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PLANAR MAPS ARE WELL LABELED TREES

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In the theory of enumeration, the part devoted to the counting of planar maps gives rather surprising results. Of special interest to the combinatorialists is the conspicuous feature of counting numbers associated with families of maps as discussed in the papers of Tutte, Brown and Mullin. These formulas are by no means easy to prove; this is also a part of their charm. We are convinced therefore that these maps possess deep combinatorial properties. We discuss such properties in this paper.

The main result of this article is the construction of a bijection between planar maps and a special family of trees: their vertices are labeled with numbers that differ by at most 1 on adjacent vertices. These trees are said to be well labeled. Combinatorialists usually claim that theorems specifying the "geometry" of objects lead to enumerating formulas. Our result obeys this general rule: we can indeed obtain a new proof of the formula counting the number of rooted planar maps with m edges. Four different proofs of this formula are known to date. The first one was given by Tutte [**14**] using a bijection from maps with m edges onto maps with m vertices all of degree four and then using the counting formula for slicings [**13**]. A second proof by Tutte [**15**] consists in solving a functional equation verified by the generating formal power series of these numbers. A third one by Lehman [**10**] is established by encoding maps with parenthesis bracket systems. The fourth proof by Cori and Richard [**3**] consists in constructing a noncommutative generating formal power series for these numbers, then establishing an equation that it satisfies and finally building a solving process for this kind of equation.

Our paper, which we have tried to make self contained, is divided into five parts. In the first part we state the combinatorial definition of maps used by Edmonds [**6**], Jacques [**8**], Tutte and some others. This approach using a permutation enables us to define trees (usually called plane trees [**4**]) and well labeled trees. In the second part, we construct a bijection from planar maps onto well labeled trees in great detail and prove our main result (Theorem A).

In order to find Tutte's formula we intend to encode well labeled trees with words that generalise parenthesis systems. Hence, the aim of the third part is to introduce essential notations and definitions on words, then to reconsider the classical construction of a tree by a parenthesis

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system using the representation of a tree by a permutation. The fourth part contains a study of the coding of well labeled trees. The reader who is familiar with formal language theory will notice that the set C of words encoding well labeled trees is a language which is the intersection of two context free languages but which is not itself context free. Finally, in the fifth section, we deduce the formula which enumerates the well labeled trees from a property of the language C (Theorem B) and hence we are able to enumerate planar maps.

1. Combinatorial maps.

1.1. *Definitions.* Let Z_m denote the set of non zero integers whose absolute value is at most m:

$$Z_m = \{-m, -m+1, \ldots, -2, -1, 1, 2, \ldots, m\}.$$

For any σ, a permutation acting on Z_m, $z(\sigma)$ denotes the number of orbits of σ and $\bar{\sigma}$ the permutation given by:

$$\bar{\sigma}(a) = \sigma(-a) \quad \text{for any } a \text{ in } Z_m.$$

A *combinatorial map* is a permutation σ such that the pair $\{\sigma, \bar{\sigma}\}$ generates a transitive group on Z_m. This could be restated in the following manner: for any a and b in Z_m, there exist integers $i_1, i_2, \ldots, i_p$ such that:

$$\sigma^{i_1}\bar{\sigma}^{i_2} \ldots \sigma^{i_{p-1}}\bar{\sigma}^{i_p}(a) = b.$$

The cycles of σ are called the *vertices* of the map, those of $\bar{\sigma}$ its *faces*. A pair $(a, -a)$ where a is in Z_m is an *edge*. The *genus* $g(\sigma)$ of a map σ is given by:

$$g(\sigma) = \tfrac{1}{2}[m + 2 - z(\sigma) - z(\bar{\sigma})].$$

This number is known to be a non negative integer. A combinatorial map (or simply a *map*) is *planar* if its genus is zero, i.e., $z(\sigma) + z(\bar{\sigma}) = m + 2$ (Euler relation).

A planar map with only one face (i.e., $z(\bar{\sigma}) = 1$) is called a *tree*. A *labeled* tree is a tree in which a natural number (*label*) is assigned to each vertex. A tree will be said *well labeled* if labels which are assigned to adjacent vertices differ by at most one. More precisely: a mapping ϵ from Z_m into $\mathcal{N}$ is a well labeling of the tree σ if $\epsilon(1) = 0$ and if, for any a in Z_m, $\epsilon(a) = \epsilon(\sigma(a))$ and $|\epsilon(a) - \epsilon(-a)| \leqq 1$.

1.2. *Intuitive map representation.* Combinatorialists draw circles in order to represent the cycles of a permutation σ. The elements a, $\sigma a, \ldots, \sigma^k a$ of each cycle are set on the circle in a conventional way (clockwise for instance). Drawing an edge between a and $-a$ for any a in Z_m will be our convention for representing the map σ. The reader familiar with the notion of topological map will find this map by con-

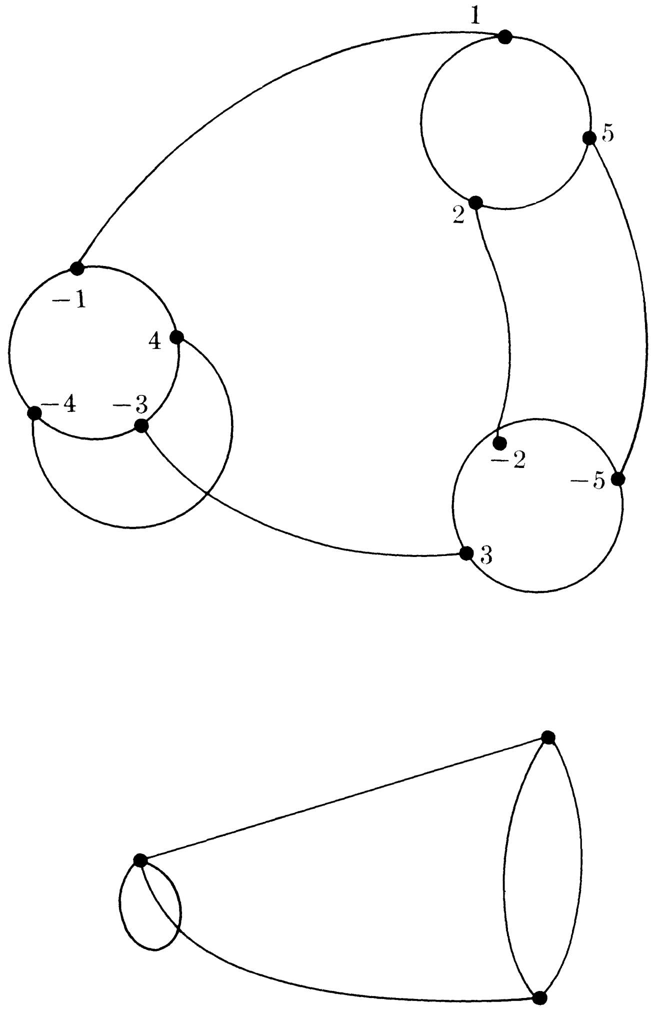

FIGURE 1. The representations of the combinatorial map: $\sigma = (1, 5, 2)(3, -2, -5)(4, -3, -4, -1)$.

tracting each circle to a point (Figure 1). Note that the underlying graph is always connected; this comes from the condition that $\{\sigma, \bar{\sigma}\}$ generates a transitive group.

