TESTS FOR THE SCALE PARAMETER OF THE TRUNCATED NORMAL

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1. Summary. This paper continues the work begun in [1] and examines the loss of power when using tests based on the assumption that the variable being sampled has a "complete" normal distribution, when in fact, sampling is from a symmetrically truncated distribution. The hypothesis considered here is the one-sided test for the variance of a normal distribution. Some tables have been computed and they show that appreciable losses in size occur. Some loss occurs in the power too, but this decreases with the alternative value of the variance and the degree of truncation.

2. Introduction. Let a normal distribution be symmetrically truncated at terminus points "a" standard deviations from the mean, i.e., its density $g(x)$ is given by

$$(2.1)\quad g(x) = \frac{c}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(x-\mu)^2/\sigma^2\right] \text{ for } |x-\mu| < a\sigma$$

$$= 0 \text{ otherwise,}$$

where c is given by

$$(2.2)\qquad \frac{1}{c} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-t^2/2}\, dt .$$

In this paper, we confine attention to symmetric truncation only, and assume that the parameter value μ is known. (The case of the parameter σ^2 assumed known is discussed in [1].)

The motivation for concern with truncation is the following.

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Very often a statistician is confronted with a hypothesis testing problem concerning the variance of a population. He all too often proceeds using the assumption that he is sampling from a population which is normally distributed, when in fact he is sampling from a population whose density function is given by (2.1). Obviously, this "assumption error" gives rise to errors in the performance characteristics of any test that he will use. In this paper we examine the influence of truncation on the size and power of the tests used when the "assumption error" is made.

3. Distribution of sums of squares. Let a sample of n independent observations be taken on X, where X has the distribution function (2.1). The sampling distributions of $S = S_n = \Sigma_{i=1}^{n} X_i^2$ can be derived for small values of n by convolution, but unfortunately no general form is available. We indicate the derivation for values of n up to 4, and for this purpose we take, without loss of generality, $\mu = 0$, $\sigma = 1$, and let g_n denote the density function of the distribution of S.

Case n = 1. Let $S = X^2$. Then from (2.1), it is easy to see that

$$g_1(s) = \frac{c}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} e^{-s/2}, \qquad 0 \leq s \leq a^2. \tag{3.1}$$

Case n = 2. Let S = X + Y, where X and Y are distributed independently, each with density (3.1). By the convolution formula the density of S, say $g_2(s)$, is given by

$$g_2(s) = \int_{-\infty}^{\infty} g_1(s-x) g_1(x) dx, \qquad 0 \leq s \leq 2a^2.$$

Since $g_1(x) \neq 0$ if $0 \leq x < a^2$ and $g_1(s-x) \neq 0$ if $s-a^2 \leq x < s$, it follows that the integrand is not zero whenever $\max(s - a^2, 0) \leq x \leq (\min(s, a^2)$. Splitting up the range of integration, we find that

$$g_2(s) = \begin{cases} g_2^*(s) = \frac{c^2}{2} e^{-s/2}, & \text{if } 0 \leq s < a^2, \\ g_2^{**}(s) = \frac{c^2}{\pi} e^{-s/2} \sin^{-1}\left(\frac{2a^2}{s} - 1\right), & \text{if } a^2 \leq s \leq 2a^2. \end{cases} \tag{3.2}$$

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Case n = 3. Here let S = X + Y, where X and Y are distributed independently, X having density function (3.2), and Y the density (3.1). Then if $g_3(s)$ denotes the density of S,

$$g_3(s) = \int_{-\infty}^{\infty} g_1(s - x) g_2(x) dx, \quad \text{for } 0 \leq s < 3a^2.$$

Since $g_1(s - x) \neq 0$ if $s - a^2 \leq x < s$, and $g_2(x) \neq 0$ if $0 \leq x < 2a^2$, it follows that the integrand is not zero if $\max(0, s - a^2) \leq x \leq \min(s, 2a^2)$. Examining figure 3.1, it is clear that

$$g_3(s) =$$

$$\int_0^s g_1(s - x)\, g_2^*(x) dx, \qquad \text{if } 0 \leq s < a^2,$$

$$\int_{s-a^2}^{s} g_1(s - x)\, g_2^*(x) dx + \int_{a^2}^{s} g_1(s - x)\, g_2^{**}(x) dx, \qquad \text{if } a^2 \leq s < 2a^2,$$

$$\int_{s-a^2}^{2a^2} g_1(s - x)\, g_2^{**}(x) dx, \qquad \text{if } 2a^2 \leq s \leq 3a^2.$$

