_X(\log D) \otimes \mathcal{O}_X(D^{-1})$ the induced filtrations from the filtration on $\mathcal{O}_X(D^{-1})$. It is easy to see that the differential $d$ preserves the filtration. Now $M(D^{-1}) = M \otimes \mathcal{O}_X(D^{-1})$ and $M \otimes \Omega^1_X(D^{-1})$ have the induced filtrations. It follows that the extended connection $\nabla : M(D^{-1}) \to M \otimes \Omega^1_X(D^{-1})$ respects the filtration. The connection $Res_{D,J}(M)$ is the graded quotient of this connection.

Let $\kappa$ be the residue field of $\mathcal{O}_K$. It was shown in [BF06] that if the Frobenius endomorphism of $(X, D)_\kappa$ locally lifts to an algebraic endomorphism of $(X, D)$ then this morphism respects the action of the Frobenius endomorphism. Indeed by [Shi02] and [CLS99] the categories $\mathcal{NC}(X - D)$ and $\mathcal{NC}(\mathcal{N}^{00}_J)$ are isomorphic to the categories of the unipotent isocrystals $\mathcal{NC}^\dagger((X - D)_\kappa) \otimes K$ and $\mathcal{NC}^\dagger((\mathcal{N}^{00}_J)_\kappa) \otimes K$ on the reductions $(X - D)_\kappa$ and $(\mathcal{N}^{00}_J)_\kappa$ and therefore admit a natural action of the Frobenius endomorphism. Choose a point $\tilde{t} \in (\mathcal{N}^{00}_J)_\kappa(\overline{\kappa})$ which is the reduction of a point $t \in \mathcal{N}^{00}_J(L)$ for some extension $L$ of $K$. The point $\tilde{t}$ defines a fiber functor $\omega_{\tilde{t}}$ from $\mathcal{NC}^\dagger((X - D)_\kappa)$ to $Vec_L$, which is Frobenius invariant if we take a high power of the Frobenius. Then following [Bes02] for any point $\tilde{x} \in (X - D)_\kappa(\overline{\kappa})$, which is the reduction of $x \in \mathcal{N}^{00}_J(L)$, we get a canonical Frobenius invariant isomorphism $\tilde{a}_{\tilde{x},\tilde{t}} : \omega_{\tilde{x}} \longrightarrow \omega_{\tilde{t}}$. The above categorical equivalence gives an isomorphism $a_{x,t} : \omega_x \longrightarrow \omega_t$. Now for any $x' \in (X - D)(L)$ and $t' \in \mathcal{N}^{00}_J(L)$ we define

$$a_{x',t'} = a_{x',x} \circ a_{x,t} \circ a_{t,t'}.$$

This is independent of the choice of $x$ and $t$. Using this we have a way (developed in [BF06, § 4]) to extend a certain type of Coleman functions (which were called Coleman functions of 'algebraic origin' in [BF06]) $(M, s, y)$ on $X - D$ to a Coleman function $(M', s', y')$ on $\mathcal{N}_J^{00}$ as follows. Let $M \in \mathcal{NC}(X - D)$ and $y$ be a compatible system over $X - D$ as before. The morphism $s : M \to \mathcal{O}_X$ induces a morphism $s_D : M(D^{-1}) \to \mathcal{O}_X(D^{-1})$ which we assume to be compatible with the filtration $F_J$. Then the Coleman function $(M', s', y')$ is defined by $M' = Res_{D,J}(M)$ as described above and the morphism $s'$ is $Gr(s_D) : Res_{D,J}M \to \mathcal{O}_{\mathcal{N}_J^{00}}$. The section $y'$ will be a collection of $y'_t$ $(t \in \mathcal{N}_J^{00}(L))$ with $y'_t = a_{x,t}(y_x)$ for some $x \in (X - D)(L)$.

## 2. The moduli space $\mathcal{M}_{0,N+3}$ and its compactification

In this section we give a quick review on some basic properties of the moduli space $\mathcal{M}_{0,N+3}$ of genus zero curves with $N+3$ distinct marked points and its stable compactification $\overline{\mathcal{M}}_{0,N+3}$. The basic references are [GHP88], [GM04] and [Man99].

The moduli space $\mathcal{M}_{0,N+3}$ can be identified with

$$\mathbb{G}_m^N - \bigcup_{1 \leqslant n \leqslant m \leqslant N} \left\{ \prod_{i=n}^{m} x_i = 1 \right\}.$$

Here $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$. The identification is given by sending $(x_1, \ldots, x_N)$ to the $N+3$ marked points on $\mathbb{P}^1$ given by $(0, x_1 \cdots x_N, x_2 \cdots x_N, \ldots, x_N, 1, \infty)$. Note that with this identification we have canonical coordinates $x_1, \ldots, x_N$ on $\mathcal{M}_{0,N+3}$.

We need to work with $\overline{\mathcal{M}}_{0,N+3}$, the stable compactification of this moduli space. There is a very concrete description of this space in [GHP88] that we now recall. Let $V_N$ be the set of all distinct ordered 4-tuples of $\{1, \ldots, N+3\}$. There is an embedding

$$r : \mathcal{M}_{0,N+3} \hookrightarrow \mathbb{A}^{V_N}$$

given by sending $(P_1, \ldots, P_{N+3})$ to all cross ratios of 4-tuples of points. To normalize the cross ratio we recall that $r(0, \infty, 1, x) = x$. Let $\lambda_v$ for $v \in V_N$ be the coordinates of $\mathbb{A}^{V_N}$. The image variety will be given by the following equations:

$$\begin{gathered} \lambda_{v_1 v_2 v_3 v_4} \lambda_{v_2 v_1 v_3 v_4} = 1, \\ \lambda_{v_1 v_2 v_3 v_4} = 1 - \lambda_{v_2 v_3 v_4 v_1}, \\ \lambda_{v_1 v_2 v_4 v_5} \lambda_{v_1 v_2 v_3 v_4} = \lambda_{v_1 v_2 v_3 v_5}, \end{gathered}$$

for all distinct 5-tuples $v_1, v_2, v_3, v_4, v_5$ in $\{1, \ldots, N+3\}$. Now the compactification $\overline{\mathcal{M}}_{0,N+3}$ is obtained simply by taking the closure of this variety inside $(\mathbb{P}^1)^{V_N}$. This means that we homogenize the equations by letting $\lambda_v = a_v / b_v$.

To give a natural stratification of $\overline{\mathcal{M}}_{0,N+3}$ which is of combinatorial origin we need to recall the notion of stable labeled trees. A stable tree is a tree (in the meaning of [Ser80, § 2] and is not oriented or planar) such that each of its vertices has valency at least 3. A stable $(N+3)$-labeled tree is a stable tree with $N+3$ external edges labeled by distinct labels from the set $\{1, \ldots, N+3\}$. Any stable curve of genus zero with $N+3$ marked points defines a stable $(N+3)$-labeled tree. This construction is standard and for details consult the above references.

Any vertex $t$ of a stable $(N+3)$-labeled tree $T$ defines an equivalence relation $\sim_t$ on the set $\{1, \ldots, N+3\}$ which can be identified with the set of external edges, as follows: $i \sim_t j$ if either the corresponding external edges have a common vertex or there is a path from $i$ to $j$ in $T$ that avoids $t$. It is easy to check that this forms an equivalence.

To each stable $(N+3)$-labeled tree $T$ we associate a closed smooth subvariety of $\overline{\mathcal{M}}_{0,N+3}$, denoted by $D(T)$. In coordinates it is defined by

$$\lambda_v = 0 \quad \text{for all } v \in V(T),$$

where $V(T)$ is the subset of $V_N$ of those quadruples $(v_1, v_2, v_3, v_4)$ such that for some internal vertex $t$ of $T$ we have $v_1 \sim_t v_4$ but $v_2 \not\sim_t v_4$ and $v_3 \not\sim_t v_4$. The main properties of this subvariety are as follows.

Proposition 2.1 [GHP88].

