EXTENSIONS OF I-BISIMPLE SEMIGROUPS

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A bisimple semigroup S is called I-bisimple if E_S, the set of idempotents of S, with its natural order is order-isomorphic to I, the set of integers, under the reverse of the usual order. In **(9)**, the author completely determined the structure of I-bisimple semigroups mod groups; in this paper, he also gave an isomorphism theorem, a homomorphism theorem, an explicit determination of the maximal group homomorphic image, and a complete determination of the congruences for these semigroups.

Clifford gave a general means of finding all possible extensions of a (weakly reductive) semigroup S by a semigroup T with zero **(2)**. As with the Schreier theory of group extensions, these means are of a theoretical nature and to carry them out for particular classes of semigroups is usually difficult **(2)**. This has been done for only three cases: (1) S is completely simple, T arbitrary; (2) S is a group and T completely 0-simple; (3) S is a Brandt semigroup and T arbitrary. The first is due to Clifford **(1)**, the second to Munn **(2**, §4.5**)**, and the third to Warne **(7)**. In **(5)**, Warne determined when the extensions of a completely 0-simple semigroup by a completely 0-simple semigroup are given by partial homomorphisms.

In this paper, we give an explicit determination of the extensions of an I-bisimple semigroup S by a completely 0-simple (Brandt) semigroup. We determine the multiplication of the translational hull $\bar{S}$ of S. All extensions of S by a semigroup T can then be described if one knows the structure of T and the partial homomorphisms of $T\backslash 0$ into $\bar{S}$ such that $AB = 0$ in T implies that $A\theta B\theta \in S$ **(2)**. (The construction of $\bar{S}$ is needed as S has no identity.)

Let S and T be disjoint semigroups, T having a zero element 0. A semigroup V will be called an (ideal) extension of S by T if it contains S as an ideal, and if the Rees factor semigroup V/S **(2)** is isomorphic with T.

Multiplication in S and T will be denoted by juxtaposition. T^* will denote the set of non-zero elements of T.

If V is an extension of a semigroup S by a semigroup T (with zero), then we say that V is determined by a partial homomorphism if there exists a partial homomorphism π: $T^* \to S$ such that for all $A, B \in T^*$, $c, d \in S$,

$$A \circ B = \begin{cases} AB & \text{if } AB \neq 0 \text{ (in } T), \\ A\pi B\pi & \text{if } AB = 0 \text{ (in } T). \end{cases}$$

$A \circ c = (A\pi)c$; $c \circ A = c(A\pi)$; $c \circ d = cd$ where $\circ$ denotes the operation in V. (π always gives an extension of S by T **(2**, p. 138, Theorem 4.19**)**).

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For all concepts and notation not defined in this paper the reader is referred to **(2)**.

THEOREM A **(8)**. *S is an I-bisimple semigroup if and only if* $S \cong G \times I \times I$, *where G is a group, I is the group of integers, under the multiplication*

$$(g, n, m)(h, p, q) = (g\alpha^{p-r}h\alpha^{m-r}, n + p - r, m + q - p), \qquad r = \min(m, p)$$

or under the multiplication

$$(g, n, m)(h, p, q) = (g\alpha^{s-m-p}h\alpha^{s-q}, n + p, s), \qquad s = \max(m + p, q),$$

where α *is an endomorphism of G and* α^0 *is the identity transformation.*

$S = (G, \delta)$ will denote an I-bisimple semigroup with structure group G and structure endomorphism δ.

LEMMA 1. *A semigroup S is left reductive and right reductive if and only if the inner right (left) translations* ρ_a *and* λ_b *of S are linked implies that* $a = b$ *for all* $a, b \in S$. *If S is a right (left) reductive semigroup* $\lambda_a(\rho_a)$ *and the right (left) translation* $\rho(\lambda)$ *are linked if and only if* $\rho = \rho_a$ ($\lambda = \lambda_a$). *An* I-bisimple *semigroup is right reductive and left reductive.*

The first two assertions are easy to prove, so we omit their proofs.