On doing the necessary integration, one finds that

$$g_3(s) =$$

$$\frac{c^3}{\sqrt{2\pi}} \sqrt{s}\, e^{-s/2}, \qquad \text{if } 0 < s \leq a^2,$$

$$\frac{c^3}{\sqrt{2\pi}}\, e^{-s/2}(3a - 2\sqrt{s}), \qquad \text{if } a^2 < s \leq 2a^2,$$

$$\frac{c^3}{\sqrt{2\pi^3}}\, e^{-s/2} \left\{ 6a \tan^{-1} \frac{3a^2 - s}{2a\sqrt{s - 2a^2}} - 4\sqrt{s} \tan^{-1} \frac{(3a^2 - s)\sqrt{s}}{\sqrt{s - 2a^2}(s + a^2)} \right\},$$
$$\text{if } 2a^2 < s \leq 3a^2.$$

Case n = 4. Using a similar procedure to the above (and consulting figure 3.2) one can verify that

$$g_4(s) =$$

$$\frac{c^4}{4}\, s\, e^{-s/2} \qquad \text{if } 0 \leq s < a^2,$$

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$$\frac{c^4}{2\pi} e^{-s/2} \left\{ s \sin^{-1} \frac{2a^2 - s}{s} - 2s \sin^{-1} \frac{\sqrt{s - a^2}}{\sqrt{s}} + 4a\sqrt{s - a^2} \right\},$$
$$\text{if } a^2 \leq s < 2a^2,$$

$$\frac{c^4}{2\pi} e^{-s/2} \left\{ 6a^2 - 6a\sqrt{s - 2a^2} - 2s \sin^{-1} \frac{\sqrt{2a^2} - \sqrt{(s-a^2)(s-2a^2)}}{s} \right.$$
$$\left. - 2a\sqrt{s - a^2} + 2\sqrt{2}\, a\sqrt{s - 2a^2} \right\}$$
$$+ \frac{c^4}{2\pi^2} e^{-s/2} \int_{2a^2}^{s} \left\{ \frac{1}{\sqrt{s - x}} 6a \tan^{-1} \frac{3a^2 - x}{2a\sqrt{x - 2a^2}} \right.$$
$$\left. - 4\sqrt{x} \tan^{-1} \frac{(3a^2 - x)\sqrt{x}}{\sqrt{x - 2a^2}(x + a^2)} \right\} dx, \qquad \text{if } 2a^2 \leq s < 3a^2,$$

$$\frac{c^4}{2\pi^2} e^{-s/2} \int_{s-a^2}^{3a^2} \frac{1}{\sqrt{s - x}} \left\{ 6a \tan^{-1} \frac{3a^2 - x}{2a\sqrt{x - 2a^2}} \right.$$
$$\left. -4\sqrt{x} \tan^{-1} \frac{(3a^2 - x)\sqrt{x}}{\sqrt{x - 2a^2}(x + a^2)} \right\} dx, \qquad \text{if } 3a^2 \leq s \leq 4a^2.$$

The functions $\phi_n(s) = \int_0^s g_n(t)dt$, $n = 1, 2, 3$ and 4, have been tabulated by the author, on the IBM 650, located at the Statistical Techniques Research Group, Princeton University. The author wishes to thank Dr. R. S. Pinkham for his generous help with the programming.

4. Tests of hypotheses under truncation. In this section, we will examine the effect of truncation on size and power of tests of hypotheses that deal with the variances of populations.

Let a sample of size n be taken from a population, whose distribution is $N(0, \sigma^2)$. Then a uniformly most powerful (UMP) test of size α for the one sided hypothesis problem

(4.1) $\qquad H: \sigma^2 = 1 \qquad \text{Alt}: \sigma^2 = \sigma_1^2 > 1$

is given by:

(4.2) Reject H if $S_n = \Sigma_{i=1}^n X_i^2 > \psi_{\alpha, n}^2$; accept H otherwise,

where $\psi_{\alpha, n}^2$ is the point exceeded with probability α using the

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Chi-Squared distribution with n degrees of freedom. Now if sampling from $N_a(0, \sigma^2)$, where $N_a(0, \sigma^2)$ is the density function (2.1) with = 0, and the test procedure (4.2) is used, the preassigned size of this 'usual' test is really not obtained. The actual size is given by

$$(4.3) \qquad \alpha' = \Pr(S_n > \psi^2_{\alpha, n})$$

where here S_n is the random variable with density function $g_n(s)$ of the last section.