(i) *The codimension of $D(T)$ is equal to the number of internal edges of $T$.*

(ii) *An inclusion $D(T) \subseteq D(T')$ holds if and only if $T'$ is obtained by contracting $T$ along certain internal edges.*

(iii) *The subvariety $D(T)$ is canonically isomorphic to $\prod_{t\in T_0} \overline{\mathcal{M}}_{0,val(t)}$. Here $T_0$ is the set of internal vertices of $T$ and $val(t)$ denotes the valency of $t$.*

(iv) *Let $D^*(T) = D(T) - \bigcup D(T')$, where the union is taken over all the trees that can be contracted to $T$ except $T$ itself. Then the set of $D^*(T)$ for all non-equivalent labeled trees gives a stratification of $\overline{\mathcal{M}}_{0,N+3}$.*

(v) *If $T$ and $T'$ are two stable trees with only one internal edge then the corresponding subvarieties $D(T)$ and $D(T')$ are codimension one divisors. Let $A_1$ and $A_2$ be the set of external edges attached to the corresponding two vertices of $T$ and similarly define $A'_1$ and $A'_2$ for $T'$. The divisors $D(T)$ and $D(T')$ intersect if and only if one of the following conditions hold:*

$$A_i \subseteq A'_j \quad \text{or} \quad A'_i \subseteq A_j$$

*for some $i, j \in \{1, 2\}$.*

We consider the following affine covering of $\overline{\mathcal{M}}_{0,N+3}$. For each labeled tree $T$ let $U(T)$ be the open subset of $\overline{\mathcal{M}}_{0,N+3}$ given by

$$\lambda_v \neq 0 \quad \text{for all } v \notin V(T).$$

Notice that for $v \in V(T)$ we should have $\lambda_v \neq \infty$ on $U(T)$. Because if $\lambda_{v_1v_2v_3v_4} = \infty$ with $(v_1, v_2, v_3, v_4) \in V(T)$ then $\lambda_{v_2v_1v_3v_4} = 0$ but $(v_2v_1v_3v_4) \notin V(T)$. This shows that $U(T)$ is affine. The basic properties of these subsets are collected in the following lemma.

Lemma 2.2 [GHP88].

(i) *If $T$ is a contraction of $T'$ along some internal edges then $U(T) \subset U(T')$. Therefore $\mathcal{M}_{0,N+3} \subset U(T)$ for all choices of $T$.*

(ii) *If the combinatorial tree associated to a marked stable curve $q \in \overline{\mathcal{M}}_{0,N+3}$ is $T$ then $q \in U(T)$. Therefore we have a covering of $\overline{\mathcal{M}}_{0,N+3}$ by the $U(T)$.*

(iii) *We have the following relation:*

$$U(T) = \overline{\mathcal{M}}_{0,N+3} - \bigcup D(T'),$$

*where the union is taken over all the labeled trees $T'$ such that $T$ cannot be contracted to $T'$.*

Finally we need to have an inductive way of constructing a coordinate system on $U(T)$. In fact we can choose $N$ elements of $v_1, \ldots, v_N \in V_N$ such that the corresponding functions $\lambda_{v_i}$ form a coordinate system on $U(T)$, i.e. the coordinate ring of $U(T)$ will be a localization of the polynomial algebra $\mathbb{Q}[\lambda_{v_1}, \ldots, \lambda_{v_N}]$. This is done in [GHP88, § 3.2]. To explain this, some preparation is needed. If $T$ is a stable $(N+3)$-labeled tree and $i$ is a label, then we define $T\backslash i$, which is a stable $(N+2)$-labeled tree, as follows. If the valency of the vertex of the external edge associated to $i$ is at least 4 then we just remove the external edge associated to $i$. If the internal vertex of the external edge associated to $i$ has valency 3 then removing the external edge makes the tree unstable. Therefore we also contract the internal edge which has a common vertex with the external edge associated to $i$. Note that the labels of $T\backslash i$ are from the set $\{1,\dots,N+3\}-\{i\}$.

The median of three distinct labels $i,j$ and $k$ in a labeled tree $T$ is the unique vertex $t$ such that removing $t$ divides $i,j$ and $k$ into different connected components.

Lemma 2.3. *Let $T$ be an $(N+3)$-labeled stable tree. Let $i$ be a label such that the valency of the vertex of the external edge associated to $i$, denoted by $v_i$, is either at least 4 or there are at least two external edges attached to $v_i$ (such a label always exists). Inductively let $\lambda_{v_1},\dots,\lambda_{v_{N-1}}$ be a coordinate system on $U(T\backslash i)$. Let $(d_1,d_2,d_3)$ be a distinct triple of labels in $\{1,\dots,N+3\}\backslash\{i\}$ with the following property: If the valency $v_i$ is at least 4 then the median of $d_1,d_2,d_3$ should be $v_i$. If the valency of $v_i$ is 3 then the median of $d_1,d_2,d_3$ should be the unique vertex connected by an internal edge to $v_i$ and $d_3$ is a external edge attached to $v_i$. Then $\lambda_{v_1},\dots,\lambda_{v_{N-1}},\lambda_{d_1d_2d_3i}$ is a coordinate system for $U(T)$.*

*Proof.* Refer to [GHP88, § 3.2]. □

## 3. The tangential base point

In this section we will select a special divisor on $\overline{\mathcal{M}}_{0,N+3}$ and define a particular line in the normal bundle of this divisor that will play a crucial role in our definition of the series regularization of $p$-adic MZVs.

Let $1\leqslant i\leqslant N$ be an integer, and define $T_i$ and $T'_i$ to be the stable $(N+3)$-labeled trees with one internal edge and two vertices $v_1$ and $v_2$. The external edges attached to $v_1$ in $T_i$ have labels $\{1,\dots,i+1\}$ and in $T'_i$ have labels $\{2,\dots,i+2\}$. The external edges of $v_2$ in $T_i$ have labels $\{i+2,\dots,N+3\}$ and in $T'_i$ have labels $\{i+3,\dots,N+3,1\}$. We let $D_i=D(T_i)$ and $D'_i=D(T'_i)$ in the notation of the previous section. Note that there is a *unique* tree, which we denote by $T$, with $N$ internal edges that can be contracted to the $T_i$, and similarly there is a *unique* tree, which we denote by $T'$, with $N$ internal edges that can be contracted to the $T'_i$. In fact $T$ is a trivalent tree with internal vertices $v_0,\dots,v_N$ where $v_i$ and $v_{i+1}$ are connected by an internal edge and the external edges of $v_0$ are $\{1,2\}$, the external edge attached to $v_i$ is $i+2$ for $0<i<N$ and the external edges of $v_N$ are $\{N+2,N+3\}$. The existence of these trees implies that the intersection of $D_1,\dots,D_N$ is a single point which we denote by $P$ and similarly the intersection of $D'_1,\dots,D'_N$ is a single point denoted by $Q$.

Lemma 3.1. *The collection of $\lambda_{1,N+3,i+2,i+1}$ for $i=1,\dots,N$ gives a coordinate system for $U(T)$. Similarly the collection $\lambda_{2,1,i+3,i+2}$ for $i=1,\dots,N$ gives a coordinate system for $U(T')$.*

*Proof.* We only prove the first part, the second part being similar. The proof is by induction on $N$. For $N=1$, $U(T)=\mathbb{P}^1-\{1,\infty\}$, and $\lambda_{1432}$ sends $(0,x,1,\infty)$ to $x$, so we have the natural coordinate. Suppose we have proved the lemma for $N-1$. Then $U(T\backslash 2)$ by induction has coordinates $\lambda_{1,N+3,i+1,i+2}$ for $i=2,\dots,N$. According to Lemma 2.3 we have to add $\lambda_{d_1,d_2,1,2}$, such that $(d_1,d_2,1)$ has median $v_1$, the vertex of the external edge associated to 3 in $T$. This can be achieved if we let $d_1=3$ and $d_2=N+3$. However since $\lambda_{3,N+3,1,2}=1-\lambda_{1,N+3,3,2}$ we can use $\lambda_{1,N+3,3,2}$ as an extra coordinate. □

Note that for the point $q=(0,x_1\cdots x_N,x_2\cdots x_N,\dots,x_N,1,\infty)$ we have the coordinates $\lambda_{1,N+3,i+2,i+1}(q)=x_i$. Furthermore

$$z_i:=\lambda_{2,1,i+3,i+2}(q)=\frac{1-x_1\dots x_i}{1-x_1\cdots x_{i+1}},\tag{3.1}$$



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where $x_{N+1} := 0$. Also notice that since $(1, N+3, i+2, i+1) \in V(T)$ so the equation of $D_i$ inside $U(T)$ is $\lambda_{1,N+3,i+2,i+1} = 0$ or more naively $x_i = 0$. Similarly the equation of $D'_i$ inside $U(T')$ is $\lambda_{2,1,i+3,i+2} = 0$ or $z_i = 0$. The divisor $D'_N$ given by $z_N = 0$ is the *special divisor* that will play an important role in defining our regularization.