1.3. *Rooted maps.* In order to avoid some difficulties in the enumeration of maps coming from their symetries, rooted maps are usually introduced. Rooted maps are the classes of the following isomorphism relation:

Two maps σ and σ' are isomorphic if there exists a permutation φ

such that:

$$\varphi(1) = 1, \varphi\sigma\varphi^{-1} = \sigma \quad \text{and, for any } a \text{ in } Z_m, \quad -\varphi(a) = \varphi(-a).$$

In a rooted map the *root vertex* and the *root edge* are the vertex and the edge which contains the element 1 of Z_m.

With this definition one finds 2 and 9 rooted planar maps with respectively 1 and 2 edges. The same number is found for well labeled trees (see Figure 2).

2. The main theorem. This section is devoted to the proof of the main theorem. We will often use two basic operations on permutations: the *closure* of a subset and the *restriction*. With these two operations we define what we call the *splitting* $\hat{\sigma}$ of a map σ. We prove that $\hat{\sigma}$ is a tree when σ is a planar map. In well labeling $\hat{\sigma}$ we obtain then a one-to-one mapping from rooted planar maps onto rooted well labeled trees. In order to prove this fact we establish first a lemma: one can reconstruct a permutation σ on a set A from its action on a subset A' and its restriction to the complement A'' of A'.

2.1. *General notations on permutations.* Let σ be a permutation acting on a set A and B be a subset of A. In the sequel σ^*B will denote the minimal subset of A containing B and closed by σ. Then the *closure* is defined by:

$$\sigma^*B = \{a \in A; \exists\, n \in \mathbf{N}, \exists\, b \in B \quad \text{such that } a = \sigma^n(b)\}.$$

If $\sigma^*B = B$, B will be said to be *saturated* by σ. Lastly, the restriction $\sigma_{/B}$ of σ to B is the permutation acting on B verifying:

$\sigma_{/B}(b)$ is the first of $\sigma(b), \sigma^2(b) \dots \sigma^n(b)$ belonging to B.

Example. Let:

$$\sigma = (1, -4, 4, 2, 5)(-2, -6, -7, 9)(8, -8, -5, -1) \times (-3, 3, 7, -9)(6)$$

be a permutation acting on Z_9. Let $B = \{-2, -6, 6, -7, -9\}$; then $\sigma^*B = \{-2, -6, -7, 9, 6, -9, -3, 3, 7\}$ and $\sigma_{/B} = (-2, -6, -7)(6)(-9)$.

2.2. *The splitting of a map.* For any map σ acting on Z_m we construct a sequence of subsets $B_0, B_1, \dots, B_i, \dots, B_p$, in the following manner:

$$B_0 = \sigma^*\{1\} \quad \text{and for any } i \geqq 0$$

$$B_{2i+1} = \sigma^*B_{2i}\backslash B_{2i} = \{b \in \sigma^*B_{2i}, b \notin B_{2i}\}$$

$$B_{2i+2} = \bar{\sigma}^*B_{2i+1}\backslash B_{2i+1}.$$

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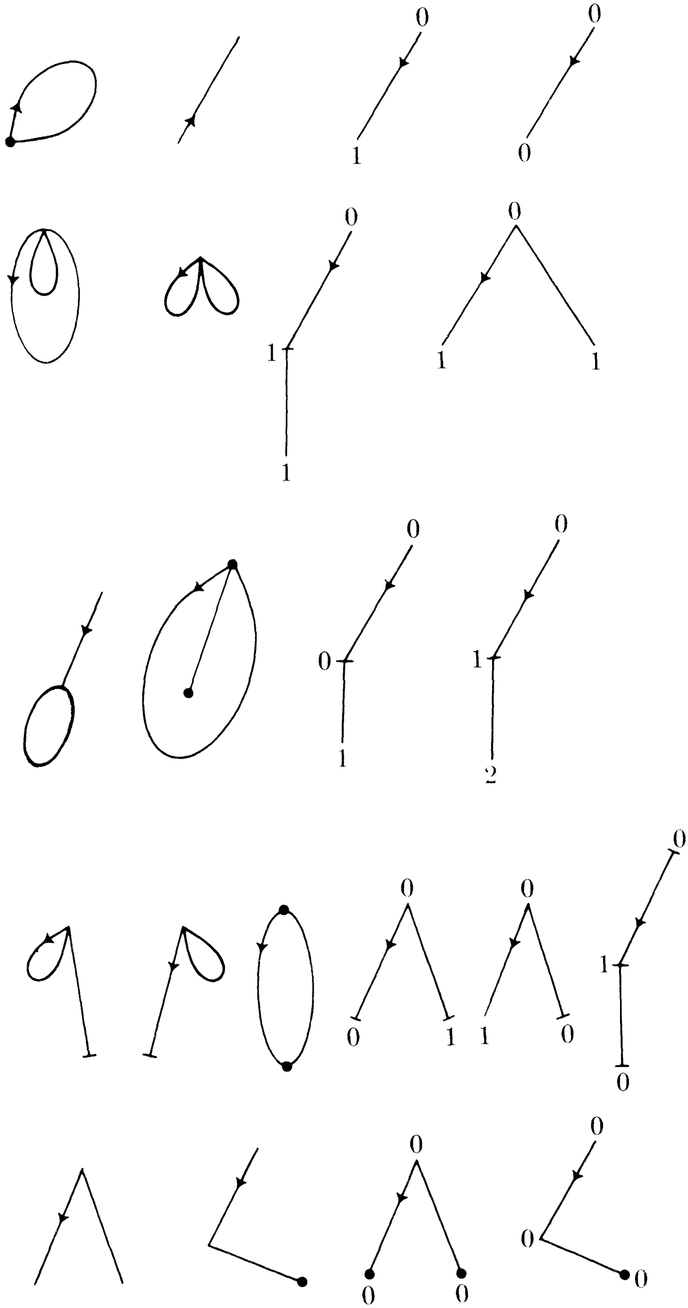

FIGURE 2. Rooted planar maps and well labeled trees with 1 and 2 edges.