Further, the usual power function of the procedure (4.2) is given by

$$(4.4) \qquad \begin{aligned} P_u(\sigma^2) &= \Pr(S_n > \psi^2_{\alpha, n} \mid S \sim \psi^2_n \sigma^2) \\ &= \Pr(S_n > \frac{\psi^2_{\alpha, n}}{\sigma^2} \mid S_n \sim \psi^2_n). \end{aligned}$$

Now if the sampling is from a truncated distribution, $N_a(0, \sigma^2)$, the 'actual' power function of the 'usual' size α test is given by

$$(4.5) \qquad \begin{aligned} P(\sigma^2, a) &= \Pr(S_n > \psi^2_{\alpha, n} \mid S_n \sim g_n(s, \sigma^2)) \\ &= \Pr(S_n > \frac{\psi^2_{\alpha, n}}{\sigma^2} \mid S_n \sim g_n(s, 1) = g_n(s)). \end{aligned}$$

Denote the difference of (4.4) and (4.5) by

$$(4.6) \qquad L(\sigma^2, a) = P_u(\sigma^2) - P(\sigma^2, a).$$

For $\sigma^2 = 1$, L equals $\alpha - \alpha'$, while for all other values of σ^2, L is the "loss of power" if the usual procedure is followed even though sampling is from a truncated distribution. Values of $P_u(\sigma^2)$, $P(\sigma^2, a)$ and the loss in power expressed as percentage of $P_u(\sigma^2)$ for different values σ^2 and terminus points 'a' are given in table I for $\alpha = .05$, n = 1, 2, 3, and 4.

Now, let us turn to the situation where sampling is from $N_a(0, \sigma^2)$. By the Neyman-Pearson Fundamental Lemma, the UMP test of (4.1) of size α is:

$$\text{(4.7)}\qquad \text{Reject } H \text{ if } \Sigma X_i^2 > K_\alpha(a, n), \text{ or } \max|x_i| > a; \text{ accept } H \text{ otherwise;}$$

where $K_\alpha(a, n)$ is the point exceeded with probability α using the distribution whose density is $g_n(s)$. Table II gives the significance points for the test (4.7) for different n, α and a. That is, if sampling from a 'truncated' normal distribution, (4.7) gives the 'correct' test for problem (4.1) and table II gives the correct significance points for this problem.

5. Conclusions. Examination of table I, shows that by far the most serious losses occur in the size of the test rather than in its power. For example, if the truncation occurs as much as two and a half standard deviations away from the mean, and n = 4, the significance level is only two percent if the usual 5 point is used. However, this increases if the truncation is three standard deviations away from the mean, and indeed the size approaches the 5% level, as the degree of truncation increases.

It is interesting to note that, although the losses in power are almost negligible as n increases, they do increase as n increases. This of course is explained by the fact that the distributions of sums of squares of variables from the truncated normal have mean and variance less than that of the distribution of $\chi^2_{(n)}$, that is, the distribution moves to the left as n increases, and gives less probability to tails on the right, and hence larger L.

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TABLE I

Values of $P_u(\sigma^2)$, $P(\sigma^2, a)$, $L(\sigma^2, a)$ and the loss in power, L, expressed as a percentage of $P_u(\sigma^2)$.

n	a	1					1.5				
	σ^2	1	1.5	2.0	2.5	3.5	1	2.5	3.5	4.5	5.5
1	P_u	.05000	.09554	.16531	.21516	.29482	.05000	.21516	.2948	.35555	.40333
	P	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	.18618	.25602	.31147
	L	.05000	.109554	.16581	.21516	.29482	.05000	.21516	.10864	.09953	.09186
	% loss	100	100	100	100	100	100	100	36.85	27.99	22.78
2	P_u	.05000	.13574	.22363	.30174	.42492	.05000	.30174	.42492	.51393	.58005
	P	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	.23378	.35256	.44064
	L	.05000	.13574	.22363	.30174	.42492	.05000	.30174	.19114	.16137	.13941
	% loss	100	100	100	100	100	100	100	44.98	31.40	24.03
3	P_u	.05000	.15707	.27165	.37264	.52548	.05000	.37264	.52548	.62884	.70066
	P	0.00000	0.00000	0.00000	0.00000	.00644	0.00000	.08949	.27004	.42913	.53966
	L	.05000	.15707	.27165	.37264	.51904	.05000	.28315	.25544	.19971	.16100
	% loss	100	100	100	100	98.77	100	75.98	48.61	31.76	22.98
4	P_u	.05000	.17613	.31459	.43443	.60732	.05000	.43493	.60732	.71583	.78615
	P	0.00000	0.00000	0.00000	.00421	.00656	0.00000	.10049	.31077	.48578	.60479
	L	.05000	.17613	.31459	.43022	.60076	.05000	.33395	.29655	.23005	.18136
	% loss	100	100	100	99.03	98.92	100	76.87	48.83	32.14	23.07