Proof. Let $S = (G, \delta)$ be an I-bisimple semigroup. If

$$(g, i, j)(e, j, j) = (h, k, l)(e, j, j),$$

$j = l + j - \min(j, l)$ and $i = k + j - l$. If $(g, i, j)(e, l, l) = (h, k, l)(e, l, l)$, $j + l - \min(j, l) = l$. Thus, $(g, i, j) = (h, k, l)$ and S is right reductive. Similarly, S is left reductive.

Let $M(I, G)$ be the full group of mappings of I into G (pointwise multiplication).

THEOREM 1. *Let* $S = (G, \alpha)$ *be an I-bisimple semigroup. Let*

$$H = \{\beta \in M(I, G)\colon (i + 1)\beta = (i\beta)\alpha \text{ for all } i\}.$$

H is a subgroup of $M(I, G)$. *Let* $\rho_i (i \in I)$ *be the inner right translation of* $(I, +)$ *determined by i.* $W = H \times I$, *under the multiplication*

$$(*) \qquad (\beta, i)(\gamma, j) = (\beta \circ \rho_i \gamma, i + j),$$

where $\circ$ *denotes the operation in H and juxtaposition denotes iteration of mappings, is a group. Let* $\bar{S}$ *be the translational hull of S. Then* $\bar{S} = W \cup S$ *under the multiplication*

$$(\beta, a) \circ (\gamma, b) = (\beta, a)(\gamma, b),$$
$$(g, i, j) \circ (h, s, t) = (g, i, j)(h, s, t)$$

where juxtaposition denotes multiplication in W and S and

$$(\beta, a) \circ (g, i, j) = ((i - a)\beta g, i - a, j),$$
$$(g, i, j) \circ (\beta, a) = (g(j\beta), i, j + a).$$

Proof. Let ρ be a right translation of S. If $(e, i, i)\rho = (h, x, y)$, e being the identity of G, then $(e, i, i)(h, x, y) = (h, x, y)$ and $x \geqslant i$. Thus

$$(e, i, i)\rho = (i\beta, i + i\rho_1, i\rho_2)$$

where $\beta(\rho_1, \rho_2)$ is a mapping of I into G (I^0, I) (I^0 is the non-negative integers). Since $(e, i+1, i+1)(e, i, i) = (e, i+1, i+1)$,

$$(e, i+1, i+1)(i\beta, i + i\rho_1, i\rho_2) = ((i+1)\beta, (i+1) + (i+1)\rho_1, (i+1)\rho_2).$$

Thus, if $i\rho_1 = 0$, $(i+1)\rho_1 = 0$, $(i+1)\rho_2 = i\rho_2 + 1$, and $(i+1)\beta = (i\beta)\alpha$, and, if $i\rho_1 \geqslant 1$, $(i+1)\rho_1 = i\rho_1 - 1$, $(i+1)\rho_2 = i\rho_2$, and $(i+1)\beta = i\beta$. Let us first consider the case $i\rho_1 = 0$ for all $i \in I$. Then, it is easy to see that $i\rho_2 = i + z$ where $0\rho_2 = z$. Hence,

$$(g, i, j)\rho = (g, i, j)((e, j, j)\rho) = (g(j\beta), i, j + z).$$

Thus, we may write

$$(1) \qquad (g, i, j)\rho_{(\beta, z)} = (g(j\beta), i, j + z)$$

where β is a mapping of I into G such that $(i+1)\beta = (i\beta)\alpha$ for all $i \in I$ and $z \in I$. Conversely it is easy to see that (1) is a right translation of S.

