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REGULAR TOPOLOGICAL DISTRIBUTIVELY GENERATED NEAR-RINGS

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In this paper we introduce regular topological distributively generated (d.g.) near-rings (distinct from d.g. regular near-rings) as the d.g. near-ring analogue of regular rings and develop a structure theory for this class of near-rings.

1. Introduction.

In [6] we showed that division near-rings which are distributively generated fail to be the d.g. near-ring analogue of division rings and in [6] and [7] we introduced the concept of division topological d.g. near-rings (distinct from d.g. division near-rings) as the d.g. near-ring analogue of division rings.

The concept of a regular near-ring was introduced by Beidleman [1] and a structure theory for regular near-rings was developed by Ligh [4] and Heatherly [3]. In this paper we introduce regular topological d.g. near-rings (distinct from d.g. regular near-rings) as the d.g. near-ring analogue of regular rings and develop a structure theory for regular topological d.g. near-rings.

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2. Preliminaries.

Throughout this paper we will assume (i) that the term near-ring refers to a non-zero right near-ring with identity; (ii) the basic definitions and notation given in [5], [6] and [7]; and (iii) that R is a topological d.g. near-ring with identity 1, S is a distributive semigroup generating R^+ topologically, $0 \in S$, $T_o(R)$ is the set of distributive elements in R, $T(R) = T_o(R)\setminus\{0\}$ and $S^* = S \setminus \{0\}$.

Let R' be a topological d.g. near-ring generated by the distributive semigroup S'.

DEFINITION 2.1. ϕ is said to be a *continuous d.g. near-ring homomorphism* from (R,S) into (R',S') if ϕ is a continuous near-ring homomorphism from R into R' with $\phi(S) \subseteq S'$. If in addition we have $\phi(R) = R'$ and $\phi(S) = S'$ then ϕ is said to be a *continuous d.g. near-ring epimorphism* from (R,S) onto $(R'.S')$.

PROPOSITION 2.1. *The topological direct product* $\prod_{\alpha \in A} R_\alpha$ *of the topological d.g. near-rings* $R_\alpha (\alpha \in A)$ *forms a topological d.g. near-ring under component-wise addition and multiplication.*

The proof is straight forward and will be omitted.

DEFINITION 2.2. R is said to be a *topological subdirect product* of the topological d.g. near-rings $R_\alpha (\alpha \in A)$ if R is a sub-near-ring of $\prod_{\alpha \in A} R_\alpha$ such that $\pi_\alpha(R) = R_\alpha$ for all α in A where π_α is the natural projection from $\prod_{\alpha \in A} R_\alpha$ onto R_α.

DEFINITION 2.3. A topological d.g. near-ring R is said to be *subdirectly irreducible* if the intersection of its non-zero closed ideals is non-zero.

The proofs of Propositions 2.2 and 2.3 are similar to the proofs of the corresponding results in ring theory and will be omitted.

PROPOSITION 2.2. *If* R *is a topological d.g. near-ring and* $K_\alpha (\alpha \in A)$ *a family of closed ideals in* R *such that* $\bigcap_{\alpha \in A} K_\alpha = 0$ *then* R *is topologically isomorphic to a topological subdirect product of the topological d.g. near-rings* R/K_α $(\alpha \in A)$.

PROPOSITION 2.3. *A topological d.g. near-ring is topologically isomorphic to a topological subdirect product of subdirectly irreducible topological d.g. near-rings.*

DEFINITION 2.4. A topological d.g. near-ring R is said to be a division topological d.g. near-ring if

(i) R has no non-trivial closed right ideals;

(ii) S^* forms a multiplicative group for some distributive semigroup S generating R^+ topologically.

3. Regular topological d.g. near-rings.

DEFINITION 3.1. A topological d.g. near-ring R is said to be a *regular topological d.g. near-ring* if there exists a distributive semigroup S generating R^+ topologically and such that

(i) every closed right ideal of R is a d.g. right (R,S)-module;

(ii) for each $t \in S$ there exists $s \in S$ such that $tst = t$.

