ON THE ENDOMORPHISM SEMIGROUP OF AN ORDERED SET

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M. E. Adams and Matthew Gould [**1**] have obtained a remarkable classification of those ordered sets P for which the monoid End P of endomorphisms (i.e. isotone maps) is *regular*, in the sense that for every $f \in \operatorname{End} P$ there exists $g \in \operatorname{End} P$ such that $fgf = f$. They show that the class of such ordered sets consists precisely of

(a) all antichains;

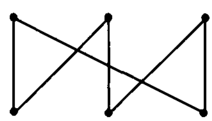
(b) all quasi-complete chains;

(c) all complete bipartite ordered sets (i.e. given non-zero cardinals α, β an ordered set $K_{\alpha,\beta}$ of height 1 having α minimal elements and β maximal elements, every minimal element being less than every maximal element);

(d) for a non-zero cardinal α the lattice M_α consisting of a smallest element 0, a biggest element 1, and α atoms;

(e) for non-zero cardinals α, β the ordered set $N_{\alpha,\beta}$ of height 1 having α minimal elements and β maximal elements in which there is a unique minimal element α_0 below all maximal elements and a unique maximal element β_0 above all minimal elements (and no further ordering);

(f) the six-element crown C_6 with Hasse diagram



A similar characterisation, which coincides with the above for sets of height at most 2 but differs for chains, was obtained by A. Ya. Aizenshtat [**2**].

Now for every ordered set P the monoid End P can be ordered by defining

$$f \leqslant g \Leftrightarrow (\forall x \in P) \quad f(x) \leqslant g(x).$$

This order is compatible with the multiplication (composition) in End P. Our purpose here is to determine precisely those ordered sets P for which the ordered semigroup End P is regular and *principally ordered* in the sense that for every $f \in \operatorname{End} P$ there exists

$$f^* = \max\{g \in \operatorname{End} P; fgf \leqslant f\}.$$

For the general properties of principally ordered regular semigroups we refer the reader to [**5**, **6**]. The main property that we shall require here is that in such a semigroup we have $ff^*f = f$, so that f^* is the biggest pre-inverse of f.

THEOREM 1. *The ordered semigroup* End P *is regular and principally ordered if and only if P is a dually well-ordered chain.*

Proof. $\Rightarrow$: Suppose that End P is regular and principally ordered. Then, by [**1**], P

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must be one of the six types mentioned above. Suppose, by way of obtaining a contradiction, that P is not a chain.

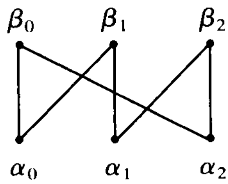
If P is an antichain or M_α or $K_{\alpha,\beta}$ then it is readily seen that, for $a, b \in P$ with $a \| b$, the mapping $f_{a,b}: P \to P$ defined by

$$f_{a,b}(x) = \begin{cases} b & \text{if } x = a; \\ x & \text{otherwise,} \end{cases}$$

belongs to End P and is idempotent. For every $g \in \text{End}\, P$ such that $f_{a,b} g f_{a,b} = f_{a,b}$ we have, applying each side to b, that $f_{a,b} g(b) = b$ from which it follows that necessarily $g(b) \in \{a, b\}$. Now, as a simple calculation reveals, $f_{b,a} \in \text{End}\, P$ is a pre-inverse of $f_{a,b}$, as is (trivially) $f_{a,b}$ itself. Consequently, $f_{b,a} \leqslant f^*_{a,b}$ and $f_{a,b} \leqslant f^*_{a,b}$. We therefore have $f^*_{a,b}(b) \in \{a, b\}$ with $f^*_{a,b}(b) \geqslant f_{b,a}(b) = a$ and $f^*_{a,b}(b) \geqslant f_{a,b}(b) = b$, which is impossible since $a \| b$. Thus P cannot be an antichain or M_α or $K_{\alpha,\beta}$.

If P is $N_{\alpha,\beta}$ then we can choose $\beta \geqslant 2$ (since $N_{\alpha,1} = K_{\alpha,1}$). In this case, let β_1 be a maximal element distinct from β_0 and consider the mapping $f_{\beta_1,\beta_0} \in \text{End}\, N_{\alpha,\beta}$. Here $f_{\beta_0,\beta_1} \notin \text{End}\, N_{\alpha,\beta}$ so consider $\text{id}_{N_{\alpha,\beta}}$ which is also a pre-inverse of f_{β_1,β_0}. From $f_{\beta_1,\beta_0} \leqslant f^*_{\beta_1,\beta_0}$ and $\text{id}_{N_{\alpha,\beta}} \leqslant f^*_{\beta_1,\beta_0}$ we obtain $f^*_{\beta_1,\beta_0}(\beta_1) \geqslant f_{\beta_1 \beta_0}(\beta_1) = \beta_0$ and $f^*_{\beta_1,\beta_0}(\beta_1) \geqslant \beta_1$, which is impossible since β_0, β_1 are maximal. Thus P cannot be $N_{\alpha,\beta}$.