Next, suppose that there exists $u \in I$ such that $u\rho_1 \neq 0$. It is easy to see that there then exists a unique $t \in I$ such that $(t + i)\rho_1 = -i$ if $i < 0$ and $(t + i)\rho_1 = 0$ if $i \geqslant 0$. If $t\rho_2 = z$, it follows that $(t + i)\rho_2 = z + i$ if $i \geqslant 0$ and $(t + i)\rho_2 = z$ if $i < 0$. If we let $t\beta = g_0$, we see that $(t + i)\beta = g_0\,\alpha^i$ if $i \geqslant 0$ and $(t + i)\beta = g_0$ if $i < 0$. Since $(e, i, i + n)(e, i, i) = (e, i, i + n)$ if $n \geqslant 0$, $(e, i, i + n)\rho = [(i\beta)\alpha^{n - i\rho_1}, i, n + i\rho_2 - i\rho_1]$ if $n \geqslant i\rho_1$ and

$$(e, i, i + n)\rho = (i\beta, i + i\rho_1 - n, i\rho_2)$$

if $i\rho_1 \geqslant n$ where $n \in I^0$. Since $(g, i + n, i + m) = (g, i + n, i)(e, i, i + m)$, we obtain

$$(2) \quad (g, i + n, i + m)\rho = \begin{cases} (g(i\beta)\alpha^{m - i\rho_1}, i + n, m + i\rho_2 - i\rho_1) & \text{if } m \geqslant i\rho_1, \\ (g\alpha^{i\rho_1 - m}(i\beta), i + n + i\rho_1 - m, i\rho_2) & \text{if } i\rho_1 \geqslant m \end{cases}$$

where $n, m \in I^0$. We note that if $(g, k, l) \in S$, $k = i + n$, $l = i + m$ $(i \in I$, $m \in I^0)$ by **(3; 4)** and Theorem A.

If we let $i = t + s$ in (2), then we obtain, if $s \geqslant 0$,

$$(3) \quad (g, t + s + n, t + s + m)\rho = (g(g_0\,\alpha^{s+m}), t + s + n, m + z + s)$$

and if $s \leqslant 0$

$$(4) \quad (g, t + s + n, t + s + m)\rho = \begin{cases} (g(g_0\,\alpha^{m+s}), t + s + n, m + z + s) & \\ & \text{if } m \geqslant -s, \\ (g\alpha^{-s-m}\,g_0, t + n - m, z) & \text{if } -s \geqslant m. \end{cases}$$

Now, let $x = t + s + n$ and $y = t + s + m$. Thus, (3) becomes

$$(5) \qquad (g, x, y)\rho = (g(g_0\,\alpha^{y-t}), x, z + y - t) \qquad \text{for } s \geqslant 0$$

and (4) becomes

$$(6) \qquad (g, x, y)\rho = \begin{cases} (g(g_0\,\alpha^{y-t}), x, z + y - t) & \text{if } m \geqslant -s, \\ (g\alpha^{t-y}\,g_0, t - y + x, z) & \text{if } -s \geqslant m \end{cases}$$

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where $s < 0$. It is easily seen that (5) and (6) are equivalent to

$$(g, x, y)\rho = \begin{cases} (g(g_0 \alpha^{y-t}), x, z + y - t) & \text{if } y \geqslant t, \\ (g\alpha^{t-y} g_0, x + (t - y), z) & \text{if } y < t. \end{cases}$$

Hence

$$\begin{aligned}(g, x, y)\rho &= (g\alpha^{t-\min(y,t)} g_0 \alpha^{y-\min(y,t)}, x + t - \min(y, t), y + z - \min(y, t)), \\ &= (g, x, y)(g_0, t, z)\end{aligned}$$

and ρ is an inner right translation.

Summarizing, every right translation of S is an inner right translation or is of the form (1).

In a similar manner, one shows that every left translation of S is an inner left translation or is of the form

$$(7) \qquad (g, i, j)\lambda_{(\delta,z)} = ((i\delta)g, i + z, j)$$

where δ is a mapping of I into G such that $(i + 1)\delta = (i\delta)\alpha$ for $i \in I$ and $z \in I$.

By Lemma 1, $\rho_{(g,i,j)}$ and $\lambda_{(h,k,l)}$ are linked if and only if $(g, i, j) = (h, k, l)$, $\rho_{(g,i,j)}$ and $\lambda_{(\delta,z)}$ are not linked, and $\rho_{(\beta,z)}$ and $\lambda_{(g,i,j)}$ are not linked. (One sees easily that $\lambda_{(\delta,z)}$ $(\rho_{(\beta,z)})$ is not an inner left (right)translation.