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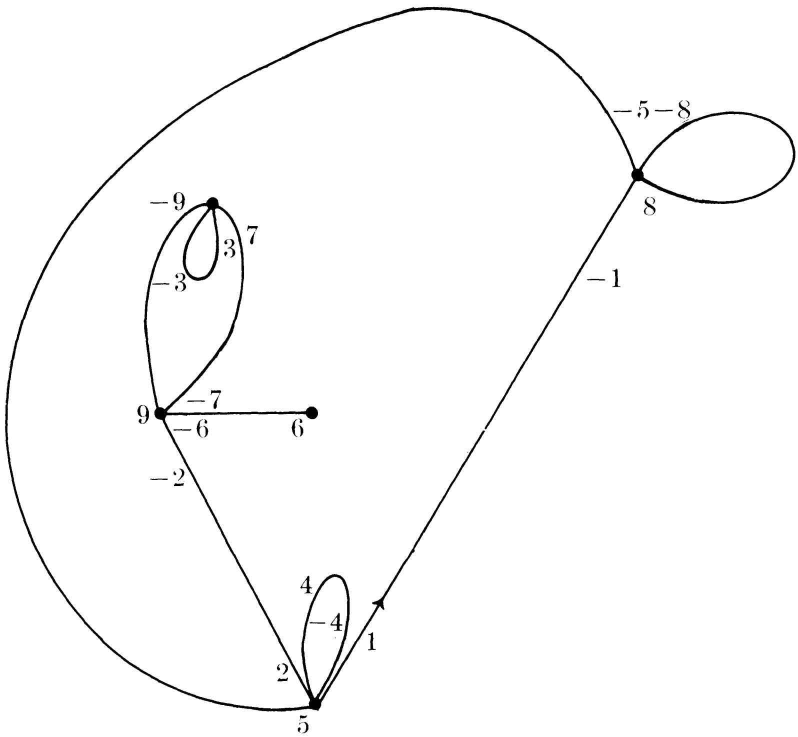

FIGURE 3. The map $\sigma = (1, -4, 4, 2, 5)\ (-2, -6, -7, 9)\ (8, -8, -5, -1)\ (-3, 3, 7, -9)\ (6)$.

Example. For the permutation σ given above, representing the map of figure 3:

$$\bar{\sigma} = (1, 8, -5)(-1, -4, 2, -6, 6, -7, -9, -2, 5)(3) \times (-3, 7, 9)(4)(-8)$$

$$B_0 = \{1, 8, -5\} \quad B_1 = \{-1, 4, -4, 5, 2, -8\}$$

$$B_2 = \{-9, -2, -7, 6, -6\} \quad B_3 = \{7, 9, 3, -3\}$$

$$B_4 = \emptyset = B_i \quad (i > 4).$$

PROPOSITION 2.1. *The permutation* σ *saturates* $B_{2i} \cup B_{2i+1}$. *Also* $B_{2i+1} \cup B_{2i+2}$ *is saturated by* $\bar{\sigma}$.

Proof. From the definition of the B_i one has:

$$\sigma^* B_{2i} = B_{2i} \cup B_{2i+1} \quad \text{and} \quad \bar{\sigma}^* B_{2i+1} = B_{2i+1} \cup B_{2i+2}.$$

The result then follows from the fact that $\sigma^*(\sigma^*(A)) = \sigma^*(A)$.

PROPOSITION 2.2. *The* B_j*'s define a partition of* Z_m.

Proof. Let us first show that the B_j's are mutually disjoint, by induction on j. As B_0 and B_1 are disjoint let us assume that $B_0, B_1, \ldots, B_j$ are disjoint and suppose that j is even (the case j odd is obtained by exchanging σ and $\bar{\sigma}$).

Let b be an element of B_{2i+1}, then $b = \sigma^k(a)$ where a belongs to B_{2i}. If b is in $B_{2i'+1}$ (or in $B_{2i'}$) with $0 \leqq i' < i$ as $a = \sigma^{2m-k}(b)$, then a would be in $\sigma^* B_{2i'+1}$ (or in $\sigma^* B_{2i'}$) $\subseteq B_{2i'} \cup B_{2i'+1}$ from Proposition 2.1, which contradicts the fact

$$(B_{2i'} \cup B_{2i'+1}) \cap B_{2i} = \emptyset.$$

Hence b does not belong to any B_p ($p \leqq 2i - 1$) no more it does to B_{2i} from the very definition of B_{2i+1}. The union of the subsets B_i is exactly Z_m: because of the transitivity of the group generated by $\{\sigma, \bar{\sigma}\}$, as for each b in Z_m there exist $i_1, i_2, \ldots, i_p$ such that if

$$b = \sigma^{i_1}\bar{\sigma}^{i_2} \ldots \sigma^{i_{p-1}}\bar{\sigma}^{i_p}(1),$$

then b belongs to some B_n with $n \leqq p$.

Definition. The splitting of a map σ is the permutation $\hat{\sigma}$ given by

$$\begin{aligned}\hat{\sigma}(a) &= \sigma_{/B_{2i}}(a) \text{ if } a \text{ belongs to } B_{2i}\\ &= \bar{\sigma}_{/B_{2i+1}}(a) \text{ if } a \text{ belongs to } B_{2i+1}.\end{aligned}$$

For the above example σ we thus obtain:

$$\hat{\sigma} = (1)(8, -5)(-1, -4, 2, 5)(4)(-8)(-6, -7, -2)(6)(-9) \times (3)(-3, 7, 9).$$

2.3. *Property of the splitting operation.*

THEOREM 2.3. *The splitting of a map is a map.*

We have to prove that the group generated by $\{\hat{\sigma}, \bar{\hat{\sigma}}\}$ acts transitively on Z_m. For this, we are going to construct for each a and b, such that $a = \sigma(b)$ or $a = \bar{\sigma}(b)$, a word f on the alphabet $\{\hat{\sigma}, \bar{\hat{\sigma}}\}$ such that $a = f(b)$; the result then follows by the transitivity of the group generated by $\{\sigma, \bar{\sigma}\}$.

1. Let us consider, first, the case where a belongs to the set B_n for which $B_{n+1} = \emptyset$, and let us suppose that n is even ($n = 2i$), the case n odd being treated in the same manner, exchanging σ and $\bar{\sigma}$. If $a = \sigma(b)$, B_{2i} is saturated by σ; as $\hat{\sigma}$ is equal to $\sigma_{/B_{2i}}$ in B_{2i} then $a = \hat{\sigma}(b)$ and thus $f = \hat{\sigma}$. If $a = \bar{\sigma}(b)$, then $b = -\sigma^{-1}(a)$. Now as above, $\sigma^{-1}(a) = \hat{\sigma}^{-1}(a)$ hence $b = -\hat{\sigma}^{-1}(a)$ and $a = \hat{\sigma}(-b)$.