Finally, suppose that P is the crown



Define $f: P \to P$ by

$$f(x) = \begin{cases} \beta_0 & \text{if } x = \beta_1; \\ \alpha_2 & \text{if } x = \alpha_1; \\ x & \text{otherwise.} \end{cases}$$

Then $f \in \text{End}\, P$ and is idempotent. A similar argument to the above gives $f^*(\beta_1) \geqslant f(\beta_1) = \beta_0$ and $f^*(\beta_1) \geqslant \text{id}_P(\beta_1) = \beta_1$, which is impossible. Thus P cannot be C_6.

It follows from these observations that P must be a chain. In this case, for every $y \in P$ define

$$\psi_y(x) = \begin{cases} y & \text{if } x \geqslant y; \\ x & \text{otherwise.} \end{cases}$$

Then $\psi_y \in \text{End}\, P$. Also, for $y, z \in P$ with $y \leqslant z$, define $\vartheta_{y,z}: P \to P$ by

$$\vartheta_{y,z}(x) = \begin{cases} z & \text{if } x \in [y, z]; \\ x & \text{otherwise.} \end{cases}$$

Then $\vartheta_{y,z} \in \text{End}\, P$.

Observe now that if $x \geqslant y$ then

$$\psi_y \vartheta_{y,z} \psi_y(x) = \psi_y \vartheta_{y,z}(y) = \psi_y(z) = y,$$

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whereas if $x<y$ then

$$\psi_y\vartheta_{y,z}\psi_y(x)=\psi_y\vartheta_{y,z}(x)=\psi_y(x)=x.$$

Consequently, $\psi_y\vartheta_{y,z}\psi_y=\psi_y$. It follows that, for all $z \geqslant y$, we have $\vartheta_{y,z}\leqslant\psi_y^*$ and therefore

$$z=\vartheta_{y,z}(y)\leqslant\psi_y^*(y).$$

Thus we see that P has a biggest element, namely $\psi_y^*(y)$ for every $y\in P$.

Now let C be an ascending chain in P. By way of obtaining a contradiction, suppose that C is infinite. Since P has a biggest element, the set $\bar{C}$ of upper bounds of C in P is not empty. For every $d\in C$ let

$$P_d=\{y\in P; y\geqslant d, y\notin\bar{C}\}.$$

Define $\varphi_d:P\to P$ by

$$\varphi_d(x)=\begin{cases} d & \text{if } x\in P_d;\\ x & \text{otherwise.}\end{cases}$$

Then $\varphi_d\in \operatorname{End} P$.

Suppose now that $z\in P_d$. Observe that if $x\in P_d$ then

$$\varphi_d\vartheta_{d,z}\varphi_d(x)=\varphi_d\vartheta_{d,z}(d)=\varphi_d(z)=d,$$

whereas if $x\notin P_d$ then

$$\varphi_d\vartheta_{d,z}\varphi_d(x)=\varphi_d\vartheta_{d,z}(x)=\varphi_d(x)=x.$$

Consequently, $\varphi_d\vartheta_{d,z}\varphi_d=\varphi_d$. It follows that, for every $z\in P_d$, we have $\vartheta_{d,z}\leqslant\varphi_d^*$ and therefore

$$z=\vartheta_{d,z}(d)\leqslant\varphi_d^*(d).$$

Thus $\varphi_d^*(d)$ is an upper bound of C and so $\varphi_d^*(d)\in\bar{C}$. On the other hand,

$$\varphi_d\varphi_d^*(d)=\varphi_d\varphi_d^*\varphi_d(P_d)=\varphi_d(P_d)=d,$$

which shows that we must have $\varphi_d^*(d)\in P_d$. From this contradiction we conclude that C cannot be infinite. Hence all ascending chains in P are finite, so every non-empty subset of P has a biggest element, so P is dually well-ordered.