If we wish to specify the distributive semigroup S then we shall speak of the regular topological d.g. near-ring (R,S).

Clearly regular rings and division topological d.g. near-rings are regular topological d.g. near-rings. Also if R is a ring and $(R,T_o(R))$ is a regular topological d.g. near-ring then R is a regular ring. In Section 4 we prove that the endomorphism d.g. near-ring of relatively free groups of certain varieties are regular.

In the Example 2 of [6], $(\overline{R},S)$ is a division topological d.g. near-ring and hence a regular topological d.g. near-ring. However by choosing a monomorphism which is not an automorphism we can prove that $(\overline{R},T_o(\overline{R}))$ does not satisfy (ii) in the above definition and hence is not regular.

PROPOSITION 3.1. *Let R,R' be topological d.g. near-rings and $\phi:(R,S) \longrightarrow (R',S')$ a continuous d.g. near-ring epimorphism. If (R,S) is regular then (R',S') is regular.*

Proof. Let I' be a closed right ideal of R'. Then $\phi^{-1}(I')$ is a closed right ideal of R and as R is regular, $\phi^{-1}(I')$ is generated by $S \cap \phi^{-1}(I')$. Thus I' is generated by $\phi(S) \cap I' = S' \cap I'$ and I' is a d.g. right (R',S')-module. Now let $t' \in S'$. Then, as ϕ is an epimorphism, there exists $t \in S$ such that $\phi(t) = t'$. Also since (R,S) is regular, there exists $s \in S$ such that $tst = t$. Hence $t' = \phi(t) = \phi(tst) = t'\phi(s)\, t'$ with $\phi(s) \in S'$ and so (R',S') is regular.

COROLLARY. *If R,R' are topological d.g. near-rings with R regular and if $\phi : R \longrightarrow R'$ is a continuous near-ring epimorphism then R' is regular.*

PROPOSITION 3.2. *(a) A topological direct product of regular topological d.g. near-rings is regular.*

(b) A topological direct sum of regular topological d.g. near-rings is regular.

Proof. Let $R = \prod_{\alpha \in A} R_\alpha$ or $\sum_{\alpha \in A} \oplus R_\alpha$.. Define $S = \{x \in R : \pi_\alpha(x) \in S_\alpha$ for all $\alpha \in A\}$ Then it is straight forward to prove that (R,S) is regular in each case.

PROPOSITION 3.3. *(R,S) is a regular topological d.g. near-ring if and only if*

(i) every closed right ideal of R is a d.g. right (R,S)-module;

(ii) for every $t \in S$, the d.g. right $(R.S)$-module tR is generated by an idempotent e in S and $tR \cap S = tS$.

Proof. Let (R,S) be a regular topological d.g. near-ring and $I = tR$. Then since (R,S) is regular, condition (i) is satisfied and there exists $s \in S$ such that $tst = t$. Now $e = ts$ is an idempotent in S and as $e \in I$ and $t = tst = et \in eR$ we have $I = eR$. Clearly $tS \subseteq tR \cap S$. Let $s_o \in tR \cap S$. Then there exists $z \in R$ such that $tz = s_o$ and consequently $s_o = tz = tstz = tss_o \in tS$. Thus $tR \cap S = tS$.

Now let (R,S) be a topological d.g. near-ring satisfying (i) and (ii) and let $t \in S$. Then there exists an idempotent e in S such that $tR = eR$. Since $tR \cap S = tS$, there exists $s,s_1 \in S$ such that $ts = e$ and $es_1 = t$. Thus $tst = et = ees_1 = es_1 = t$ and so (R,S) is regular.