2. Let us suppose that the result is true if a belongs to B_j, for all $j > 1$. Let $a = \sigma b$ be in B_i, and let $i = 2k$. One can deal in the same way with the case $a \in B_{2k+1}$. If b is in B_{2k}, then $a = \hat{\sigma} b$ and the result is proved. So,

let b be in B_{2k+1} and let us compute the sequence:

$$b_1 = b, b_2 = \sigma^{-1}b_1, \ldots, b_{l+1} = \sigma^{-1}b_l;$$

all the elements of this sequence are in $B_{2k} \cup B_{2k+1}$; the first one is in B_{2k+1} and the last one in B_{2k}; then there exists p such that:

$$b_1, b_2, \ldots, b_{p-1} \in B_{2k+1}, \quad b_p \in B_{2k}.$$

Then we have $\hat{\sigma}(b_p) = a$ (as $\hat{\sigma} = \sigma_{/B_{2k}}$ for the elements of B_{2k}). By induction there exists f_l such that $b_{l+1} = f_l(b)$ (where $1 \leqq l \leqq p - 1$). Thus

$$a = \hat{\sigma}f_{p-1} \ldots f_1(b).$$

If $a = \bar{\sigma}(b)$ we have $a = \sigma(-b)$ with a in B_{2k}; since we have proved in the first part that there exists f such that $a = f(-b)$, the result is true with f.

2.4. *Splitting a planar map.* For any a in Z_m, let $\epsilon(a)$ be equal to j if a belongs to B_j; then ϵ defines a labelling of σ; moreover we have:

PROPOSITION 2.4. *The splitting of a planar map is a tree, well labeled by* ϵ.

Proof. (a) Let us first determine the number of cycles of $\hat{\sigma}$. From its definition each B_j is saturated by $\hat{\sigma}$; further:

$$z(\hat{\sigma}/_{B_{2i}}) = z(\sigma/_{B_{2i}}) = z(\sigma/_{B_{2i} \cup B_{2i+1}})$$
$$z(\hat{\sigma}/_{B_{2i+1}}) = z(\bar{\sigma}/_{B_{2i+1}}) = z(\bar{\sigma}/_{B_{2i+1} \cup B_{2i+2}}).$$

Taking the sum of these relations we obtain:

$$z(\sigma) = \sum_{i=0}^{[n/2]} z(\sigma/_{B_{2i} \cup B_{2i+1}}) + \sum_{i=1}^{[n/2]} z(\bar{\sigma}/_{B_{2i-1} \cup B_{2i}}).$$

Then, $z(\hat{\sigma}) = z(\sigma) + z(\bar{\sigma}) - 1$, as the second sum begins with $i = 1$ and B_0 is a cycle of $\bar{\sigma}$.

(b) The genus $g(\hat{\sigma})$ is now given by:

$$g(\hat{\sigma}) = \tfrac{1}{2}(2 + m - z(\hat{\sigma}) - z(\hat{\sigma}))$$
$$g(\hat{\sigma}) = g(\sigma) + \tfrac{1}{2}[1 - z(\hat{\sigma})].$$

The map σ being planar, $g(\sigma) = 0$; moreover the genus of a map is a natural number [**7**], thus $z(\hat{\sigma})$ is necessarily equal to 1 and $\hat{\sigma}$ is a tree, thus it is a planar map.

(c) To prove that this tree is well labeled by ϵ we must verify that if b belongs to B_j, then $-b$ is in one of the subsets B_{j-1}, B_j, B_{j+1}.

As $B_{2i} \cup B_{2i-1}$ is saturated by $\bar{\sigma}$ one has:

$$\bar{\sigma}B_{2i} \subset B_{2i-1} \cup B_{2i};$$

then, applying σ^{-1} to these sets and using the fact that $B_{2k} \cup B_{2k+1}$ is saturated by σ (then also by σ^{-1}) we obtain:

(1) $\quad -B_{2i} \subset B_{2i-2} \cup B_{2i-1} \cup B_{2i} \cup B_{2i+1}.$

Starting with $\sigma B_{2i} \subset B_{2i} \cup B_{2i+1}$ and applying $\bar{\sigma}^{-1}$ we obtain:

(2) $\quad -B_{2i} \subset B_{2i-1} \cup B_{2i-1} \cup B_{2i} \cup B_{2i+2}.$

The result for j even now follows by comparing (1) and (2). The same kind of proof can be given for j odd.

2.5. *A useful lemma.* We have associated to each rooted planar map a rooted well labeled tree having the same number of edges. We have now to show that this correspondance is one-to-one. For this we need the following lemma which allows us to "extend" a permutation.

Definition. Let φ be a mapping from A to A. φ *admits a cycle* if there exist $a_1, a_2, \ldots, a_p$ such that $\varphi(a_i) = a_{i+1}$ $(1 \leqq i < p)$ and $\varphi(a_p) = a_1$.

LEMMA. *Let A' and A'' be two disjoint subsets the union of which is A. Let φ be a one-to-one mapping from A' into A which admits no cycle in A'. Let α be a permutation acting on A''. Then there exists a unique permutation*

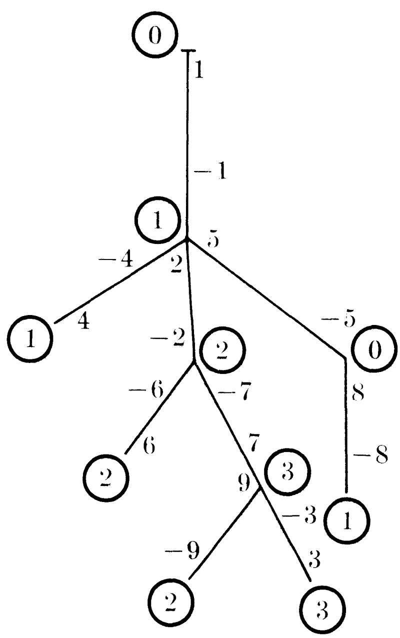

FIGURE 4. The well labeled tree $\hat{\sigma}$ associated to the map σ of figure 3.

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θ acting on A such that:

(i) $\theta/_{A''} = \alpha$,
(ii) $\theta(a) = \varphi(a)$ *for every a in A'.*

In the sequel we will denote by $T_{A'',A'}(\alpha, \varphi)$ the permutation θ which may be considered as an extension of α by φ.

Proof of the Lemma. (by induction on the number of elements of A')
If A' is empty then $\theta = \alpha$ is the unique permutation.

If A' is not empty let b be an element of A' the image of which by φ is not in A' (such an element exists as φ admits no cycle). By induction, we obtain a unique permutation θ' on $A \backslash \{b\}$ such that $\theta'/_{A''} = \alpha$ and $\theta'(a) = \varphi(a)$ for $a \in A'$, $a \neq b$. Let x be the image of b by φ. $y = \theta'^{-1}(x)$ is necessarily an element of A'' as φ is one-to-one. The permutation defined by:

$$\theta(b) = x, \theta(y) = b, \theta(a) = \theta'(a) \quad \text{for} \quad a \neq b, a \neq y$$

is clearly the unique permutation which satisfies the conditions of the lemma.

2.6. THEOREM A. *The mapping $\sigma \to (\hat{\sigma}, \epsilon)$ is a one-to-one correspondance between rooted planer maps and rooted well labeled trees.*

Proof. We have to reconstruct σ from $(\hat{\sigma}, \epsilon)$; for this, we use the following remarks.

