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# Orthogonality relations and orthomodularity

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An abstract orthogonality relation is defined, a closure operation and a corresponding lattice of closed sets are associated with it. Necessary and sufficient conditions are obtained for the orthomodularity of a sub-ortholattice of the lattice of closed sets.

## 1. Introduction

A binary relation $\perp$ on a non-empty set $I$ is said to be an orthogonality relation when

(1) $x \perp y$ implies $y \perp x$ ,

(2) $x \perp x$ implies $x \perp y$ for all $y$ in $I$ .

Note that a particular case of (2) occurs when $\perp$ is anti-reflexive, that is $x \perp x$ for no $x$ in $I$ .

REMARK. In [3] MacLaren defines an orthogonality relation by requiring that

(3) ($z \perp x$ if and only if $z \perp y$) implies $x = y$ ,

in addition to (1) and (2) above. However, we make no use of (3) and so we omit it from the definition of an orthogonality relation.

An orthogonality relation is a special kind of polarity in the sense of Birkhoff [1], p. 122-3, and it follows from the results given there that $X \to X^{\perp\perp}$ is a closure operation on the subsets of $I$ , and that $X \to X^{\perp}$ is an orthocomplementation of the lattice of closed subsets of

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$I$ . Here, of course,

$$X^{\perp} = \{y : y \perp x , \quad \forall x \in X\} .$$

We write $I^{\perp} = 0$ . Note that $0$ is the empty set when $\perp$ is anti-reflexive; when $0$ is not the empty set one has

$$0 = \{x : x \in L \ \& \ x \perp y , \quad \forall y \in L\} .$$

A subset $S$ of $I$ is said to be an orthogonal set when $x \perp y$ for any $x$, $y$ in $S$ with $x \neq y$ . In particular the set $0$ and the one element subsets of $I$ are orthogonal sets. Let $A$ be a subset of $I$ and let $P$ be an orthogonal subset of $A$ ; a straight-forward argument establishes that the set of all orthogonal subsets of $A$ which contain $P$ has at least one element which is maximal with respect to set inclusion.

Let $\mathcal{L}$ denote the lattice of closed subsets of $I$ . By a sub-ortholattice of $\mathcal{L}$ we mean a sublattice $L$ of $\mathcal{L}$ which contains $I$ and has the property that $X^{\perp}$ is in $L$ whenever $X$ is in $L$ . A sub-ortholattice $L$ of $\mathcal{L}$ is clearly orthocomplemented, our interest here is the determination of necessary and sufficient conditions for its orthomodularity, that it has the property (cf. Birkhoff [1])

$$X \subseteq Y^{\perp} \ \& \ X \vee Y = I \Rightarrow X = Y^{\perp} \tag{1}$$

for any $X$, $Y$ in $L$ . Note that orthocomplementation of (1) gives

$$X \supseteq Y^{\perp} \ \& \ X \cap Y = 0 \Rightarrow X = Y^{\perp} , \tag{2}$$

we use this fact below.

## 2. Orthomodularity in a sub-ortholattice

We say that a sub-ortholattice $L$ of $\mathcal{L}$ has the $B$-property when, for each $X$ in $L$ and any maximal orthogonal subset $M$ of $X$ one has $M^{\perp\perp} = X$ . We note firstly,

LEMMA. *If* $X$ *is a closed subset and* $M$ *is a maximal orthogonal subset of* $X$ *then* $0 \subseteq M$ .

Proof. Since $X$ is closed it contains $0$ , if $P \subseteq X$ is orthogonal then $P \cup 0 \subseteq X$ is also orthogonal.

We are now able to prove our main result, namely

THEOREM. *A sub-ortholattice* $L$ *of* $\mathcal{L}$ *is orthomodular if and only if it has the B-property.*

Proof. Suppose that $L$ does have the $B$-property and assume the antecedent in the implication (1). We prove that $X = Y^{\perp}$. To do so let $M$ be a maximal orthogonal subset of $X$ and let $P$ be a maximal orthogonal subset of $Y$. Then

$$I = X \vee Y = \left(M^{\perp\perp} \cup P^{\perp\perp}\right) = (M \cup P)^{\perp\perp} .$$

It follows that $M \cup P$ is a maximal orthogonal subset of $I$ for

$$x \perp M \cup P \Rightarrow x \in (M \cup P)^{\perp} = 0$$

and $0 \subseteq M \cup P$ by the lemma.

Let $N \supseteq M$ be a maximal orthogonal subset of $Y^{\perp}$ which contains $M$, then $N \perp P$ and

$$(N \cup P)^{\perp\perp} = \left(Y^{\perp} \cup Y\right)^{\perp\perp} = I .$$

Since $M \cup P$ is maximal and $M \cup P \subseteq N \cup P$ we must have $M = N$, that is $X = M^{\perp\perp} = N^{\perp\perp} = Y$. This establishes that $L$ is orthomodular. Conversely assume that $L$ is orthomodular, we establish that it has the $B$-property. Let $X$ be in $L$ and let $M$ be a maximal orthogonal subset of $X$. Since $M$ is maximal we have $M^{\perp} \cap X = 0$, but $X \supseteq M^{\perp\perp}$ and so, using orthomodularity in the form (2), we have $X = M^{\perp\perp}$, that is $L$ has the $B$-property.

REMARK. Note that the theorem remains meaningful when $L = \mathcal{L}$. In fact a study of the proof shows that we do not need to assume that $L$ is a lattice, the theorem remains valid when $L$ is a sub-orthoposet of $\mathcal{L}$ in which orthogonal joins exist. A special case of the theorem, namely when $L = \mathcal{L}$ and $\mathcal{L}$ is the completion by cuts of an orthoposet was established in Finch [2].

## References

[1] Garrett Birkhoff, *Lattice theory* (Colloquium Publ. 25, Amer. Math. Soc., Providence, 3rd ed., 1967).

[2] P.D. Finch, "On orthomodular posets", *J. Austral. Math. Soc.* (to appear).

[3] M. Donald MacLaren, "Atomic orthocomplemented lattices", *Pacific J. Math.* 14 (1964), 597-612.

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