Let $E_N$ be the Zariski open subset of $\mathcal{M}_{0,N+3}$ defined by

$$E_N := \mathbb{G}_m^N - \bigcup_{I \subseteq \{1,\ldots,N\}} \Big\{ \prod_{i\in I} x_i = 1 \Big\}.$$

For a subset $I$ of $\{1, \ldots, N\}$ consisting of non-consecutive numbers we have the divisor $\prod_{i\in I} x_i = 1$ inside $\mathcal{M}_{0,N+3}$. Its closure inside $\overline{\mathcal{M}}_{0,N+3}$ is denoted by $D(I)$.

Lemma 3.2. *The line $\ell = \{(t, t, \ldots, t)\}$ inside $\mathcal{M}_{0,N+3}$ has a limit in $\overline{\mathcal{M}}_{0,N+3}$ when $t$ approaches 1; we denote this point by $R$. This point lies on the divisor $D'_N$ given by $z_N = 0$. Its coordinates using the coordinate system $z_1, \ldots, z_N$ are given by $(\frac{1}{2}, \frac{2}{3}, \ldots, (N-1)/N, 0)$. The point $R$ does not lie on any other component of $\overline{\mathcal{M}}_{0,N+3} - E_N$.*

*Proof.* As is explained in § 2, all the divisors of $\overline{\mathcal{M}}_{0,N+3} - \mathcal{M}_{0,N+3}$ are in one-to-one correspondence with unordered partitions of $\{1, \ldots, N+3\}$ into two subsets, where each subset has at least two elements. For a given partition $A \cup B$, the equation of the divisor associated to it inside $(\mathbb{P}^1)^{V_N}$ is given by $\lambda_{v_1 v_2 v_3 v_4} = 0$ for all quadruples such that the sets $\{v_1, v_4\}$ and $\{v_2, v_3\}$ are separated by $A$ and $B$. Now since the cross ratio of $(0, t^k, t^l, t^i)$ has the limit $(l-k)/(l-i)$ when $t$ approaches 1 and the cross ratio $(0, t^k, \infty, t^i)$ has the limit 1, it follows that the limit of $(0, t^N, \ldots, t, 1, \infty)$ when $t$ approaches 1 will not lie on any divisor other than the one obtained by the partition $\{1, N+3\} \cup \{2, \ldots, N+2\}$. In fact since the cross ratio of $(0, t^i, t^j, \infty)$ approaches 0, the limit point lies on this divisor. This also shows that $R$ belongs to $U(T')$ using the above notation. The coordinates of the point $(0, t^N, \ldots, t, 1, \infty)$ in terms of $z_i$ are

$$z_i = \frac{1-t^i}{1-t^{i+1}} \quad (i < N), \quad z_N = 1 - t^N.$$

So when $t$ approaches 1 we get the desired coordinates of the lemma. If $I$ is a subset of $\{1, \ldots, N\}$ then the divisor $D(I) \cap U(T')$ lies inside the divisor

$$1 - \prod_{i\in I} \frac{1 - z_i \cdots z_N}{1 - z_{i-1} \cdots z_N} = 0. \tag{3.2}$$

An easy inspection shows that the above divisor has $z_N = 0$ as a component. If we remove this component then the point $R$ does not lie on the remaining components. The reason is that substituting $z_i = i/(i+1)$ for $i < N$ in (3.2) we get

$$\frac{\prod_{i\in I}(1 - [(i-1)/N]z_N) - \prod_{i\in I}(1 - (i/N)z_N)}{\prod_{i\in I}(1 - [(i-1)/N]z_N)},$$

The numerator is $(|I|/N)z_N + \cdots$ where the remaining factors are divisible by $z_N^2$. Now if we divide by $z_N$ and let $z_N = 0$ we get $|I|/N$. □

Let $\mathcal{N}^{00}(N)$ be the normal bundle of $D'_N$ minus the zero section and restricted to $D'^{0}_N = D'_N - \bigcup D$ where $D$ runs over all the divisors of $\overline{\mathcal{M}}_{0,N+3} - \mathcal{M}_{0,N+3}$ other than $D'_N$. This is a $\mathbb{G}_m$-bundle. According to the lemma above we have the following embedding:

$$\iota_N : \mathbb{G}_m = \mathit{Spec}\,\mathbb{Q}\Big[t, \frac{1}{t}\Big] \hookrightarrow \mathcal{N}^{00}(N),$$
$$t \mapsto \Big(\frac{1}{2}, \ldots, \frac{N-1}{N}, Nt\Big).$$

By this we identify $\mathcal{N}^{00}(N)$ with $D'^{0}_N \times \mathbb{G}_m$.

LEMMA 3.3. *The composition*

$$\mathbb{G}_m \xrightarrow{\iota_N} \mathcal{N}^{00}(N) \xrightarrow{\pi_N} \mathcal{N}^{00}(N-1)$$

*is $\iota_{N-1}$. Here $\pi_N$ is the projection induced from $\overline{\mathcal{M}}_{0,N+3} \to \overline{\mathcal{M}}_{0,N+2}$ obtained by neglecting the $(N+2)$nd marked point.*

*Proof.* Note that $\pi_N$ sends the special divisor of $\overline{\mathcal{M}}_{0,N+3}$ to the special divisor of $\overline{\mathcal{M}}_{0,N+2}$. Using the coordinates $z_i$ the equation of $\pi_N$ will become $(z_1,\ldots,z_N) \longrightarrow (z_1,\ldots,z_{N-2},z_{N-1}\cdot z_N)$. (This can be easily derived from the obvious description of the map in $x_i$ coordinates which is $(x_1,\ldots,x_N) \longrightarrow (x_1,\ldots,x_{N-1})$ and a change of variables.) Therefore the point $(\frac{1}{2},\ldots,[(N-1)/N],Nt)$ will map to $(\frac{1}{2},\ldots,[(N-2)/(N-1)],(N-1)t)$ which is by definition $\iota_{N-1}(t)$. $\square$

## 4. Series regularization of $p$-adic multiple zeta values

Let $1 \leqslant l \leqslant m$. The MPL is the series defined by

$$Li_{n_1,\ldots,n_m}(x_l,\ldots,x_m) = \sum_{0<k_1<\cdots<k_m} \frac{x_l^{k_l}\cdots x_m^{k_m}}{k_1^{n_1}\cdots k_m^{n_m}}.$$

Notice that the number of variables could be smaller than the depth $m$. This satisfies the differential equation

$$dLi_{n_1,\ldots,n_m}(x_l,\ldots,x_m) = \sum_{i=l}^{m} \partial_i Li_{n_1,\ldots,n_m}(x_l,\ldots,x_m),$$

where $\partial_i Li_{n_1,\ldots,n_m}(x_l,\ldots,x_m)$ is given by the following formula:

$$\begin{aligned} &Li_{n_1,\ldots,n_i-1,\ldots,n_m}(x_l,\ldots,x_m)d\log x_i, && \text{if } n_i>1,\\ &Li_{n_1,\ldots,\widehat{n_i},\ldots,n_m}(x_l,\ldots,x_{i-1}x_i,\ldots,x_m)d\log(1-x_i) &&\\ &\qquad - Li_{n_1,\ldots,\widehat{n_i},\ldots,n_N}(x_l,\ldots,x_ix_{i+1},\ldots,x_m)d\log x_i(1-x_i), && \text{if } n_i=1. \end{aligned}$$

Here by convention $x_{l-1}=1$ and in the case where $n_m=1$ and $i=m$ we omit the last line, i.e. formally let $x_{m+1}=0$ and assume $Li_{n_1,\ldots,n_m}(x_l,\ldots,x_{m-1},0)=0$. By iterated integrations we get a Coleman function on $\mathcal{M}_{0,N+3}$ (for $N \geqslant m$) satisfying the differential equation and having the above expansion near the origin. We denote the corresponding Coleman function by (we may sometimes omit the '$a$') $Li^a_{n_1,\ldots,n_m}(x_l,\ldots,x_m)$ in accordance with a branch $a \in \mathbb{Q}_p$ of the $p$-adic logarithm. We remark that the way we have parameterized $\mathcal{M}_{0,N+3}$ is specially useful to see that MPL is a Coleman function.

The following proposition describes the behavior of the MPL functions around the divisor $D'_N$ which is essential for our definition of regularized $p$-adic MZV.