(a) The subsets $B_0, B_1, \ldots, B_n$ are determined by:

$$a \in B_j \Leftrightarrow \epsilon(a) = j.$$

(b) On B_n, $\hat{\sigma}$ is equal to σ or $\bar{\sigma}$ according to the parity of n. For instance, if n is even then B_n is saturated by σ and $\sigma(a) = \hat{\sigma}(a)$ for every a in B_n.
(c) For any a, $\sigma^{-1}(a) = \bar{\sigma}^{-1}(a)$.
(d) For any i, $\sigma^{-1}/_{B_{2i}} = \hat{\sigma}^{-1}/_{B_{2i}}$ and $\bar{\sigma}^{-1}/_{B_{2i-1}} = \hat{\sigma}^{-1}/_{B_{2i-1}}$.

Let us now describe the reconstruction process if n is even (replace σ by $\bar{\sigma}$ if n is odd).

σ is first determined in B_n by Remark 2(b).
Remarks (c) and (d) imply that:

$$\bar{\sigma}^{-1}/_{B_{n-1} \cup B_n}' = T_{B_{n-1},B_n}(\hat{\sigma}^{-1}/_{B_{n-1}}, -\sigma^{-1}),$$

then the lemma of Section 2.5 allows the construction of $\bar{\sigma}^{-1}$ on $B_{n-1} \cup B_n$.
Then remarks (c) and (d) again imply that:

$$\sigma^{-1}/_{B_{n-2} \cup B_{n-1}} = T_{B_{n-2},B_{n-1}}(\hat{\sigma}^{-1}/_{B_{n-2}}, -\bar{\sigma}^{-1}),$$

which allows the construction of σ^{-1} on $B_{2n-2} \cup B_{2n-1}$, and so on. We can thus claim that σ and $\bar{\sigma}$ are uniquely determined by:

$$\hat{\sigma}(a) = \sigma(a) \quad \text{if} \quad a \in B_n \quad \text{and}$$
$$\bar{\sigma}^{-1}/_{B_{2i-1} \cup B_{2i}} = T_{B_{2i-1} \cup B_{2i}}(\hat{\sigma}^{-1}/_{B_{2i-1}}, -\sigma^{-1}),$$
$$\sigma^{-1}/_{B_{2i} \cup B_{2i+1}} = T_{B_{2i} \cup B_{2i+1}}(\hat{\sigma}^{-1}/_{B_{2i}}, -\bar{\sigma}^{-1}).$$

3. Coding trees by parenthesis systems. In this part we begin by stating our notations concerning words written over an alphabet. We shall use this notion for the classical construction of the code of a rooted plane tree [**9**], [**4**]. We will briefly state the coding algorithm using the combinatorial definition of a tree.

3.1. *Parenthesis systems.* Let X be a finite set called *alphabet*; a *word* is a mapping from $[n]$ (where $[n] = \{1, 2, \ldots, n\}$) into X. We denote by $|f|$ the *length* n of f. One generally writes a word f by $f = f(1)f(2) \ldots f(n)$; for example if $X = \{a, b\}$ and $f(2i + 2) = a, f(2i + 1) = b$ for $i = 0, 1$ then f is written $f = baba$. The *product* (or concatenation) of two words f and g of respective lengths n and m is the word h of length $n + m$ given by: $h(i) = f(i)$ if $i \leqq n$, and $h(i) = g(i - n)$ if $n + 1 \leqq i \leqq n + m$. If $f = aba$, $g = ba$ is written $ababa$. The set X^* of all the words written on the alphabet X has a structure of free monoid the neutral element of which is the empty word 1. A word g is a *prefix* of the word f if one can find h such that $f = gh$ (where h is a word, possibly empty).

Let us consider the alphabet $X = \{x, \bar{x}\}$ and let δ be the morphism from X^* into Z (the set of integers) given by:

$$\delta(x) = 1, \delta(\bar{x}) = -1 \quad \text{and} \quad \delta(f) = \sum_{i=1}^{n} \delta(f(i)).$$

Definition. A word f is a *parenthesis system* if:

(i) $\delta(f) = 0$
(ii) $\delta(g) \geqq 0$ for every prefix g of f.

Let us denote by D the set of all parenthesis systems.

3.2. *Counting parenthesis systems.* Let $D\bar{x}$ be the set of words obtained from those of D by right product of each one with $\bar{x}$. The following proposition [**11**] enables the counting of the words of length $2n$ of D.

PROPOSITION 3.1. *For each word f in X^* such that $\delta(f) = -p$ $(p > 0)$ there exist exactly p pairs (g_i, h_i) such that $f = g_ih_i$, $h_i \neq 1$ and h_ig_i is a product of p elements of $D\bar{x}$.*

Proof. For every j $(1 \leqq j \leqq q)$ let f_j be the shortest prefix of the word f such that $\delta(f_j) = -j$; then f may be written: $f = u_1u_2 \ldots u_qv$ with $f_i = u_i \ldots u_i$. The pairs g_i, h_i are then given by

$$g_i = u_1 \ldots u_{q-p+1} \quad h_i = u_{q-p+i+1} \ldots u_qv.$$

PROPOSITION 3.2. *The number of p-uples of parenthesis systems whose lengths sum to 2n is:*

$$\frac{p}{2n+p}\frac{(2n+p)!}{n!(n+p)!}.$$

Proof. From Proposition 3.1, the number $a_{n,p}$ of words of $(D\bar{x})^p$ of length $2n + p$ is equal to $p/(2n + p)$ times the number of words f in X^* such that $\delta(f) = -p$. The fact that this last set contains all the words with n "x" and $(n + p)$ "$\bar{x}$" implies

$$a_{n,p} = \frac{p}{2n+p}\frac{(2n+p)!}{n!(n+p)!}.$$

The correspondance

$$(f_1, f_2, \ldots, f_p) \rightarrow (f_1\bar{x}f_2\bar{x}f_3 \ldots f_p\bar{x})$$

is one-to-one, thus the proof of property follows from the fact that

$$|f_1\bar{x}f_2\bar{x} \ldots f_p\bar{x}| = \sum_{i=1}^{p} |f_i| + p.$$

3.3. *Coding rooted trees.* Usually this coding is presented in the following intuitive manner: "Walk around the tree starting from the root and turning counter clockwise. Write x when going along an edge for the first time, $\bar{x}$ when going along it again" (see figure 5). If a tree is defined as a combinatorial map σ such that $z(\sigma) = m + 1$, $z(\bar{\sigma}) = 1$ then the above algorithm should be made precise: first a canonical tree σ_0 (isomorphic to σ) is constructed from σ; then the coding is obtained from σ_0.

Canonical tree: a tree σ is *canonical* if the permutation $\bar{\sigma}$ satisfies:

(1) If i, j are such that $1 \leqq i < j \leqq n$ and $\bar{\sigma}^i(1) > 0$, $\bar{\sigma}^j(1) > 0$, then $\bar{\sigma}^i(1) > \bar{\sigma}^j(1)$.

