*Proceedings of the Edinburgh Mathematical Society* (1998) 41, 429–432 ©

# BOOK REVIEWS

HUXLEY, M. N. *Area, lattice points and exponential sums* (London Mathematical Society Monographs New Series No. 13, Clarendon Press, Oxford, 1996), xii+494 pp., 0 19 853466 3, (hardback) £85.

As its title hints, a theme in this book is the problem of finding upper bounds for the difference between the area of a given plane region and the number of integer points (points with both Cartesian coordinates integer-valued) lying in that region: call this difference the discrepancy. Typically one is approximating $y = f(m) = TF(m/M)$ by $[y] + 1/2$, for $M \le m < 2M$, where $T$ and $M$ are large parameters (usually $T^c \le M \le T$, for some positive constant $c$) and $F(x)$ is a fixed function that is well-behaved in the sense that a certain finite set of its derivatives, and of rational functions of them, all exist and are continuous for $1 \le x \le 2$. As the approximation error $\rho(y) = [y] - y + 1/2$ has the Fourier series

$$\rho(y) = \sum_{\substack{h=-\infty \\ h\neq 0}}^{\infty} \frac{\exp(2\pi i h y)}{2\pi i h} \qquad (y \notin \mathbb{Z}),$$

one can bound the total effect of approximation errors in terms of exponential sums of the shape

$$S(H, M) = \sum_{H \le h < 2H} \frac{1}{h} \sum_{M \le m < 2M} \exp(2\pi i h f(m))$$

(in practice the Fourier series is truncated at the optimal point, so that one always has $H$ to be significantly smaller than $M$). The past dozen years have seen significant advances in the estimation of such exponential sums through the powerful new method found by Bombieri and Iwaniec in 1985. The book under review, written by an expert in the field, is the first to address this new method in depth. It provides an up-to-date, systematic and virtually comprehensive description of the method and its applications.

The book is divided into six parts. The first, a sort of hors d'oeuvre, deals with elementary methods not involving exponential sums. It includes results on the discrepancy of a polygon and the construction of Jarník's polygon, as well as upper bounds for the number of points on, or within a prescribed distance of, a given curve (an iterative method and Swinnerton-Dyer's method): the remarkable work of Bombieri and Pila is briefly sketched in the first section of Part 6 of the book. The main course comes in Parts 2 to 5, which deal with the application of the Bombieri-Iwaniec method to various sorts of exponential sums including $S(H, M)$ and $S(1, M)$. Also covered is Jutila's method, coeval with that of Bombieri and Iwaniec, for bounding sums of the form

$$\sum_{M \le m < 2M} b(m) \exp(2\pi i f(m)),$$

where $b(m)$ denotes the $m$-th Fourier coefficient of a cusp form for the full modular group. The two methods were developed independently, but share important features: notably the rôle played by rational values of either $f'(x)$ or $f'(x)$, which echoes the relevance to the discrepancy for a plane region of tangents with rational gradient. The main results (on exponential sums, on the discrepancy of plane regions, on sums with modular form coefficients, on Riemann's zeta-function and on prime numbers) are collected together in Part 5. In Part 6 the author discusses related work (outside the scope of the Bombieri-Iwaniec method), as well as several open problems and ideas for future researchers: the above-mentioned rôle of rational values means that the full modular group comes into the picture, hence (as Bombieri and Iwaniec noticed) one avenue for progress may be through the application of deep results from the theory of non-holomorphic modular forms.

The great wealth of material in the book is appropriately distributed amongst the six parts and has been further subdivided into sections and subsections (sections do not exceed 30 pages). For ease of location, results and important displayed formulae are numbered by section, subsection and place in subsection (e.g. 'Lemma 4.2.1'). The style is not exactly leisurely, but there are frequent substantial passages explaining the ideas behind the bare mathematics.

Apart from some quoted results on Fourier coefficients of modular forms, the treatment of the main topics in this book is self-contained: proofs assuming only a little real and complex analysis (and some algebra known to Newton) are given for each result. References are given where necessary (over 150 in all). There are no exercises.

This book is going to be useful for researchers in the field of exponential sums and their applications. It could be used either for learning the Bombieri-Iwaniec method, or as a reference. Sections 8, 13, 14 and 18 (with some reference to Sections 1, 5 and 7) would be enough to prepare a postgraduate course on the use of the Bombieri-Iwaniec method to bound the discrepancy of a plane region, but to reach the 'state-of-the-art' one would have to include Sections 15 and 16, which are the deepest and most demanding in the book.

N. WATT

KANNAN, R. and KRUEGER, C. K., *Advanced analysis on the real line* (Universitext, Springer-Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo–Hong Kong, 1996) 260 pp., 0 387 94642 X, softcover DM68.

A century ago, according to Kline's *Mathematical thought from ancient to modern times* (Oxford, 1972), mathematicians of the eminence of Poincaré were objecting to the then current research into discontinuous functions, non-differentiable functions and similar topics. They would no doubt be horrified by the mere existence of this book – just as there are few of today's undergraduates who would be happy with the material. However, we should not fall into the trap of equating a popularity poll with a judgement of quality.

This book is exactly what it claims to be: it is an analysis of real functions of real variables – the 'awkward' topics such as absolute continuity, Dini derivatives, bounded variation, singular functions etc. The authors have drawn their material from diverse sources and have made a good job of synthesising it into a coherent whole. Although virtually all of the content is technical (and this applies to the statements of the results as well as the proofs), it is elegantly put together, relying heavily on the Vitali Covering Theorem.

Those who work in the area know (and perhaps love) the content. For most who need to use real analysis beyond dealing with our traditional 'sufficiently well-behaved' functions, this collection will save some time by giving efficient proofs of the odd results we need from time to time. For example, just how much do you *need* to know about $f'$ to be able to conclude that $f$ is increasing?

The book is well written and has a decent set of problems at the end of most chapters. Your reviewer would have welcomed more non-routine problems, but that is a personal view – the