PROPOSITION 4.1.

(i) *On the region $]D'_N[\cap\mathcal{M}_{0,N+3}(\mathbb{C}_p)$ the Coleman function $Li^a_{n_1,\ldots,n_m}(x_l,\ldots,x_m)$ can be uniquely expressed as $\sum_{i=0}^{M} f_i^a(z_1,\ldots,z_N)\cdot(\log^a z_N)^i$ for some $M\in\mathbb{N}$, where $f_i^a$ is a locally analytic function on the region, and can be extended to $D_N'^0$ (refer to the previous section for the definition of the coordinates $z_i$ and $D'_N$).*

(ii) *The analytic continuation of $Li^a_{n_1,\ldots,n_m}(x_l,\ldots,x_m)$ to $\mathcal{N}^{00}_{D'_N}$, which can be identified with $D_N'^0\times\mathbb{G}_m$, is given by $\sum_{i=0}^{M} f_i^a(z_1,\ldots,z_{N-1},0)\cdot(\log^a \overline{z}_N)^i$.*

*Proof.* The proof is given by induction on the weight. Both statements are clear for weight 1. Using the formula given before for the differential of MPL we see that by the induction hypothesis the differential of $Li_{n_1,\ldots,n_m}(x_l,\ldots,x_m)$ can be expressed as $\sum g_i\omega_i$ where $g_i$ are Coleman functions of the desired (logarithmic) form and $\omega_i$ are holomorphic forms on $\mathcal{M}_{0,N+3}$ with logarithmic singularity along $z_N = 0$ (and other divisors which we are not interested in). Integrating such a form will give the desired logarithmic expansion. This proves the first part of the proposition. The second part of the proposition follows from the definition of the analytic continuation explained in [BF06, § 4]. $\square$

We are now ready to give the following definition for the series regularized $p$-adic MZV, which is more accurately an element of $\mathbb{Q}_p[T]$. Its definition *a priori* depends on the choice of a branch of $p$-adic logarithm, i.e. a choice of $a \in \mathbb{Q}_p$ for the value of this logarithm at $p$.

Definition 4.2. The *series regularized p-adic MZV* $\zeta_p^S(n_1, \ldots, n_m) \in \mathbb{Q}_p[T]$ is defined as follows. The analytic continuation of $Li^a_{n_1,\ldots,n_m}(x_1, \ldots, x_m)$ to $N^{00}_{D'_N}$ for $N \geqslant m$ can be restricted to the line defined by $\iota_N : \mathbb{G}_m \to \mathcal{N}^{00}_{D'_N}$; this way we get a Coleman function in $\mathbb{Q}_p[\log^a t]$. If we replace $\log^a t$ by $T$ and use the notation of the previous proposition, we get

$$\begin{aligned}\zeta_p^S(n_1, \ldots, n_m) &= \tau_N^* Li^{a(D'_N)}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) \\ &= \sum_{i=0}^{M} f_i^a\left(\frac{1}{2}, \ldots, \frac{N-1}{N}, 0\right) \cdot (T + \log^a N)^i \in \mathbb{Q}_p[T].\end{aligned}$$

This is independent of the choice of $N \geqslant m$ which follows from Lemma 3.3. This regularization will be independent of the choice of the branch $a$. But this will be proved later in Theorem 6.4.

Proposition 4.3. *The MPL $Li^a_{n_1,\ldots n_m}(t, \ldots, t)$ is uniquely expressed in a neighborhood of $t = 1$ as*

$$\sum_{i=0}^{M} g_i^a(t) \log^a(1-t)^i, \tag{4.1}$$

*where $g_i^a$ are locally analytic functions that can be extended to $t = 1$, then $\zeta_p^S(n_1, \ldots, n_m) = \sum_{i=0}^{M} g_i^a(1) T^i$.*

*Proof.* The uniqueness is clear. With the notation of Proposition 4.1 we have

$$g_r^a(t) = \sum_{i=r}^{N} \binom{i}{r} f_i^a\left(\frac{1-t}{1-t^2}, \ldots, \frac{1-t^{N-1}}{1-t^N}, 1-t^N\right)\left(\log^a \frac{1-t^N}{1-t}\right)^{i-r}.$$

By letting $t = 1$ and comparing it with the definition of regularized MZV, the claim follows. $\square$

## 5. Series shuffle relation

We now describe the series shuffle relation for multiple polylogarithms. To do this we need the notion of generalized shuffles of order $r$ and $s$, denoted by

$$\begin{aligned}Sh^{\leqslant}(r, s) := \bigcup_N \{\sigma : \{1, \ldots, r+s\} \to \{1, \ldots, N\} \mid \sigma \text{ is onto}, \\ \sigma(1) < \cdots < \sigma(r), \sigma(r+1) < \cdots < \sigma(r+s)\}.\end{aligned}$$

We recall the definition from [Gon02, § 7.1]. Let

$$\mathbb{Z}^m_{++} = \{(k_1, \ldots, k_m) \in \mathbb{Z}^m_+ \mid 0 < k_1 < \cdots < k_m\} \subset \mathbb{Z}^m.$$

There is a natural decomposition

$$\mathbb{Z}^r_{++} \times \mathbb{Z}^s_{++} = \bigcup_{\sigma \in Sh^{\leqslant}(r,s)} \mathbb{Z}^{\sigma}_{++},$$

where

$$\mathbb{Z}^{\sigma}_{++} := \{(k_1, \ldots, k_{r+s}) \in \mathbb{Z}^{r+s}_{++} \mid k_i < k_j \text{ if } \sigma(i) < \sigma(j),\ k_i = k_j \text{ if } \sigma(i) = \sigma(j)\}.$$

For example for $r = s = 1$ we have

$$\{k_1 > 0\} \times \{k_2 > 0\} = \{0 < k_1 < k_2\} \cup \{0 < k_1 = k_2\} \cup \{0 < k_2 < k_1\}.$$

We define the permuted multiple polylogarithm by

$$Li^{\sigma}_{n_1,\ldots,n_{r+s}}(x_1, \ldots, x_{r+s}) = \sum_{(k_1,\ldots,k_{r+s}) \in \mathbb{Z}^{\sigma}_{++}} \frac{x_1^{k_1} \cdots x_{r+s}^{k_{r+s}}}{k_1^{n_1} \cdots k_{r+s}^{n_{r+s}}}.$$

Then formally we have

$$\begin{aligned} &Li_{n_1,\ldots,n_r}(x_1, \ldots, x_r) Li_{n_{r+1},\ldots,n_{r+s}}(x_{r+1}, \ldots, x_{r+s}) \\ &\quad = \sum_{\sigma \in Sh^{\leqslant}(r,s)} Li^{\sigma}_{n_1,\ldots,n_{r+s}}(x_1, \ldots, x_{r+s}). \end{aligned} \tag{5.1}$$

In fact $Li^{\sigma}_{n_1,\ldots,n_{r+s}}(x_1, \ldots, x_{r+s})$ is of the form $Li_{q_1,\ldots,q_l}(y_1, \ldots, y_l)$ with the same weight and the $y_i$ are either one of the $x_j$ or the product of two of the $x_j$. If we let $N = r + s$, notice that this function *cannot* be considered in general as a Coleman function on $\mathcal{M}_{0,N+3}$. This follows from the fact that the parameterization of $\mathcal{M}_{0,N+3}$ is not symmetric with respect to the permutation of the coordinates $x_1, \ldots, x_N$. Recall that we only remove the product of the consecutive coordinates equaling 1. However, all of these functions can be considered as Coleman functions on $E_N$ which was defined in § 3.

Since the shuffle formula given above is formal, its validity can be extended if we regard the functions as Coleman functions on $E_N$. The idea of the proof of series shuffle relation for MZV is to restrict to the line $(t, t, \ldots, t)$ and use the asymptotic expansion around $t = 1$. The crucial step is the following proposition, which was inspired by Proposition 7.7 of [Gon02].