(2) If i,j are such that $\bar{\sigma}^i 1 = -\bar{\sigma}^j 1 > 0$ then $i < j$.

PROPOSITION 3.3. *For every rooted tree σ there exists a unique canonical tree σ_0 isomorphic to σ.*

This isomorphism is obtained by renaming the elements of Z_m using $\bar{\sigma}$. For example, let us consider the tree $\hat{\sigma}$ constructed in Section 2:

$$\hat{\sigma} = (1)(8, -5)(-1, -4, 2, 5)(4)(-8)(-6, -7, -2)(6)(-9)(3)(-3, 7, 9).$$

One finds:

$$\hat{\bar{\sigma}} = (1, -4, 4, 2, -6, 6, -7, 9, -9, -3, 3, 7, -2, 5, 8, -8, -5, -1).$$

Hence,

$$\hat{\bar{\sigma}}_0 = (1, 2, -2, 3, 4, -4, 5, 6, -6, 7, -7, -5, -3, 8, 9, -9, -8, -1)$$

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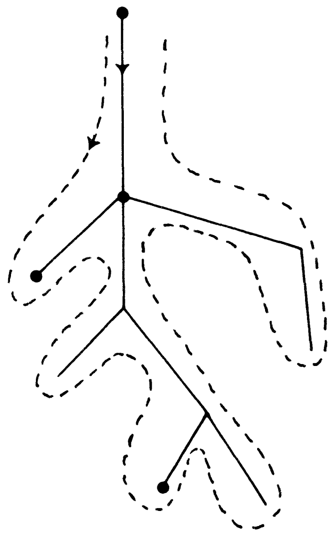


FIGURE 5. Coding: $xx\bar{x}xx\bar{x}xx\bar{x}x\bar{x}\bar{x}\bar{x}xx\bar{x}\bar{x}\bar{x}$ of the tree $(1), (8, -5)(-1, -4, 2, 5)(4)(-8)(-6, -7, -2)(6)(-9)(3)(-3, 7, 9)$.

and

$$\hat{\sigma}_0 = (1)(2, 3, 8, -1)(-2)(-3, 4, 5)(-4)(-5, 6, 7)(-6)(-7) \times (-8, 9)(-9).$$

Coding a canonical tree. Let σ be a canonical tree. The coding $\gamma(\sigma)$ of σ is the word f defined by:

$$f(i) = x \text{ if } \bar{\sigma}^{i-1}(1) \text{ is positive}$$
$$f(i) = \bar{x} \text{ if } \bar{\sigma}^{i-1}(1) \text{ is negative.}$$

Clearly condition (2) on a canonical tree implies that f belongs to D. Conversely given a word f in D we associate to it a tree σ (such that $\gamma(\sigma) = f$) using the two following algorithms:

A1. For every prefix f of a word belonging to D we associate the subset $A(f)$ of $[n]$ (where n is the length of f) by the recursive procedure:

$$A(1) = \emptyset$$
$$A(fx) = A(f) \cup \{|f|_x\}$$

(where $|g|_x$ denotes the number of "x" occurring in g)

$$A(f\bar{x}) = A(f) \setminus \operatorname{Max} A(f).$$

If $f = x\bar{x}xxx\bar{x}x$ then $A(f) = \{2, 3, 5\}$.

A2. For every f in D, $\gamma^{-1}(f)$ is the permutation $\bar{\theta}$ acting on Z_m (where $2m = |f|$) such that θ is the circular permutation defined by:

$$\theta^i(1) = \operatorname{Max} A(f_{i+1}) \quad \text{if} \quad f(i+1) = x$$
$$\theta^i(1) = -\operatorname{Max} A(f_i) \quad \text{if} \quad f(i+1) = \bar{x}$$

(where f_i denotes the prefix of length i of f).

4. The coding of well labeled trees.

4.1. *Code description.* In the last section we defined the way to code a rooted tree; in the case of well labeled trees some information has to be added in the code word in order to find the labels again. This could be done by giving subscripts to the letters x and $\bar{x}$ appearing in the code word. More precisely, walking around the infinite face of the tree (as for rooted trees) one of the letters x_1, x_2, x_3, $\bar{x}_1$, $\bar{x}_2$ or $\bar{x}_3$ is written according to the following rules:

— x_1 (resp. x_2, x_3) is written if the edge is traversed for the first time and if the initial vertex label is greater than (resp. equal to, less than) the terminal vertex one.

— $\bar{x}$ ($i = 1, 2$ or 3) is written if the edge is traversed for the second time and when x_i was written in the first encounter.

See figure for example.

Formally, the coding of a well labeled rooted tree is given by the following:

To each canonical well labeled rooted tree (σ, ϵ) is associated the word $f = \tilde{\gamma}(\sigma, \epsilon)$ on the alphabet $X_3 = \{x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ in the following way: let $b_i = \bar{\sigma}^{i-1}1$; then

$$f(i) = x_{2+\epsilon(b_i)-\epsilon(-b_i)} \quad \text{if} \quad b_i > 0$$
$$f(i) = \bar{x}_{2+\epsilon(-b_i)-\epsilon(b_i)} \quad \text{if} \quad b_i < 0.$$

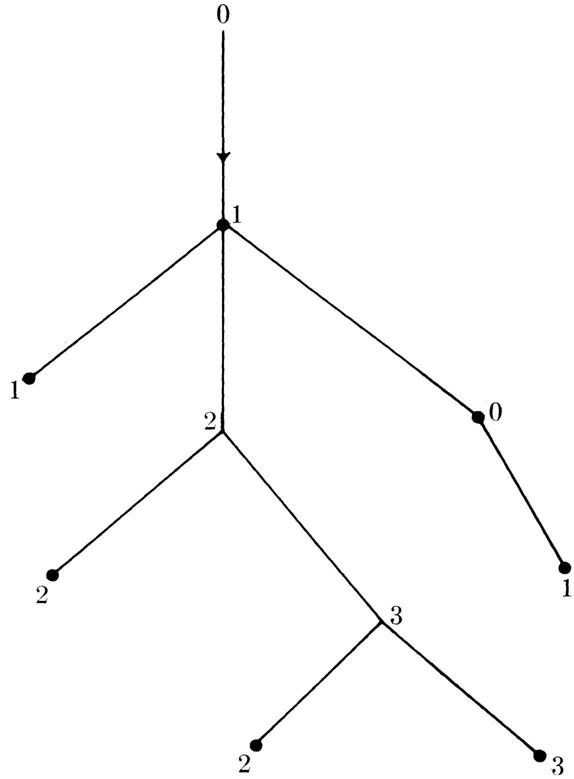
4.2. *The D_3 language.* A first glance at the codes for well labeled rooted trees shows that they contain an equal number of x_i and $\bar{x}_i$; moreover each x_i has a matching $\bar{x}_i$. This property comes from the fact that when walking around the tree and traversing the edge a_1 and further the edge a_2, then a_2 will be traversed again before a_1 will be.