COROLLARY. *A ring R is regular (that is, $(R,T_o(R))$ regular) if and only if every principal right ideal of R is generated by an idempotent.*

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PROPOSITION 3.4. *If (R,S) is a regular topological d.g. near-ring, I a closed ideal of R and J a closed two sided I-module such that $J \cap S^*$ generates J then J is a closed two sided R-module.*

Proof. Clearly J is closed in R. Now let $t \in J \cap S^*$ and $x \in R$. Since R is regular there exists $s \in S^*$ such that $tst = t$. Hence $tx = tstx = tsy_1 = ty_2 \in J$ where $y_1, y_2 \in I$ and $xt = xtst = y_3st = y_4t \in J$ where $y_3, y_4 \in I$. Hence J is a two sided R-module.

PROPOSITION 3.5. *If R is a regular topological d.g. near-ring then R has no non-zero nilpotent closed right ideals.*

Proof. Suppose I is a non-zero nilpotent closed right ideal of R and $I^n = 0$. Since R is regular, we have $I \cap S^* \neq \Phi$ and let $t \in I \cap S^*$. Then there exists $s \in S^*$ such that $tst = t$. Now $e = ts$ is a non-zero idempotent in S^* and as I is a right ideal we have $e \in I$. Thus $e^n = e \in I$ and so $I^n \neq 0$. This is a contradiction and the result follows.

PROPOSITION 3.6. *Let (R,S) be a topologocal d.g. near-ring having every closed right ideal as a d.g. right (R,S)-module. If S has no non-zero nilpotent elements then $\text{Ann}_r t = \{x \in R: tx = 0\}$ is a closed ideal in R for all $t \in S$.*

Proof. Clearly $\text{Ann}_r t$ is a closed right ideal and hence a d.g. right (R,S)-module. Let $s \in S^* \cap \text{Ann}_r t$ and $s_1 \in S^*$. Then $ts = 0$ and $(st)^2 = stst = 0$. Since S has non-zero nilpotent elements we have $st = 0$. Consequently $(ts_1s)^2 = ts_1sts_1s = 0$ and $ts_1s = 0$. Thus $s_1s \in \text{Ann}_r t$ for all $s_1 \in S^*$ and all $s \in S^* \cap \text{Ann}_r t$. Now $\text{Ann}_r t$ is a d.g. right (R,S)-module and so we have $s_1x \in \text{Ann}_r t$ for all $x \in \text{Ann}_r t$ and $s_1 \in S^*$. Hence $\text{Ann}_r t$ is a left ideal and the result follows.

COROLLARY. *If (R,S) is a regular topological d.g. near-ring and S has no non-zero nilpotent elements then $\text{Ann}_r t$ is a closed ideal in R for all $t \in S$.*

PROPOSITION 3.7. *If (R,S) is a regular topological d.g. near-ring whose idempotents in S are central then S has no non-zero nilpotent elements.*

Proof. Suppose t is a non-zero nilpotent element in S and $n(> 1)$ is the least positive integer such that $t^n = 0$. Since R is regular there exists $s \in S$ such that $tst = t$. Now as ts is an idempotent in S we have ts central and $t^{n-1} = t^{n-2}.t = t^{n-2}.tst = t^{n-2}.tts = t^n s = 0$. This is a contradiction and the result follows.

PROPOSITION 3.8. *Let (R,S') be a topological d.g. near-ring having every closed right ideal as a d.g. right (R,S')-module and let S be a distributive semigroup in R such that $S \supseteq S'$. If S has no non-zero nilpotent elements then every idempotent in S is centrel.*