Proposition 5.1. *Let $1 = p_1 \leqslant p_2 \leqslant \cdots \leqslant p_m \lneqq p_{m+1} = N + 1$ be integers and let $y_i = \prod_{j=p_i}^{p_{i+1}-1} x_j$ (the empty product is defined to be 1). Assume below that $n_k > 1$ and $k + l = m$. Let $F$ be the following Coleman function on $\mathcal{M}_{0,N+3}$: for $l > 0$, $F$ is*

$$Li^a_{n_1,\ldots,n_k,\underbrace{1,\ldots,1}_{l}}(y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+l}) - Li^a_{n_1,\ldots,n_k,\underbrace{1,\ldots,1}_{l}}(y_{k+1}, \ldots, y_{k+l})$$

*and for $l = 0$, $F$ is*

$$Li^a_{n_1,\ldots,n_k}(y_1, \ldots, y_k) - Li^a_{n_1,\ldots,n_k}(y_k).$$

*For a divisor $D$ in $\overline{\mathcal{M}}_{0,N+3} - \mathcal{M}_{0,N+3}$ let $F^{(D)}$ denote the extension of $F$ to the normal bundle $\mathcal{N}^{00}_D$. Then $F^{(D)} = 0$ for $D = D'_N, D'_{N-1}$ and $D_N$.*

*Proof.* Notice that $D'_N$ intersects $D'_{N-1}$ and $D'_{N-1}$ intersects $D_N$. In fact we saw in § 2 that all the divisors $D'_i$ intersect at a single point $Q$. To see that $D'_{N-1}$ and $D_N$ intersect we can use part (v) of Proposition 2.1.

We will prove below that $F^{(D_N)}$ is zero and $F^{(D'_N)}$ and $F^{(D'_{N-1})}$ are constant. Now since $D'_{N-1}$ and $D_N$ intersect if we apply Proposition 3.6 of [BF06] it follows that $F^{(D'_{N-1})}$ is zero. A similar argument using the divisors $D'_{N-1}$ and $D'_N$ implies that $F^{(D'_N)}$ vanishes as well.

Lemma 5.2. *The extension of $Li^a_{n_1,\ldots,n_m}(y_1, \ldots, y_m)$ to $\mathcal{N}^{00}_{D_N}$ is zero.*

*Proof.* The constant term of MPL at the origin, i.e. the intersection of the $D_i$ for $i = 1, \ldots, N$, is zero. This follows from the fact that in the neighborhood of the origin we have the power series



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expansion without constant term. We now calculate the differential of the MPL and take its residue at $x_N = 0$. The differential which contributes is the differential with respect to the last parameter $y_m$. If $n_m = 1$ it is $Li_{n_1,\dots,n_{m-1}}(y_1,\dots,y_{m-2},y_{m-1}y_m)d\log(1-y_m)$; however, its residue along $x_N = 0$ is zero. If $n_m > 1$ it is $Li_{n_1,\dots,n_{m-1},n_m-1}(y_1,\dots,y_m)d\log y_m$ and its residue along $x_N = 0$ is $Li_{n_1,\dots,n_{m-1},n_m-1}(y_1,\dots,y_m)$, which is an MPL of weight one smaller than the original MPL. Hence by using induction we can deduce that the extension of MPL to $\mathcal{N}^{00}_{D_N}$ will be zero. This finishes the proof of the lemma. □

Let us now show that $F^{(D'_j)}$ is constant for $j = N-1, N$. Recall that the coordinates $z_i$ for the divisors $D'_i$ are related to the original coordinates $x_i$ by

$$x_i = \frac{1 - z_i \cdots z_N}{1 - z_{i-1} \cdots z_N}.$$

The residue of $d\log x_i$ at $z_N = 0$ and $z_{N-1} = 0$ is zero for $i < N$. This implies that the differentials with respect to those indices $i$ for which $n_i > 1$ do not contribute. If $n_i = 1$ for $i < k$ then the differential of $F$ with respect to $y_i$ (we are assuming $y_i \not\equiv 1$) is

$$Li_{n_1,\dots,\hat{n_i},\dots,n_k,\underbrace{1,\dots,1}_{l}}(y_1,\dots,y_{i-1}y_i,\dots,y_{k+l})d\log(1-y_i)$$
$$- Li_{n_1,\dots,\hat{n_i},\dots,n_k,\underbrace{1,\dots,1}_{l}}(y_1,\dots,y_iy_{i+1},\dots,y_{k+l})d\log y_i(1-y_i).$$

Since $d\log y_i$ has residue zero along $z_N = 0$ or $z_{N-1} = 0$ an induction on the weight shows that this difference is zero when the residue is taken. So the only variables that are left are those $y_{k+1},\dots,y_{k+l}$ that are not identically 1 when $l > 0$ and $y_k$ when $l = 0$. The induction implies that these also do not have any contribution. We provide the details for the case when $l > 0$, the case $l = 0$ being similar and even simpler. The differential with respect to $y_i$ when $i > k$ is given by

$$Li_{n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1}}(y_1,\dots,y_{i-1}y_i,\dots,y_{k+l})d\log(1-y_i)$$
$$- Li_{n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1}}(y_{k+1},\dots,y_{i-1}y_i,\dots,y_{k+l})d\log(1-y_i)$$
$$- Li_{n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1}}(y_1,\dots,y_iy_{i+1},\dots,y_{k+l})d\log y_i(1-y_i)$$
$$+ Li_{n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1}}(y_{k+1},\dots,y_{i-1}y_i,\dots,y_{k+l})d\log y_i(1-y_i).$$

Now it is clear that induction on the weights implies that the first two and the last two terms will cancel each other after taking the residues. This finishes the proof of Proposition 5.1. □

Corollary 5.3. *With the notation of Proposition 5.1 and $n_k > 1$, the analytic continuation of $Li^a_{n_1,\dots,n_k}(y_1,\dots,y_k)$ to $\mathcal{N}^{00}_{D'_N}$ is constant and is equal to $\zeta_p(n_1,\dots,n_k)$. Therefore in this case $\zeta^S_p(n_1,\dots,n_m)$ coincides with the p-adic MZV in [Fur04].*

*Proof.* By Proposition 5.1 the analytic continuation is the same as the analytic continuation of $Li^a_{n_1,\dots,n_k}(y_k)$. If $y_k = x_m$ then the claim follows from Lemma 3.3 and the definition of $p$-adic MZV. If $y_k = x_i \cdots x_m$, a similar argument as above using the differential equation of MPL shows that the analytic continuation of $Li^a_{n_1,\dots,n_k}(y_k) - Li^a_{n_1,\dots,n_k}(x_m)$ to $D'_N$ and $D'_{N-1}$ is constant and it is zero if it is continued to $D_N$. This finishes the proof. □



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## 6. Proof of the main theorems

We will derive the validity of series shuffle relations from (5.1). The main idea is to get a nice means of analytic continuation to give series shuffle relations. A difficulty is that the terms of (5.1) are not Coleman functions of $\mathcal{M}_{0,N+3}$ $(N = r+s)$ but Coleman functions of $E_N$ which generally no longer have a good reduction. Hence we are not able to use directly the method of analytic continuation to normal bundle explained in § 1 and 4. In order to achieve this we introduce a subfamily $\mathrm{Col}'^a$ inside $\mathrm{Col}^a(E_N)$ which contains all terms of (5.1) and in purely algebraic way we extend the methods of analytic continuation into $\mathrm{Col}'^a$ to give series shuffle relations.

Let $\tau \in \mathfrak{S}_m$. Put $\mathcal{M}^\tau_{0,N+3} = \mathcal{M}_{0,N+3} \times_{\mathbb{A}^N,\tau} \mathbb{A}^N$ where $\tau : \mathbb{A}^N \to \mathbb{A}^N$ is a map sending $x_i \to x_\tau(i)$, i.e.

$$\mathcal{M}^\tau_{0,N+3} = \mathbb{G}_m^N - \bigcup_{1\leqslant n\leqslant m\leqslant N} \Big\{\prod_{i=n}^m x_{\tau(i)} = 1\Big\}.$$

Denote its coordinate along $D'^\tau_N$ by

$$z_i^\tau = \frac{1 - x_{\tau(1)}\cdots x_{\tau(i)}}{1 - x_{\tau(1)}\cdots x_{\tau(i-1)}}.$$

Let $\mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})(\log D'^\tau_N)$ be the subalgebra of $\mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})$ each of which has an expansion $\sum_{i=0}^M f_i^a(z_1^\tau, \ldots, z_N^\tau)\cdot(\log^a z_N^\tau)^i$ as Proposition 4.1(i). Let $\mathrm{Col}'^a$ to be the subalgebra inside $\mathrm{Col}^a(E_N)$ generated by all $\mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})(\log D'^\tau_N)$ for $\tau \in \mathfrak{S}_N$. Note that $E_N \subset \mathcal{M}^\tau_{0,N+3}$ and hence $\mathrm{Col}^a(E_N)$ contains $\mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})$ for all $\tau$. For each $f \in \mathrm{Col}'^a$ its restriction into $\ell = \{(t, \ldots, t)\}$ has an expansion $\sum_{i=0}^M g_i^a(t)\log^a(1-t)^i$ like (4.1). By the algebraic prescription described in Proposition 4.3 we get an algebra homomorphism

$$\widetilde{t}^a_N : \mathrm{Col}'^a \to \mathbb{Q}_p[[T]]$$

sending it to $\sum_{i=0}^M g_i^a(1)\log^a(1-t)^i$. Its restriction to $\mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})(\log D'^\tau_N)$ agrees with

$$t_N^{a,\tau} : \mathrm{Col}^a(\mathcal{M}^\tau_{0,N+3})(\log D'^\tau_N) \to \mathbb{Q}_p[[T]],$$

which is the analytic continuation into $\mathcal{N}^{00}_{D'_N}$ and the restriction into $\tau_N(\mathbb{G}_m)$.