More formally, we have the following definition and proposition.

Definition. Let $\rightarrow$ be the relation defined on the words of $X_3{}^*$ by:

$$f \rightarrow g \Leftrightarrow f = f_1 x_i \bar{x}_i f_2 \quad \text{and} \quad g = f_1 f_2.$$

Let $\xrightarrow{*}$ be the transitive closure of $\rightarrow$. We will say that f reduces to g if $f \xrightarrow{*} g$.



$x_1x_2\bar{x}_2x_1x_2\bar{x}_2x_1x_3\bar{x}_3x_2\bar{x}_2\bar{x}_1\bar{x}_1x_3x_1\bar{x}_1\bar{x}_3\bar{x}_1.$

FIGURE 6. The code word of the labeled tree given in figure 4.

PROPOSITION 4.1. *If f is a code of a well labeled rooted tree, then f reduces to the empty word* $||$.

Let D_3 denote the set of words of $X_3{}^*$ reducing to the empty word, let C be the set of codes of well labeled rooted trees. Then C is strictly included in D_3 as $x_3\bar{x}_3$ belongs to D_3 but not to C.

The characteristic properties of the words in C will be given later. Let us establish first some properties of D_3.

Definition. A word f in D_3 is prime if it has no prefix in D_3 (other than $||$ or f).

PROPOSITION 4.2. *Every word of D_3 different from $||$ has a unique factorisation as a product of prime words.*

Proof. If f is not prime $f = f_1g$ where f_1 is prime; it is clear that g reduces to $||$ and repeating this decomposition to g one finds $f = f_1f_2 \ldots f_p$.

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PROPOSITION 4.3. *If f is prime then $f = x_i g \bar{x}_i$ and g belongs to D_3.*

Proof. As f belongs to D_3, its first letter is necessarily an x_i; this x_i has a matching $\bar{x}_i$ and so $f = x_i f_1 \bar{x}_i f_2$ where f_1 and f_2 reduce to $\|$. Clearly $f_2 = \|$ otherwise $x_i f_1 \bar{x}_i$ would be a left factor of f in D_3.

PROPOSITION 4.4. *The number of p-tuples of words in D_3 whose lengths sum to $2n$ is given by*

$$3^n \frac{p}{2n+p} \frac{(2n+p)!}{n!(n+p)!}.$$

Proof. The elements of $D_3{}^p$ are obtained from those of D^p by choosing subscripts (1, 2, or 3) for each x, those of the $\bar{x}$ being deduced from the x_i matching them. Then one gets the formula enumerating $D_3{}^p$ multiplying by 3^n the number of elements of D^p given by Proposition 3.2.

4.3. *Characterisation of the words of C.* Let Δ be the mapping of X_3 in Z given by $\Delta(x_1) = \Delta(\bar{x}_3) = 1$, $\Delta(x_2) = \Delta(\bar{x}_2) = 0$, $\Delta(\bar{x}_1) = \Delta(x_3) = -1$. This mapping can be extended to a morphism of $X_3{}^*$ into $\mathbf{Z}$ (considered as a monoid for addition), denoted also by Δ in the following way:

$$-\Delta(\|) = 0$$
$$-\Delta(fy) = \Delta(f) + \Delta(y) \qquad y \in X_3.$$

PROPOSITION 4.5. *A word f belongs to C if and only if:*

(1) $f \in D_3$

(2) $\Delta(f') \geqq 0$ *for every prefix f' of f.*

Proof. The code $\tilde{\gamma}(\sigma, \epsilon)$ is obtained from $\gamma(\sigma) = g$ in giving subscripts to the letters appearing in g. This way to give subscripts is in relation with the labeling ϵ by:

$$\Delta(f') = \epsilon(\bar{\sigma}^{|f'|}1) \text{ for every prefix } f' \text{ of } f = \tilde{\gamma}(\sigma, \epsilon).$$

This relation enables the construction of ϵ from the code f using $\Delta(f')$. The property $\Delta(f') \geqq 0$ ensures that labels are positive or null.

5. Counting well labeled trees.

5.1. *Main results.* The coding of well labeled trees enables us to find now the formula counting them. More precisely, we first state a combinatorial result (Theorem B) on the words of D_3 which do not belong to C; we then deduce as a corollary the enumerating formula. In order to prove Theorem B we establish four lemmas.

THEOREM B. *There exists a bijection β of $D_3 \backslash C$ onto $D_3 \times D_3 \times D_3$ satisfying:*

$$\text{if } \beta(f) = (f_1, f_2, f_3) \quad \text{then} \quad |f| = |f_1| + |f_2| + |f_3| + 2.$$

COROLLARY. *The number a_m of well labeled rooted trees is given by:*

$$a_m = \frac{2.3^m (2m)!}{m!(m+2)!}.$$

This is also the number of rooted planar map with m edges.

Proof (of the corollary). a_m is also the number of words of C with length $2m$. From Theorem B this is the difference between the number of words of D_3 with length $2m$ and the number of triples of words in D_3 whose lengths sum to $2m - 2$. These two numbers have been determined in Section 4 (Proposition 4.4). They are respectively equal to:

$$\frac{3^m (2m)!}{m!(m+1)!} \quad \text{and} \quad \frac{3^m}{2m+1} \frac{(2m+1)!}{(m-1)!(m+2)!}.$$

The first part of the corollary is then obtained by the computation of their difference. The second part is a direct consequence of Theorem A (Section 2).

In order to prove Theorem B we have to associate three words of D_3 to each word f of $D_3 \backslash C$. Lemma 1 will give us a decomposition of f as a product of three words. This decomposition is not a bijection onto $D_3 \times D_3 \times D_3$, but we may construct (Lemma 2) a bijection β_1 of $D_3 \backslash C$ onto $D_3 \times D_3 \times D_3 \times \{1, 2, 3\}$. Lemma 3 and Lemma 4 will then give combinatorial results on $\mathbf{N}^3$ which will allow to prove the theorem.

4.2. *The mapping β_1.*

Definition. Let ∂ be the mapping from X^* into Z given by:

$$\partial(f) = \operatorname{Min}\,(\Delta f' \mid f' \text{ is a prefix of } f).$$

As the empty word 1 is always a prefix of f and as $\Delta || = 0$ we have that ∂f is a negative or zero number; moreover ∂f equals zero if and only if f belongs to C.

LEMMA 5.1. *Every word f in $D_3 \backslash C$ has a unique factorisation as $f = uvw$ where u, v, w are three words in D_3 satisfying:*

(1) *v is a prime word*

(2) $\partial u > \partial v$

(3) $\partial w \geqq \partial v$.

Proof. From Proposition 4.2, f can be decomposed as $f = f_1 f_2 \ldots f_p$, where each f_i is a prime word. Let $v = f_i$ be the leftmost with minimal image by ∂: we have thus either $\partial f_j > \partial f_i$ or $\partial f_j = \delta f_i$ and $j \geqq i$ in this case.