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TABLE I continued

Values of $P_u(\sigma^2)$, $P(\sigma^2, a)$, $L(\sigma^2, a)$ and the loss in power, L, expressed as a percentage of $P_u(\sigma^2)$.

n	a	2.0					2.5				
	σ^2	1	2.5	3.5	4.5	5.5	1	2.5	3.5	4.5	5.5
1	P_u	.05000	.21516	.29482	.35555	.40333	.05000	.21516	.29482	.35555	.40333
	P	0.00000	.17421	.26132	.32470	.37503	0.00000	.20535	.28606	.34732	.39616
	L	.05000	.04095	.03350	.03085	.02830	.05000	.00981	.00876	.008	.06717
	% loss	100	19.03	11.36	8.677	7.017	100	4.559	2.971	2.315	1.778
2	P_u	.05000	.30174	.42492	.51393	.58005	.05000	.30174	.42492	.51393	.58005
	P	.00407	.23365	.36895	.46658	.53915	.02597	.24413	.41030	.50172	.56951
	L	.04593	.06809	.05597	.04735	.04090	.02403	.01761	.01462	.01221	.01054
	% loss	91.86	22.57	13.17	9.213	7.051	48.06	5.836	3.441	2.376	1.817
3	P_u	.05000	.37204	.52548	.62884	.70066	.05000	.37264	.52548	.62884	.70066
	P	.00266	.27884	.45434	.57009	.65114	.02149	.34864	.50735	.61456	.68919
	L	.04734	.09380	.07114	.05875	.04955	.02851	.02400	.01813	.01428	.01147
	% loss	94.68	25.17	13.54	9.342	7.072	57.02	6.441	3.450	2.271	1.637
4	P_u	.05000	.43443	.60732	.71583	.78615	.05000	.43443	.60732	.71583	.78615
	P	.00178	.31867	.51689	.64773	.73039	.02000	.40548	.58607	.69931	.77323
	L	.04922	.11576	.09043	.06810	.05576	.03000	.02895	.02125	.01652	.01292
	% loss	98.40	26.65	14.89	9.513	7.092	60	6.664	3.499	2.308	1.644

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TABLE I continued

Values of $P_u(\sigma^2)$, $P(\sigma^2, a)$, $L(\sigma^2, a)$ and the loss in power, L, expressed as a percentage of $P_u(\sigma^2)$.

n	a	3.0				
	σ^2	1	2.5	3.5	4.5	5.5
1	P_u	.05000	.21516	.29482	.35555	.40333
	P	0.00000	.21309	.29302	.35379	.40185
	L	.05000	.00207	.00180	.00176	.00148
	% loss	100	.8621	.6105	.4950	.3669
2	P_u	.05000	.30174	.42492	.51393	.58005
	P	.04486	.29801	.42174	.51138	.57786
	L	.00514	.00373	.00318	.00255	.00219
	% loss	10.8	1.236	.7487	.4962	.3776
3	P_u	.05000	.32264	.52548	.62884	.70066
	P	.04226	.36750	.52153	.62550	.69799
	L	.00774	.00514	.00395	.00334	.00267
	% loss	13.48	1.379	.7512	.5311	.3813
4	P_u	.05000	.43443	.60732	.71583	.78615
	P	.03975	.42832	.60275	.71199	.78385
	L	.01025	.00611	.00457	.00384	.00334
	% loss	20.50	1.406	.7518	.5369	.4197

TABLE II

Upper 100α % points of $g_n(s)$.

α	a \ n	1	2	3	4
.10	1.0	.753	1.126	1.545	1.926
	1.5	1.503	2.252	3.045	3.799
	2.0	2.168	3.429	4.489	5.574
	2.5	2.537	4.204	5.594	6.865
	3.0	2.667	4.510	6.091	7.553
	∞	2.706	4.605	6.251	7.779
.05	1.0	.862	1.319	1.755	2.073
	1.5	1.823	2.635	3.521	4.318
	2.0	2.818	4.014	5.266	6.449
	2.5	3.489	5.222	6.641	8.054
	3.0	3.759	5.796	7.495	9.063
	∞	3.841	5.991	7.815	9.488
.025	1.0	.932	1.484	1.929	2.268
	1.5	2.020	3.008	3.927	4.716
	2.0	3.298	4.881	5.978	7.225
	2.5	4.346	6.029	7.609	9.052
	3.0	4.851	6.996	8.745	10.369
	∞	5.074	7.378	9.348	11.143
.01	1.0	.972	1.657	2.140	2.547
	1.5	2.153	3.435	4.381	5.218
	2.0	3.631	5.330	6.808	8.329
	2.5	5.823	6.953	8.820	10.482
	3.0	6.215	8.355	10.200	12.018
	∞	6.635	9.210	11.345	13.277
.005	1.0	.986	1.755	2.287	2.733
	1.5	2.201	3.697	4.701	5.496
	2.0	3.833	5.848	7.358	9.426
	2.5	5.660	7.696	9.675	11.587
	3.0	7.106	9.158	11.269	13.203
	∞	7.879	10.597	12.838	14.860