We note that $Li_{n_1,\ldots,n_m}(x_{\tau(1)}, \ldots, x_{\tau(m)}) \in \mathrm{Col}'^a$ for $\tau \in \mathfrak{S}_m$ by Proposition 4.3.

Lemma 6.1. *We have* $\widetilde{t}^a_m(Li_{n_1,\ldots,n_m}(x_{\tau(1)}, \ldots, x_{\tau(m)})) = \zeta_p^S(n_1, \ldots, n_m)$ *for any* $\tau \in \mathfrak{S}_m$.

*Proof.* The proof is clear because $Li_{n_1,\ldots,n_m}(x_1, \ldots, x_m)|_\ell = Li_{n_1,\ldots,n_m}(x_{\tau(1)}, \ldots, x_{\tau(m)})|_\ell$. □

Lemma 6.2. *With the notation of formula (5.1), the image of* $Li^\sigma_{n_1,\ldots,n_{r+s}}(x_1, \ldots, x_{r+s})$ $(\sigma \in Sh^\leqslant(r,s))$ *under the tangential morphism* $\widetilde{t}^a_{r+s}$ *is* $\zeta_p^S(\sigma(n_1, \ldots, n_{r+s}))$. *Here* $\sigma(n_1, \ldots, n_{r+s}) = (c_1, \ldots, c_N)$ *where $N$ is the cardinality of the image of $\sigma$ and*

$$c_i = \begin{cases} n_m + n_l, & \text{if } \sigma^{-1}(i) = \{m, l\},\\ n_m, & \text{if } \sigma^{-1}(i) = \{m\}.\end{cases}$$

*Proof.* Note that

$$Li^\sigma_{n_1,\ldots,n_{r+s}}(x_1, \ldots, x_{r+s}) = Li_{c_1,\ldots,c_N}(y_1, \ldots, y_N)$$

where

$$y_i = \begin{cases} x_m x_l, & \text{if } \sigma^{-1}(i) = \{m, l\},\\ x_m, & \text{if } \sigma^{-1}(i) = \{m\}.\end{cases}$$

Now since the tangential morphism $t^a_{r+s}$ is invariant under the action of the symmetric group, with a permutation of the parameters we can assume that we are in the situation of Proposition 5.1.



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Therefore if $c_N > 1$ the result follows from Corollary 5.3. If for some $k$ we have $c_{k-1} > 1$ and $c_k = \cdots = c_N = 1$ then we necessarily have $y_k = x_i, \ldots, y_N = x_{r+s}$ with $i = r+s+k-N$. Proposition 5.1 implies that the extension of

$$Li_{c_1,\ldots,c_{k-1},\underbrace{1,\ldots,1}_{N-k+1}}(y_1,\ldots,y_N)$$

to $\mathcal{N}^{00}_{D'_{r+s}}$ is the same as the extension of

$$Li_{c_1,\ldots,c_{k-1},\underbrace{1,\ldots,1}_{N-k+1}}(x_i,\ldots,x_{r+s}) \in \mathrm{Col}^a(\mathcal{M}_{0,r+s})$$

to $\mathcal{N}^{00}_{D'_{r+s}}$. Another application of the same proposition implies that this extension is the same as the analytic continuation of

$$Li_{c_1,\ldots,c_{k-1},\underbrace{1,\ldots,1}_{N-k+1}}(x_{r+s-N+1},\ldots,x_{r+s})$$

to $\mathcal{N}^{00}_{D'_{r+s}}$. This is a Coleman function on $\mathcal{M}_{0,r+s}$ with variables $(x_1,\ldots,x_{r+s})$. Using Lemma 6.1 to calculate its image of $\widetilde{t}^a_{r+s}$ we may replace it by

$$Li_{c_1,\ldots,c_{k-1},\underbrace{1,\ldots,1}_{N-k+1}}(x_1,\ldots,x_N) \in \mathrm{Col}^a(\mathcal{M}_{0,r+s}).$$

By Lemma 3.3 the image of this function under the map $t^a_{r+s}$ is the same if we consider it as a function on $\mathcal{M}_{0,N}$ and apply the map $t^a_N$. By definition therefore the image under $t^a_N$ of this function as a Coleman function on $\mathcal{M}_{0,N}$ is $\zeta^S_p(c_1,\ldots,c_N)$. □

Hence we have $\widetilde{t}^a_{r+s}(Li^\sigma_{n_1,\ldots,n_{r+s}}(x_1,\ldots,x_{r+s})) = \zeta^S_p(\sigma(n_1,\ldots,n_{r+s}))$.

Theorem 6.3. *Series shuffle relations for series regularized $p$-adic MZV hold, i.e.*

$$\zeta^S_p(n_1,\ldots,n_r)\cdot\zeta^S_p(n_{r+1},\ldots,n_{r+s}) = \sum_{\sigma\in Sh^{\leqslant}(r,s)} \zeta^S_p(\sigma(n_1,\ldots,n_{m+p}))$$

*holds for all $r,s,n_1,\ldots,n_{r+s} \geqslant 1$.*

*Proof.* Apply the homomorphism $\widetilde{t}^a_{r+s}$ to both sides of the identity (5.1) and use the previous lemma. □

It is interesting that this theorem together with the observation that $\zeta^S_p(1) = -T$ as a polynomial in $\mathbb{Q}_p[T]$, which follows from the fact that $Li^a_1(z) = -\log^a(1-z)$, implies that the definition of the series regularization is independent of the branch of the $p$-adic logarithm.

Theorem 6.4. *The definition of series regularized $p$-adic MZV $\zeta^S_p(n_1,\ldots,n_m)$ does not depend on the choice of a branch $a \in \mathbb{Q}_p$ of the $p$-adic logarithm.*

*Proof.* This is clear if $n_m > 1$, since by Corollary 5.3 we have $\zeta^S_p(n_1,\ldots,n_m) = \zeta_p(n_1,\ldots,n_m)$, which is independent of the branch (cf. [Fur04]). Now assume that $n_k > 1$ and $n_{k+1} = \cdots = n_{k+l} = 1$



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where $k+l=m$. Then the series shuffle relation implies that

$$\begin{aligned}
&\zeta_p^S(1)\zeta_p^S(n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1})\\
&\quad = l\zeta_p^S(n_1,\dots,n_k,\underbrace{1,\dots,1}_{l})\\
&\qquad + \zeta_p^S(n_1,\dots,n_{k-1},1,n_k,\underbrace{1,\dots,1}_{l-1}) + \dots + \zeta_p^S(1,n_1,\dots,n_k,\underbrace{1,\dots,1}_{l-1})\\
&\qquad + \zeta_p^S(n_1+1,\dots,n_k,\underbrace{1,\dots,1}_{l-1}) + \dots + \zeta_p^S(n_1,\dots,n_k,1,\dots,1,2).
\end{aligned}$$