$\Leftarrow$: Suppose, conversely, that P is a dually well-ordered chain. Since P is quasi-complete in the sense of **[1]** it follows that End P is regular. In what follows we shall use the notation

$$x^{\downarrow}=\{y\in P; y\leqslant x\},\qquad x^{\uparrow}=\{y\in P; y\geqslant x\}.$$

Consider $f\in\operatorname{End} P$. For every $x\in P$ such that $x^{\downarrow}\cap\operatorname{Im} f\neq\varnothing$ define

$$f^+(x)=\max\{y\in P; f(y)\leqslant x\}.$$

Let Θ_f be the equivalence relation given by $(x,y)\in\Theta_f$ if and only if $f(x)=f(y)$. Then for every $x\in P$ we have

$$f^+f(x)=\max\{y\in P; f(y)\leqslant f(x)\}=\max\{y\in P; f(y)=f(x)\}=\max[x]\Theta_f$$

and therefore

$$(\forall x \in P) \qquad ff^*f(x)=f(x).$$

As shown in [**1**, p. 197], for every $x \in P$ either $x^{\downarrow} \cap \operatorname{Im} f$ has a biggest element x_I or $x^{\uparrow} \cap \operatorname{Im} f$ has a smallest element x_F. Moreover, if for every $x \in \operatorname{Im} f$ we choose and fix an element $h(x) \in f^{-1}\{x\}$ then $h:\operatorname{Im} f \to P$ so defined is an isotone injection and the mapping $g:P\to P$ given by

$$g(x)=\begin{cases} h(x_I) & \text{if } x_I \text{ exists;} \\ h(x_F) & \text{otherwise,} \end{cases}$$

is an isotone pre-inverse of f. For our purpose here, we observe that the same is true of the mapping $g':P\to P$ given by

$$g'(x)=\begin{cases} h(x_F) & \text{if } x_F \text{ exists;} \\ h(x_I) & \text{otherwise.} \end{cases}$$

Now since P is dually well-ordered we can choose $h(x)=f^+(x)$ for every $x \in \operatorname{Im} f$ and thereby obtain from g' the mapping $f^*:P\to P$ given by

$$f^*(x)=\begin{cases} f^+(x_F) & \text{if } x_F \text{ exists;} \\ f^*(x_I) & \text{otherwise.} \end{cases}$$

Then $f^* \in \operatorname{End} P$ and is a pre-inverse of f. Our objective is to show that if $k \in \operatorname{End} P$ is such that $fkf \leqslant f$ then $k \leqslant f^*$.

For this purpose, observe that if $x_I = \max(x^{\downarrow} \cap \operatorname{Im} f)$ exists then $f(y) \leqslant x$ implies $f(y) \leqslant x_I$ so that we have

$$f^+(x_I)=f^+(x).$$

Also, for every $x \in P$, we have

$$fkf(x) \leqslant f(x) \Rightarrow kf(x) \leqslant f^+f(x).$$

There are two cases to consider.

(a) *x_F exists.*

In this case we have $x_F = f(z)$ for some $z \in P$ and so

$$k(x) \leqslant k(x_F) = kf(z) \leqslant f^+f(z) = f^+(x_F) = f^*(x).$$

(b) *x_F does not exist.*

Here there are two sub-cases:

(b_1) $x^{\uparrow} \cap \operatorname{Im} f = \varnothing$. In this case, denoting by 1 the biggest element of P, we have $f(1) \leqslant x$ in which case $f^+(x)=1$ and consequently

$$k(x) \leqslant 1 = f^+(x) = f^+(x_I) = f^*(x).$$

(b_2) $x^{\uparrow} \cap \operatorname{Im} f \neq \varnothing$. In this case $x \leqslant \inf\{f(y); f(y) \geqslant x\}$, the latter existing since P is join-complete and therefore the addition of a smallest element produces a complete

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chain. Observe that the existence of $x_I \in \operatorname{Im} f$ with $x_I \leqslant \inf\{f(y); f(y) \geqslant x\}$ implies the existence of $f^+(\inf\{f(y); f(y) \geqslant x\})$. Since

$$z \leqslant f^+(\inf\{f(y); f(y) \geqslant x\}) \Leftrightarrow f(z) \leqslant \inf\{f(y); f(y) \geqslant x\}$$
$$\Leftrightarrow z \leqslant \inf\{f^+f(y); f(y) \geqslant x\}$$

it follows that

$$f^+(\inf\{f(y); f(y) \geqslant x\}) = \inf\{f^+f(y); f(y) \geqslant x\}.$$

We thus have

$$\begin{aligned} k(x) &\leqslant k(\inf\{f(y); f(y) \geqslant x\}) \\ &\leqslant \inf\{kf(y); f(y) \geqslant x\} \\ &\leqslant \inf\{f^+f(y); f(y) \geqslant x\} \\ &= f^+(\inf f(y); f(y) \geqslant x\}). \end{aligned}$$