Next, suppose that $\rho_{(\beta,a)}$ and $\lambda_{(\delta,b)}$ are linked. Thus,

$$\begin{aligned}((e, 0, 0)\rho_{(\beta,a)})(e, 0, 0) = (0\beta, 0, a)(e, 0, 0) &= (e, 0, 0)((e, 0, 0)\lambda_{(\delta,b)}) \\ &= (e, 0, 0)(0\delta, b, 0).\end{aligned}$$

Hence by a straightforward calculation $b = -a$.

Now,

$$\begin{aligned}((e, i, i)\rho_{(\beta,a)})(e, i + a, i) &= (i\beta, i, i + a)(e, i + a, i) = (i\beta, i, i) \\ &= (e, i, i)((e, i + a, i)\lambda_{(\delta,-a)}) = ((i + a)\delta, i, i).\end{aligned}$$

Hence $i\beta = (i + a)\delta$. By a straightforward calculation, these conditions are sufficient for linkage.

Thus, since there is one and only one left translation of S linked with each right translation of S, it follows that $\bar{S}$ is isomorphic with the semigroup of right translations of S.† Hence

$$\bar{S}\backslash S = \{\rho_{(\beta,a)} : (i + 1)\beta = (i\beta)\alpha \text{ for all } i \in I\}.$$

Thus, if $\rho_{(\beta,a)}, \rho_{(\gamma,b)}$ in $\bar{S}\backslash S$, $\rho_{(\beta,a)}\, \rho_{(\gamma,b)} = \rho_{(\beta o \rho_a \gamma, a+b)}$ and $\bar{S}\backslash S$ is a semigroup.

$H = \{\beta \in M(I, G)/(i + 1)\beta = (i\beta)\alpha\}$ is a subgroup of $M(I, G)$. We have just shown that $(\beta, a)\phi = (\lambda_{(\rho_{-a}\beta,-a)}, \rho_{(\beta,a)})$ defines an isomorphism of $W = H \times I$ under the multiplication $(*)$ onto $\bar{S}\backslash S$. It is easy to see that W under $(*)$ is a group with identity $(\beta_1, 0)$ where $i\beta_1 = e$, the identity of G, for all $i \in I$ and the inverse of (β, a) is $(\beta', -a)$ where $i\beta' = ((i - a)\beta)^{-1}$ for all $i \in I$.

†We thank the referee for this remark.

By (2, Lemma 1-2) and (7),

$$(\beta, a) \circ (g, i, j) = (\lambda_{(\rho_{-a}\beta,-a)}, \rho_{(\beta,a)})(g, i, j) = (g, i, j)\lambda_{(\rho_{-a}\beta,-a)}$$
$$= ((i\rho_{-a})\beta g, i - a, j).$$

$(g, i, j) \circ (\beta, a)$ is determined in a similar manner.

As usual (2), $T = M^0(H, K, \Lambda, P)$ will denote a completely O-simple semigroup with structure group H and $\Lambda \times K$ sandwich matrix P and

$$M^0(H, K, K, \Delta)$$

will denote a Brandt semigroup (2, Theorem 3.9, p. 102).

THEOREM 2. *Let $S = (G, \delta)$ and $T = M^0(H; K, \Lambda; P)$. Let I be the group of integers under addition, and let the following functions be given:*

$$\psi\colon K \to I, \qquad \theta\colon \Lambda \to I, \qquad \alpha\colon K \to G, \qquad \beta\colon \Lambda \to G,$$

and γ be a homomorphism of H into G such that $p_{\lambda i} \neq 0$ implies $\lambda\theta = i\psi$ and $(\lambda\beta)(i\alpha) = p_{\lambda i}\gamma$. Then ϕ, defined on T^* by

$$(a; i, \lambda)\phi = ((i\alpha)(a\gamma)(\lambda\beta), i\psi, \lambda\theta),$$

is a partial homomorphism of T^* into S Conversely, every partial homomorphism of T^* into S is obtained in this fashion.