Now an induction on $l$ proves the theorem. $\square$

Let us now explain the integral regularization of $p$-adic MZVs. The one variable MPL

$$Li_{n_1,\dots,n_m}(z) = \sum_{0<k_1<\dots<k_m} \frac{z^{k_m}}{k_1^{n_1}\cdots k_m^{n_m}}$$

can be viewed as a Coleman function on $\mathcal{M}_{0,4}=E_1$. Its image under the tangential morphism $t_1^a$ is the integral regularization of the $p$-adic MZV and we use the notation $\zeta_p^I(k_1,\dots,k_m)$. This is an element of $\mathbb{Q}_p[T]$ where $T=\log^a(1-z)$. By the $p$-adic iterated integral expression of $p$-adic MPL, the first author in [Fur04] deduced an integral shuffle product formula that we now explain. For $\mathbf{k}=(k_1,\dots,k_m)$ and $\mathbf{k}'=(k'_1,\dots,k'_{m'})$ with $k_i,k'_j\geqslant 1$ the following formula holds for $p$-adic MPLs:

$$Li_{\mathbf{k}}(z)Li_{\mathbf{k}'}(z) = \sum_{\tau\in Sh(N,N')} Li_{\mathbf{a}_{\tau(W_{\mathbf{k}},W_{\mathbf{k}'})}}(z). \tag{6.1}$$

Here $N=k_1+\dots+k_m$, $N'=k'_1+\dots+k'_{m'}$ and

$$\begin{aligned}Sh(N,N') := \{\tau:\{1,\dots,N+N'\}\to\{1,\dots,N+N'\} \mid \tau \text{ is bijective},\\ \tau(1)<\dots<\tau(N), \tau(N+1)<\dots\tau(N+N')\}.\end{aligned}$$

For $W=X_1\cdots X_k$, $W'=X_{k+1}\cdots X_{k+l}$ with $X_i\in\{A,B\}$ and $\tau\in Sh(k,l)$, the symbol $\tau(W,W')$ stands for $Z_1\cdots Z_{k+l}$ with $Z_i=X_{\tau^{-1}(i)}$. For $\mathbf{a}=(a_1,\dots,a_l)$ with $l,a_1,\dots,a_l\geqslant 1$ the symbol $W_{\mathbf{a}}$ means a word $A^{k_l-1}BA^{k_{l-1}-1}B\cdots A^{k_1-1}B$ and for such $W$ we denote its corresponding index by $\mathbf{a}_W$.

Each term in (6.1) lies in $\mathrm{Col}^a(\mathcal{M}_{0,4})$. Applying the morphism $t_1^a$ to identity (6.1) gives the following result.

PROPOSITION 6.5. *The integral series shuffle relation for integral regularized $p$-adic MZVs holds, i.e.*

$$\zeta_p^I(\mathbf{k})\zeta_p^I(\mathbf{k}') = \sum_{\tau\in Sh(N,N')} \zeta_p^I(\mathbf{a}_{\tau(W_{\mathbf{k}},W_{\mathbf{k}'})})$$

*holds for $\mathbf{k}=(k_1,\dots,k_m)$ and $\mathbf{k}'=(k'_1,\dots,k'_{m'})$.*

Note that $\zeta_p^I(1)=-T$ and therefore the integral shuffle relation implies that

$$\zeta_p^I(\underbrace{1,\dots,1}_{n}) = \frac{(-T)^n}{n!}. \tag{6.2}$$

The proof of the regularization relation is a $p$-adic analogue of the proof given in § 7 of [Gon02].



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Theorem 6.6. *The regularization relation holds. Namely for $n_1,\dots,n_m \geqslant 1$*

$$\zeta_p^S(n_1,\dots,n_m) = \mathbb{L}_p(\zeta_p^I(n_1,\dots,n_m)) \tag{6.3}$$

*where $\mathbb{L}_p : \mathbb{Q}_p[T] \longrightarrow \mathbb{Q}_p[T]$ is a linear map that is defined by*

$$\sum_{n=1}^{\infty} \mathbb{L}_p(T^n)\frac{u^n}{n!} = \exp\biggl(-\sum_{n=1}^{\infty}\frac{\zeta_p^I(n)}{n}u^n\biggr).$$

*Proof.* The validity of (6.3) is clear if $n_m > 1$. The following special case for $(n_1,\dots,n_m) = (1,\dots,1)$ can be proved exactly as in Lemma 7.9 of [Gon02]:

$$\sum_{n=1}^{\infty}\zeta_p^S(\underbrace{1,\dots,1}_{n})u^n = \exp\biggl(-\sum_{n=1}^{\infty}\frac{\zeta_p^I(n)}{n}(-u)^n\biggr). \tag{6.4}$$

Assume that $n_k > 1$ and $n_{k+1} = \cdots = n_m = 1$. We prove the regularization formula by induction on $m-k$. Note that by (5.1) we have

$$\begin{aligned} &Li_{n_1,\dots,n_k}(x_1,\dots,x_k)Li_{\underbrace{1,\dots,1}_{m-k}}(x_{k+1},\dots,x_m) \\ &\quad = Li_{n_1,\dots,n_m,\underbrace{1,\dots,1}_{m-k}}(x_1,\dots,x_m) + \text{other terms}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} &Li_{n_1,\dots,n_k}(\underbrace{1,\dots,1}_{k-1},x)Li_{\underbrace{1,\dots,1}_{m-k}}(\underbrace{1,\dots,1}_{m-k-1},y) \\ &\quad = Li_{n_1,\dots,n_m,\underbrace{1,\dots,1}_{m-k}}(\underbrace{1,\dots,1}_{k-1},x,\underbrace{1,\dots,1}_{m-k-1},y) + \text{other terms}, \end{aligned} \tag{6.6}$$

where 'other terms' above means the collection of terms whose number of ones at the end is less than $m-k$. We apply the morphism $\widetilde{t}_N^a : \mathrm{Col}'^a \longrightarrow \mathrm{Col}^a(\mathbb{G}_m)$ to (6.5) and the tangential map $t_2^a : \mathrm{Col}^a(\mathcal{M}_{0,5}) \longrightarrow \mathrm{Col}^a(\mathbb{G}_m)$ to (6.6). Now if we use Proposition 5.1 it follows that the first equation gives the series regularized $p$-adic MZVs and the second will give the integral regularized $p$-adic MZVs. We therefore have

$$\zeta_p^S(n_1,\dots,n_k)\zeta_p^S(\underbrace{1,\dots,1}_{m-k}) = \zeta_p^S(n_1,\dots,n_m,\underbrace{1,\dots,1}_{m-k}) + \text{other terms}, \tag{6.7}$$

$$\zeta_p^I(n_1,\dots,n_k)\zeta_p^I(\underbrace{1,\dots,1}_{m-k}) = \zeta_p^I(n_1,\dots,n_m,\underbrace{1,\dots,1}_{m-k}) + \text{other terms}. \tag{6.8}$$

The left-hand side of (6.8) after applying the map $\mathbb{L}_p$ coincides with the left-hand side of (6.7). This follows from (6.2) and (6.4). Also note that the terms which are not written in (6.7) and (6.8) have less than $m-k$ ones at the end, so by induction after applying $\mathbb{L}_p$ will match. This finishes the proof. □

## 7. Deligne's problem on double shuffle relations

In [Del02], Deligne proposed the following definition for $p$-adic MZVs. Let $X = \mathbb{P}^1\backslash\{0,1,\infty\}$ and $\pi^{\mathrm{DR}}(X,\overrightarrow{01})$ denote the de Rham fundamental group of $X$ with the tangential base point $\overrightarrow{01}$ at 0. This can be identified as the group-like elements with constant term 1 of the non-commutative power series Hopf algebra $\mathbb{Q}\langle\langle A,B\rangle\rangle$, where $A$ corresponds to the loop around 0 and $B$ to the loop around 1. The coproduct is defined by $\Delta A = A\otimes 1 + 1\otimes A$ and similarly for $B$. Since $X$ and the base point have a good reduction modulo $p$, we have an action of the Frobenius endomorphism $\phi$

on this fundamental group tensored with $\mathbb{Q}_p$ which can be extended to $\mathbb{Q}_p\langle\langle A, B\rangle\rangle$. It can be shown that under this endomorphism[2]

$$A \mapsto \frac{A}{p},$$
$$B \mapsto (\Phi^p_{\mathrm{De}})^{-1}\left(\frac{B}{p}\right)\Phi^p_{\mathrm{De}},$$

for a certain group-like element $\Phi^p_{\mathrm{De}}$ of $\mathbb{Q}_p\langle\langle A, B\rangle\rangle$ with constant term 1 and the coefficients of $B^n$ equal zero. Deligne defines $(-1)^m\zeta_p^{\mathrm{De}}(n_1, \dots, n_m)$ to be the coefficient of $A^{n_m-1}B\cdots A^{n_m-1}B$ in $\Phi^p_{\mathrm{De}}$.