Then, denoting $u = f_1 f_2 \ldots f_{i-1}$ and $w = f_{i+1} \ldots f_p$ we obtain the decomposition of the lemma. The fact that $\partial u > \partial v$ follows from $\partial u f_i = \partial f_i$ as $\Delta u = 0$ and as ∂f_i is minimal and strictly negative. The unicity is clear.

Let f be a word in $D_3\backslash C$. From Lemma 5.1, f can be written as uvw and as v is a prime word $v = x_i v' \bar{x}_i$ ($v' \in D_3$). Let us denote $\beta_1(f) = (u, v', w, i)$. Lemma 5.1 could be stated in the following way:

LEMMA 5.2. *β_1 is a bijection from $D_3\backslash C$ onto $\tilde{U}$, the subset of elements (u, v, w, i) of $D_3 \times D_3 \times D_3\{1, 2, 3\}$ satisfying $\partial u > \partial(x_i v \bar{x}_i)$ and $\partial w \geqq \partial(x_i v \bar{x}_i)$.*

5.3. *A combinatorial property of* $\mathbf{N}^3$.

LEMMA 5.3. *Let $\mathbf{N}$ be the set of natural numbers E_1, E_2, E_3 the subsets of $\mathbf{N}^3$ defined by:*

$$E_1 = \{(a, b, c) | c + 1 < a\} \quad E_2 = \{(a, b, c) | a \leqq b\} \quad E_3 = \{(a, b, c) | b \leqq c\}.$$

Then the following subsets $A = E_1 \cap \bar{E}_2$ $B = E_2 \cap \bar{E}_3$ $C = E_3 \cap \bar{E}_1$ form a partition of $\mathbf{N}^3$ (where for any E, $\bar{E} = \{x \in \mathbf{N}^3 | x \notin E\}$).

Proof. The fact that A, B, C are mutually disjoint could easily be verified. Now $E_1 \cap E_2 \cap E_3 = \emptyset$ as $(a, b, c) \in E_1 \cap E_2 \cap E_3$ implies $c < a \leqq b \leqq c$. Similarly $\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 = \emptyset$. The classical relation

$$\mathbf{N}^3 = (\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) \cup (E_1 \cap E_2 \cap E_3) \cup (E_1 \cap \bar{E}_2) \cup (\bar{E}_1 \cap E_3) \cup (E_2 \cap \bar{E}_3)$$

finally gives the result.

LEMMA 5.4. *Let U be the subset of $\mathbf{N}^4$ given by $U = U_1 \cup U_2 \cup U_3$ where:*

$$U_1 = \{(a, b, c, d) | d = 1, a < b - 1, c \leqq b - 1\}$$
$$U_2 = \{(a, b, c, d) | d = 2, a < b, c \leqq b\}$$
$$U_3 = \{(a, b, c, d) | d = 3, a < b + 1, c \leqq b + 1\}.$$

Then the mapping φ defined below is one-to-one from U onto $\mathbf{N}^3$:

$$\varphi(a, b, c, 1) = (b, c, a); \varphi(a, b, c, 2) = (c, b, a)$$
$$\varphi(a, b, c, 3) = (c, a, b).$$

Proof. One verifies first that φ restricted to each of U_1, U_2, U_3 is one-to-one. The relations $\varphi(U_1) = A$, $\varphi(U_2) = B$, $\varphi(U_3) = C$ and Lemma 5.3 give the result.

5.4. *Proof of Theroem* B. This proof uses the following diagram:

$$\begin{array}{ccccc} D_3\backslash C & \xrightarrow{\beta_1} & D_3 \times D_3 \times D_3 \times [3] & \xrightarrow{\tilde{\varphi}} & D_3 \times D_3 \times D_3 \\ & & \Big\downarrow (-\partial)^3 \times \mathrm{id} & & \Big\downarrow (-\partial)^3 \\ & & \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times [3] & \xrightarrow{\varphi} & \mathbf{N} \times \mathbf{N} \times \mathbf{N} \end{array}$$

where notations are:

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a) ∂^3 is the mapping defined by

$$-\partial^3(f_1, f_2, f_3) = (-\partial(f_1), -\partial(f_2), -\partial(f_3))$$

b) $\partial^3 \times \mathrm{id}$ is defined by:

$$-\partial^3 \times \mathrm{id}\ (f_1, f_2, f_3, i) = (-\partial(f_1), -\partial(f_2), -\partial(f_3), i)$$

c) $\tilde{\varphi}$ is the mapping given by:

$$\tilde{\varphi}(f_1, f_2, f_3, 1) = (f_2, f_3, f_1), \varphi(f_1, f_2, f_3, 2) = (f_3, f_2, f_1)$$

$$\varphi(f_1, f_2, f_3, 3) = (f_3, f_1, f_2).$$

The proof is then divided in four steps:

1. The diagram commutes as an immediate consequence of the definition of φ and $\tilde{\varphi}$

2. $\tilde{\varphi}$ is from $\tilde{U}$ onto $D_3 \times D_3 \times D_3$:

Let (f_1, f_2, f_3) be an element of $(D_3)^3$. $(-\partial f, -\partial f_2, -\partial f_3)$ is in $\mathbf{N}^3$ and from Lemma 5.4 there exists $(n_{i_1}, n_{i_2}, n_{i_3}, i)$ in U such that:

i_1, i_2, i_3 is a permutation of $(1, 2, 3)$

$n_i = -\partial f_i$ for $i = 1, 2, 3$

$\varphi(n_{i_1}, n_{i_2}, n_{i_3}, i) = (-\partial f_1, -\partial f_2, -\partial f_3).$

We then easily verify that the element $(f_{i_1}, f_{i_2}, f_{i_3}, i)$ of $\tilde{U}$ has (f_1, f_2, f_3) as image by $\tilde{\varphi}$.

3. $\tilde{\varphi}$ is one-to-one: Let $F = (f_1, f_2, f_3, i)$, $G = (g_1, g_2, g_3, j)$ be such that $\tilde{\varphi}(F) = \tilde{\varphi}(G)$. Then $\partial^3(\tilde{\varphi}(F)) = \partial^3(\tilde{\varphi}(G))$, and as the diagram commutes we obtain:

$$\varphi(\partial^3 \times \mathrm{id}(F)) = \varphi(\partial^3 \times \mathrm{id}(G)).$$

Now φ is a bijection, and so $i = j$ and

$$\partial^3(f_1, f_2, f_3) = \partial^3(g_1, g_2, g_3).$$

By the construction of $\tilde{\varphi}$ this implies $F = G$.

4. Theorem B is now established using for β the composition of β_1 (a one-to-one mapping from $D_3 \backslash C$ onto $\tilde{U}$) and $\tilde{\varphi}$ (a one-to-one mapping from $\tilde{U}$ onto $D_3 \times D_3 \times D_3$).

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