Proof. For the direct part, we have

$$(a; i, \lambda)\phi(b; k, u)\phi = ((i\alpha)(a\gamma)(\lambda\beta), i\psi, \lambda\theta)((k\alpha)(b\gamma)(u\beta), k\psi, u\theta);$$

If $p_{\lambda k} \neq 0$, then

$$(a; i, \lambda)(b; k, u)\phi = (ap_{\lambda k}b; i, u)\phi = ((i\alpha)(a\gamma)(p_{\lambda k}\gamma)(b\gamma)(u\beta), i\psi, u\theta).$$

Since $p_{\lambda k} \neq 0$, $k\psi = \lambda\theta$ and $(\lambda\beta)(k\alpha) = p_{\lambda k}\gamma$. Thus

$$((i\alpha)(a\gamma)(\lambda\beta), i\psi, \lambda\theta)((k\alpha)(b\gamma)(u\beta), k\psi, u\theta)$$
$$= ((i\alpha)(a\gamma)(\lambda\beta)(k\alpha)(b\gamma)(u\beta), i\psi, u\theta)$$
$$= ((i\alpha)(a\gamma)(p_{\lambda k}\gamma)(b\gamma)(u\beta), i\psi, u\theta)$$

and hence ϕ is a partial homomorphism of T^* into S.

Conversely, let ϕ be a partial homomorphism of T^* into S. Since ϕ maps $\mathfrak{R}$- ($\mathfrak{L}$-)classes into $\mathfrak{R}$- ($\mathfrak{L}$-)classes, $(a; i, j)\phi = (g; i\psi, j\theta)$, where $g \in G$ and $\psi(\theta)$ is a mapping of $K(\Lambda)$ into I. We suppose that $K \cap \Lambda = 1$ and that $p_{11} = e$, the identity of H. Now, $(e; i, 1)\phi = (i\alpha, i\psi, 1\theta)$ and

$$(e; 1, \lambda)\phi = (\lambda\beta; 1\psi, \lambda\theta),$$

where $\alpha(\beta)$ is a mapping of $K(\Lambda)$ into G. Since $(e; 1, 1)$ is idempotent, $(e; 1, 1)\phi = (E, 1\psi, 1\theta)$, where E is the identity of G, and $1\psi = 1\theta$. Define γ on H by $(a; 1, 1)\phi = (a\gamma, 1\psi, 1\theta)$. Since $(a; 1, 1)(b; 1, 1) = (ab; 1, 1)$, $(a\gamma, 1\psi, 1\theta)(b\gamma; 1\psi, 1\theta) = ((ab)\gamma, 1\psi, 1\theta)$ and $(a\gamma)(b\gamma) = (ab)\gamma$, i.e. γ is a homomorphism of H into G. Since $(e; 1, \lambda)(e; i, 1) = (p_{\lambda i}; 1, 1)$, $p_{\lambda i} \neq 0$

implies $(\lambda\beta, 1\psi, \lambda\theta)(i\alpha, i\psi, 1\theta) = (p_{\lambda i}\gamma, 1\psi, 1\theta)$. Hence it easily follows that $p_{\lambda i} \neq 0$ implies $i\psi = \lambda\theta$ and $(\lambda\beta)(i\alpha) = p_{\lambda i}\gamma$. Finally

$$\begin{aligned}(a; i, \lambda)\phi &= (e; i, 1)\phi(a; 1, 1)\phi(e; 1, \lambda)\phi \\ &= (i\alpha, i\psi, 1\theta)(a\gamma, 1\psi, 1\theta)(\lambda\beta, 1\psi, \lambda\theta) = ((i\alpha)(a\gamma)(\lambda\beta), i\psi, \lambda\theta).\end{aligned}$$

COROLLARY 1. *Let $S = (G, \delta)$, $T = M^0(H; K; K; \Delta)$, and ψ, α, γ be as in Theorem 2. Then ϕ, defined on T^* by*

$$(a; i, j)\phi = ((i\alpha)(a\gamma)(j\alpha)^{-1}, i\psi, j\psi),$$

is a partial homomorphism of T^* into S. Conversely, every partial homomorphism of T^* into S is obtained in this fashion.