It was asked in [Del02] and [DG05] to prove the validity of the generalized double shuffle relation for these $p$-adic MZVs. This is achieved using the results of this paper and [Fur07]. In fact in the language of Racinet in [Rac02] we need to show that $\Phi^p_{\mathrm{De}}(A, -B) \in \underline{DMR}_0(\mathbb{Q}_p)$. We recall his machinery briefly. The group scheme $\underline{DMR}_0$ has $k$ (a field of characteristic 0) valued points which is a subset of power series $k\langle\langle A, B\rangle\rangle$ of those power series $g = \sum c_W W$, where $W$ runs over all words in $A$ and $B$, such that the following hold.

(i) The constant term $c_\emptyset = 1$ and $c_A = c_B = 0$.

(ii) The series $g$ is group-like with respect to the coproduct defined above, i.e. $\Delta g = g \otimes g$. This is a concise way of saying that the coefficients of $g$ satisfy the integral shuffle relation.

(iii) Let $\pi_y : k\langle\langle A, B\rangle\rangle \longrightarrow k\langle\langle y_1, y_2, \dots\rangle\rangle$ be defined as a linear map that sends all the words ending $A$ to zero and the word $A^{n_1-1}B\cdots A^{n_m-1}B$ to $y_{n_1}\cdots y_{n_m}$. Define the coproduct $\Delta_*$ on $k\langle\langle y_1, y_2, \dots\rangle\rangle$ by

$$\Delta_* y_n = \sum_{i=0}^{n} y_i \otimes y_{n-i}, \quad y_0 := 1.$$

Also define

$$g_* = \exp\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}c_{A^{n-1}B}y_1^n\right)\pi_y(g).$$

The last condition is that $\Delta_* g_* = g_* \otimes g_*$. This is a concise way of saying that the coefficients of $g_*$ satisfy the series shuffle relation.

In [Fur04] a fundamental solution, denoted by $G_0(z)(A, B)$, for the $p$-adic KZ equation

$$dG(z) = \left(A\frac{dz}{z} + B\frac{dz}{z-1}\right)G(z), \quad z \in \mathbb{P}^1(\mathbb{C}_p)\backslash\{0, 1, \infty\},$$

was constructed. Its coefficients are Coleman functions on $\mathbb{P}^1\backslash\{0, 1, \infty\}$. If we analytically continue this function to the tangent vector 1 at $z = 1$, we get a power series $\Phi^p_{\mathrm{KZ}}(A, B) \in \mathbb{Q}_p\langle\langle A, B\rangle\rangle$, called the $p$-adic Drinfel'd associator, whose coefficient for $A_{n_m-1}B\cdots A^{n_1-1}B$ is the integral regularization $(-1)^m\zeta_p^I(n_1, \dots, n_m)$ evaluated at $T = 0$. The main result of this paper says that in the language of Racinet

$$\Phi^p_{\mathrm{KZ}}(A, -B) \in \underline{DMR}_0(\mathbb{Q}_p).$$

Now we can prove the following theorem.

THEOREM 7.1. *Deligne's $p$-adic MZV satisfies the generalized double shuffle relation.*

*Proof.* It is shown in Theorem 2.7 of [Fur07] that

$$\Phi^p_{\mathrm{KZ}}(A, -B) = \Phi^p_{\mathrm{De}}(A, -B)\cdot\Phi^p_{\mathrm{KZ}}\left(\frac{A}{p}, \Phi^p_{\mathrm{De}}(A, -B)^{-1}\frac{B}{p}\Phi^p_{\mathrm{De}}(A, -B)\right).$$

[2] We are using the inverse of the usual Frobenius as opposed to [Del02].



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This can be rewritten using the product $\circledast$ of $\underline{DMR}_0$:

$$\Phi^p_{\mathrm{KZ}}(A,-B)=\Phi_{\mathrm{KZ}}\left(\frac{A}{p},-\frac{B}{p}\right)\circledast\Phi^p_{\mathrm{De}}(A,-B).$$

The set $\underline{DMR}_0(\mathbb{Q}_p)$ and also $\mathbb{Q}_p\langle\langle A,B\rangle\rangle$ form a group under this product (see [Rac02]) and the two elements $\Phi^p_{\mathrm{KZ}}(A,-B)$ and $\Phi^p_{\mathrm{KZ}}(A/p,-B/p)$ belong to this group, hence $\Phi^p_{\mathrm{De}}(A,-B)\in\underline{DMR}_0(\mathbb{Q}_p)$. Let

$$\tilde{\Phi}^p_{\mathrm{De}}(A,B):=\exp(BT)\Phi^p_{\mathrm{De}}(A,B).$$

The coefficient of $A^{n_m-1}B\cdots A^{n_1-1}B$ in $\tilde{\Phi}^p_{\mathrm{De}}$ is denoted by $(-1)^m\zeta_p^{\mathrm{De},I}(n_1,\dots,n_m)$. This is $(-1)^m\zeta_p^{\mathrm{De}}(n_1,\dots,n_m)$ if $n_m>1$, and if $n_m=1$ this is a polynomial in terms of $T$ for which if we let $T=0$ we get $(-1)^m\zeta_p^{\mathrm{De}}(n_1,\dots,n_m)$. The series regularization is obtained by applying $\mathbb{L}_p$, i.e. the coefficient of $A^{n_m-1}B\cdots A^{n_1-1}B$ in $\mathbb{L}_p(\tilde{\Phi}^p_{\mathrm{De}})$ is defined to be $(-1)^m\zeta_p^{\mathrm{De},S}(n_1,\dots,n_m)$. The fact that $\Phi^p_{\mathrm{De}}\in\underline{DMR}_0(\mathbb{Q}_p)$ implies that $\zeta_p^{\mathrm{De},I}(n_1,\dots,n_m)$ will satisfy the integral shuffle relations and $\zeta_p^{\mathrm{De},S}(n_1,\dots,n_m)$ will satisfy the series shuffle relations. Note that the relation between the two regularizations automatically holds by the way we have defined the second regularization. $\square$


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## References


Bes02 A. Besser, *Coleman integration using the Tannakian formalism*, Math. Ann. **322** (2002), 19–48.

Bes05 A. Besser, $p$-adic Arakelov theory, *J. Number Theory* **111** (2005), 318–371.

BF06 A. Besser and H. Furusho, *The double shuffle relations for p-adic multiple zeta values*, Contemporary Mathematics, vol. 416 (American Mathematical Society, Providence, RI, 2006), 9–29.

CLS99 B. Chiarellotto and B. Le Stum, *F-isocristaux unipotents*, Compositio Math. **116** (1999), 81–110.

Del89 P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, in *Galois groups over Q*, Berkeley, CA, 1987, Math. Sci. Res. Inst. Publ., vol. 16 (Springer, New York, 1989), 79–297.

Del02 P. Deligne, Arizona Winter School 2002, Course and project description, http://swc.math.arizona.edu/notes.

DG05 P. Deligne and A. B. Goncharov, *Groupes fondamentaux motivique de Tate mixte*, Ann. Sci. École Norm. Sup. (4) **38** (2005), 1–56.

Fur04 H. Furusho, *p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation*, Invent. Math. **155** (2004), 253–286.

Fur07 H. Furusho, *p-adic multiple zeta values II. Tannakian interpretations*, Amer. J. Math. **129** (2007), 1105–1144.

GHP88 L. Gerritzen, F. Herrlich and M. van der Put, *Stable n-pointed trees of projective lines*, Indag. Math. **50** (1988), 131–163.

Gon02 A. B. Goncharov, *Periods and mixed motives*, Preprint (2002), math.AG/0202154.

GM04 A. B. Goncharov and Yu. I. Manin, *Multiple zeta-motives and moduli spaces* $M_{0,n}$, Compositio Math. **140** (2004), 1–14.

IKZ06 K. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), 307–338.

Man99 Yu. I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, vol. 47 (American Mathematical Society, Providence, RI, 1999).

Rac02 G. Racinet, *Doubles mélanges des polylogarithmes multiples aux racines de l'unité*, Publ. Math. Inst. Hautes Études Sci. **95** (2002), 185–231.

Ser80 J.-P. Serre, *Trees*, translated from the French by J. Stillwell (Springer, Berlin, 1980).

Shi02 A. Shiho, *Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo **9** (2002), 1–163.

Vol03 V. Vologodsky, *Hodge structure on the fundamental group and its application to p-adic integration*, Mosc. Math. J. **3** (2003), 205–247.

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