Now if $f(z) \leqslant \inf\{f(y); f(y) \geqslant x\}$ then we cannot have $f(z) > x$, for this would imply that x_F exists. Hence $f(z) \leqslant x$ and consequently

$$f^+(\inf\{f(y); f(y) \geqslant x\} = f^+(x) = f^+(x_I) = f^*(x).$$

It therefore follows that $k(x) \leqslant f^*(x)$.

Thus we see that in all cases $k(x) \leqslant f^*(x)$ and therefore $k \leqslant f^*$. Consequently, End P is principally ordered. □

In a principally ordered regular semigroup S every element has a biggest inverse [**5**], that of $x \in S$ being $x^0 = x^*xx^*$. Since $xx^0x = x$ we have, in general, $x^0 \leqslant x^*$. We say that S is *compact* if $x^0 = x^*$ for every $x \in S$. For example, if S is completely simple then S is compact; in fact, by [**7**, Theorem IV.2.4], if S is completely simple then $x^* \in V(x)$ so that $x^* \leqslant x^0$, whence $x^0 = x^*$.

THEOREM 2. *If P is a dually well-ordered chain then the principally ordered regular semigroup* End P *is compact.*

Proof. Since (whenever they exist) $x_I, x_F \in \operatorname{Im} f$, and since $ff^+f(x) = f(x)$ for every $x \in P$, it follows from the definition of f^* that

$$ff^*(x) = \begin{cases} x_F & \text{if } x_F \text{ exists;} \\ x_I & \text{otherwise.} \end{cases}$$

Consequently, we have

$$f^0(x) = f^*ff^*(x) = \begin{cases} f^+(x_F) & \text{if } x_F \text{ exists;} \\ f^+(x_I) & \text{otherwise,} \end{cases}$$

and therefore $f^0 = f^*$ for every $f \in \operatorname{End} P$. □

Suppose now that P is a dually well-ordered chain. Then since End P has a biggest element, namely the constant map $\pi : P \to P$ given by $\pi(x) = 1$ for every $x \in P$, the regular semigroup End P is trivially strong Dubreil–Jacotin [**3**]. We close by using End P to settle a question concerning perfect elements in such semigroups. Specifically, it is shown in [**4**] that if S is an ordered regular semigroup that is strong Dubreil–Jacotin then the subset

$P(S)$ of perfect elements (i.e. those $x \in S$ such that $x = x(\xi:x)x$ where ξ is the bimaximum element of S) is a regular subsemigroup which is orthodox under certain conditions. We show as follows, using End P where P is a dually well-ordered chain, that it is possible for $P(S)$ to be orthodox while S is not.

The perfect elements of End P are those isotone maps $f: P \to P$ such that $f = f\pi f$, i.e. those isotone maps f for which $f(x) = f(1)$ for every $x \in P$, i.e. the constant maps. It follows that the perfect elements of End P form a left zero semigroup which is therefore orthodox. But End P itself is not orthodox. To see this, note that since P is dually well-ordered every $p \in P$ has a predecessor, namely

$$p_1 = \max\{x \in P; x < p\}.$$

Define recursively $p_2 = (p_1)_1, \ldots, p_k = (p_{k-1})_1, \ldots$, and let $f_{p,k}: P \to P$ be given by

$$f_{p,k}(x) = \begin{cases} p & \text{if } x \geqslant p; \\ p_k & \text{otherwise.} \end{cases}$$

Then each $f_{p,k} \in$ End P and is idempotent.

Now choose $p, q \in P$ and $k \geqslant 2$ such that $q_k < p < q$. Then, for every $x \in P$,

$$\begin{aligned} f_{p,1} f_{q,k}(x) &= f_{p,1} \begin{cases} q & \text{if } x \geqslant q; \\ q_k & \text{otherwise,} \end{cases} \\ &= \begin{cases} p & \text{if } x \geqslant q; \\ p_1 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $f_{p,1} f_{q,k}(q) = p$ and $(f_{p,1} f_{q,k})^2(q) = f_{p,1} f_{q,k}(p) = p_1$, we see that $f_{p,1} f_{q,k}$ is not idempotent. Hence End P is not orthodox.

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