Remark 1. If one replaces I by I^0, the semigroup of non-negative integers under addition in Theorem 2 (Corollary 1), the theorem will give all partial homomorphisms of a completely 0-simple (Brandt) semigroup into an ω-bisimple semigroup (a bisimple semigroup S such that E_S is order isomorphic to the non-negative integers under the reverse of the usual order (see **(6)** for example). Since such semigroups have an identity, all their extensions are determined by partial homomorphisms **(2**, p. 138, Theorem 4.19**)**. We note that the proofs of Theorem 2 for the ω-bisimple and I-bisimple cases are almost identical.

Remark 2. The statements of Remark 1 are valid for a class of bisimple semigroups S for which E_S is n-lexicographically ordered **(6)**.

THEOREM 3. *Let S be a weakly reductive semigroup and let $\bar{S}$ be its translational hull. Let T be a 0-bisimple semigroup having proper divisors of zero. If $S = \bar{S}$ or $\bar{S}\backslash S$ is a subsemigroup of S, then every extension of S by T is given by a partial homomorphism.*

Proof. If $S = \bar{S}$, S has an identity, and the result follows from **(2**, p. 138, Theorem 4.19**)**. Thus, assume that $\bar{S}\backslash S$ is a semigroup. Let $(V, \circ)$ be an extension of S by T. Then, there exists an extension $(\bar{V}, *)$ of $\bar{S}$ by T such that $(V, \circ)$ is a subsemigroup of $(\bar{V}, *)$ **(2**, p. 139, Theorem 4.20**)**. By **(2**, p. 138, Theorem 4.19**)**, $\bar{V}$ is determined by a partial homomorphism θ. Since $T\backslash 0$ is a $\mathfrak{D}$-class, $(T\backslash 0)\theta$ is contained in a $\mathfrak{D}$-class of $\bar{S}$. Thus, $(T\backslash 0)\theta$ is contained in $\bar{S}\backslash S$ or S since S is an ideal of $\bar{S}$ **(2**, p. 11, Lemma 1.2**)**. If $A, B \in T\backslash 0$ and $AB = 0$ (in T), $A\theta B\theta \in S$. Thus, since $\bar{S}\backslash S$ is a semigroup, $(T\backslash 0)\theta \subseteq S$ and $(V, \circ)$ is given by a partial homomorphism.

COROLLARY 2. *Let S be an I-bisimple semigroup and T be a 0-bisimple semigroup with proper divisors of zero. Every extension of S by T is given by a partial homomorphism. In particular, if T is a completely 0-simple (Brandt) semigroup with proper divisors of zero, then every extension of S by T is given by a partial homomorphism. An explicit multiplication for these extensions is thus given by employing Theorem 2 (Corollary 1).*

Proof. The first part of the theorem is a consequence of Lemma 1, Theorem 1 and Theorem 3 while the second part follows from the first part (**2**, p. 79, Theorem 2.51 and p. 102, Theorem 3.9).

THEOREM 4. *Let $S = (G, \delta)$ be an I-bisimple semigroup and let $T = M^0(R, K, \Lambda; P)$ be a completely 0-simple semigroup without proper divisors of zero. Let V be an extension of S by T. Then, either V is given by a partial homomorphism and an explicit multiplication for V is thus given by employing Theorem 2 or $V = (T\backslash 0)US$ under the multiplication*

$$(8)\quad (a; s, \lambda) * (g, i, j) = ((i\rho_{-(k_s+i_a+t_\lambda)})(\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda))g, i - k_a - i_a - t_\lambda, j),$$

$$(9)\quad (g, i, j) * (a; s, \lambda) = (g(j(\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_{\lambda},)), i, j + k_s + i_a + t_\lambda)$$