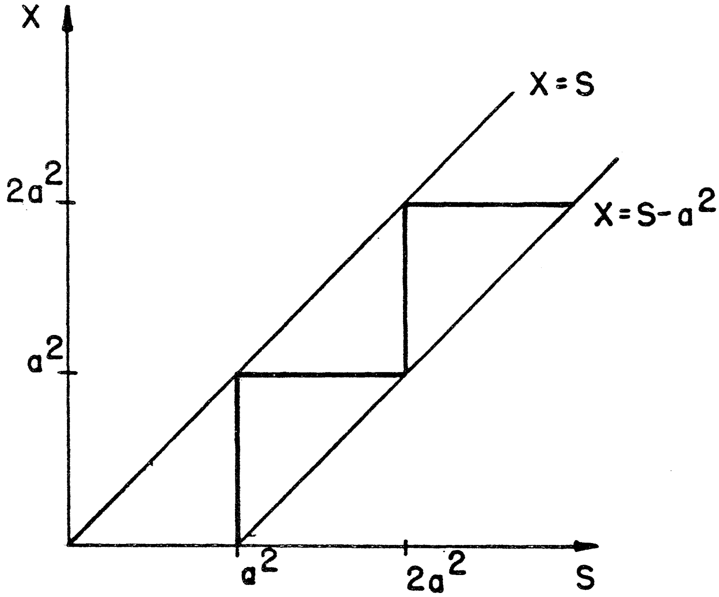

FIG. 3.1

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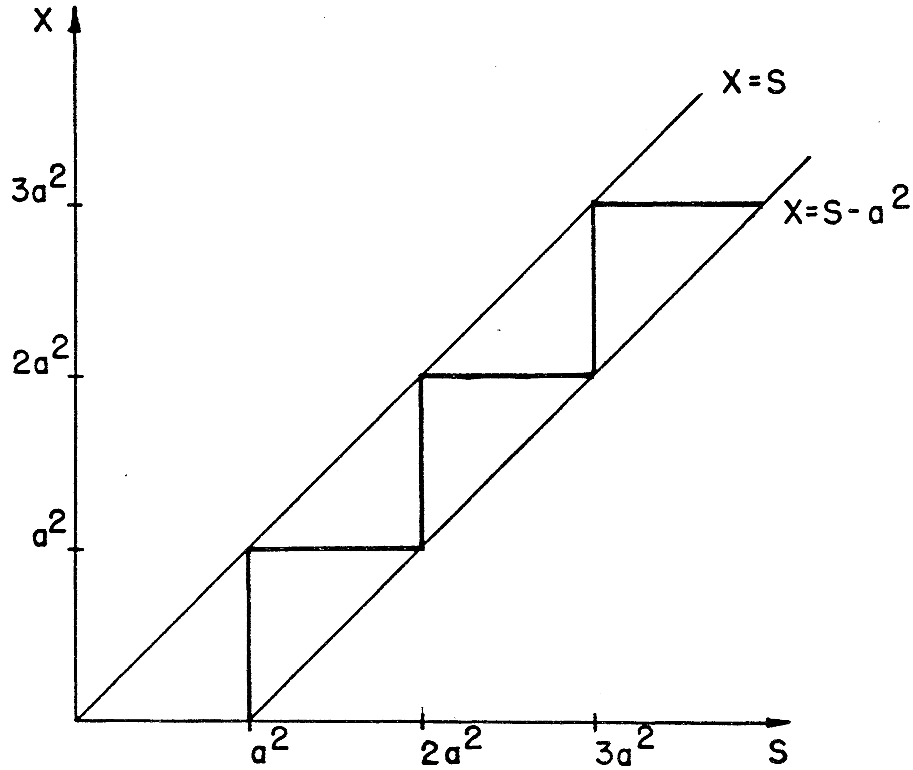

FIG. 3.2