where $(g, i, j) \in S$ and $(a; s, \lambda) \in T\backslash 0$, $\circ$ denotes the product in $M(I, G)$, $a \to i_a$ is a homomorphism of R into $(I, +)$, $a \to \theta_a$ is mapping of R into H (see statement of Theorem 1) such that $\theta_{ab} = \theta_a \circ \rho_{i_a}\theta_b$ for all a, b in R, $s \to \beta_s$ is a mapping of K into H, $s \to k_s$ is a mapping of K into I, $\lambda \to \gamma_\lambda$ is a mapping of Λ into H, and $\lambda \to t_\lambda$ is a mapping of Λ into I such that $i_{p_{\lambda s}} = t_\lambda + k_s$ and $\theta_{p_{\lambda s}} = \gamma_\lambda \circ \rho_{t_\lambda}\beta_s$. Conversely, (8) *and* (9) *define an extension of S by T.*

Proof. Let $\bar{S}$ be the translational hull of S. As in the proof of Theorem 3, there exists an extension $(\bar{V}, x)$ of $\bar{S}$ by T given by a partial homomorphism θ such that $(V, *)$ is a subsemigroup of $(\bar{V}, x)$. Clearly, $(T\backslash 0)\theta \subseteq S$ or $(\bar{S}\backslash S)$. In the first case V is given by the partial homomorphism θ. Thus, in the notation of Theorem 1, it is only necessary to consider the homomorphisms θ of T^* into W. By (**2**, p. 143, Theorem 4.22), we have that $(a; s, \lambda)\theta = u_s(a\omega)v_\lambda$ where $s \to u_s$ and $\lambda \to v_\lambda$ are mappings of K into W and Λ into W, respectively, and ω is a homomorphism of R into W such that $p_{\lambda s}\omega = v_\lambda u_s$ for all s in K and λ in Λ. Let H be as in the statement of Theorem 1. It is easily seen that $a\omega = (\theta_a, i_a)$ where $a \to i_a$ is a homomorphism of R into $(I, +)$ and $a \to \theta_a$ is a mapping of R into H such that $\theta_{ab} = \theta_a \circ \rho_{i_a}\theta_b$ for all $a, b \in R$. We may write $u_s = (\beta_s, k_s)$ where $s \to \beta_s$ is a mapping of K into H and $s \to k_s$ is a mapping of K into I and $v_\lambda = (\gamma_\lambda, t_\lambda)$ where $\lambda \to \gamma_\lambda$ is a mapping of Λ into H and $\lambda \to t_\lambda$ is a mapping of Λ into I. Thus, $(\theta_{p_{\lambda s}}, i_{p_{\lambda s}}) = (\gamma_\lambda, t_\lambda)(\beta_s, k_s)$. Hence $i_{p_{\lambda s}} = t_\lambda + k_s$ and $\theta_{p_{\lambda s}} = \gamma_\lambda \circ \rho_{t_\lambda}\beta_s$. Thus,

$$(a; s, \lambda)\theta = (\beta_s, k_s)(\theta_a, i_a)(\gamma_\lambda, t_\lambda) = (\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda, k_s + i_a + t_\lambda).$$

Hence, by (**2**, p. 11, Lemma 1.2) and the proof of Theorem 1, if

$$(g, i, j) \in S, \quad (a; s, \lambda) * (g, i, j) = (g, i, j)\lambda_{(\rho_{-z}\delta, -z)} = (i(\rho_{-z}\delta)g, i - z, j)$$

where $\delta = \beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda$ and $z = k_s + i_a + t_\lambda$ and (8) follows. One obtains (9) in a similar manner. Reversing our steps and using (**2**, Theorem 4.22), we see that the last statement of the theorem is valid.

Remark 3. In the special case $T\backslash 0$ is a group R, V is either given by a partial homomorphism or (8) and (9) become

$$a * (g, i, j) = ((i\rho_{-i_a}\theta_a)g, i - i_a, j)$$

and

$$(g, i, j) * a = (g(j\theta_a), i, j + i_a)$$

respectively.

Remark 4. If T is a 0-bisimple semigroup without proper divisors of zero, an extension of $S = (G, \delta)$ by T is either given by a partial homomorphism or the equations in Remark 3 with $a \rightarrow \theta_a$ a mapping of $T\backslash 0$ into H.

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