Proof. Clearly every closed right ideal is a d.g. right (R,S)-module. Let e be an idempotent in S. Suppose $t,s \in S$ and $ts = 0$. Then $(st)^2 = stst = 0$ and as S has no non-zero nilpotent elements we have $st = 0$. Now $e(xe - ese) = 0$ and so $xe - exe \in \text{Ann}_r e$ for all $x \in R$. But $\text{Ann}_r e$ is generated by $S^* \cap \text{Ann}_r e$ and since $e \in S$ and $et = 0$ for all $t \in S^* \cap \text{Ann}_r e$ we have $te = 0$ for all $t \in S^* \cap \text{Ann}_r e$ and consequently $(xe - exe)e = 0$. Thus $xe = exe$ for all $x \in R$. Now Re is a topological d.g. near-ring. Let $\phi : R \longrightarrow Re$ be defined by $\phi(x) = xe$. Then $\phi(x + y) = (x + y)e = xe + ye = \phi(x) + \phi(y)$ and $\phi(xy) = xye = x(ye) = x(eye) = xe.ye = \phi(x)\,\phi(y)$. Thus ϕ is a continuous near-ring epimorphism and ker ϕ being a closed ideal of R is generated by $S^* \cap \ker \phi$ and $te = 0$ for all $t \in S^* \cap \ker \phi$. Consequently since $e \in S$, we have $et = 0$ for all $t \in S^* \cap \ker \phi$. Now $(xe - ex)e = xe - exe = 0$ and so $xe - ex \in \ker \phi$ for all $x \in R$. Hence $exe - ex = e(xe - ex) = 0$ for all $x \in R$. Thus $ex = exe = xe$ and so e is central.

COROLLARY 1. *If (R,S) is a regular topological d.g. near-ring with no non-zero nilpotent elements in S then every idempotent in S is central.*

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By Proposition 3.1 and Corollary 1 we have

COROLLARY 2. *Let* (R,S) *be a regular topological d.g. near-ring. Then* S *has no non-zero nilpotent elements if and only if every idempotent in* S *is central.*

COROLLARY 3. *If* R *is a regular topological d.g. near-ring with no non-zero nilpotent distributive elements then every distributive idempotent in* R *is central.*

COROLLARY 4. *If* R *is a regular ring with no non-zero nilpotent elements then every idempotent of* R *is central.*

COROLLARY 5. *If* (R,S) *is a regular topological d.g. near-ring and* S *has no non-zero nilpotent elements then* $tR = Rt$ *for all* $t \in S$.

Proof. Let $t \in S$. Then there exists $s \in S$ such that $tst = t$. Then st and ts are central idempotents and so $ty = tsty = tyst \in Rt$ and $yt = ytst = tsyt \in tR$ for all $y \in R$. Then $tR = Rt$.

PROPOSITION 3.9. *Suppose* R, R' *are topological d.g. near-rings and* $\phi : (R,S) \longrightarrow (R',S')$ *is a continuous d.g. near-ring epimorphism. If* (R,S) *is regular and every idempotent of* S *is central then* (R',S') *is regular and every idempotent of* S' *is central.*

Proof. By Proposition 3.1, (R',S') is regular. Now let e' be an idempotent in S and $x' \in R'$. Then there exist $t \in S$ and $x \in R$ such that $\phi(t) = e'$ and $\phi(x) = x'$. Since R is regular, there exists $s \in S$ such that $tst = t$. Now $e = ts$ is an idempotent in S and so is central. Thus $t = tst = et = te$ and $e' = \phi(t) = \phi(te) = \phi(t^2s)$ $= (\phi(t))^2\phi(s) = \phi(t)\phi(s) = \phi(ts) = \phi(e)$. Hence $e'x' = \phi(e)\phi(x) = \phi(ex)$ $= \phi(xe) = \phi(x)\phi(e) = x'e'$ and so e' is central.

COROLLARY. *Let* $\phi : (R,S) \longrightarrow (R',S')$ *be a continuous d.g. near-ring epimorphism. If* (R,S) *is regular and* S *has no non-zero nilpotent elements then* (R',S') *is regular and* S' *has no non-zero nilpotent elements.*

THEOREM 1. *If* (R,S) *is a regular topological d.g. near-ring and* S^* *forms a multiplicative group then* (R,S) *is a division topological d.g. near-ring.*

Proof. Let I be a non-zero closed right ideal of R. Then I is a non-zero d.g. right (R,S)-module and so $I \cap S^* \neq \Phi$. Now any element of S^* is invertible and hence $I = R$. Thus R has no non-trivial closed right ideals and the result follows.

THEOREM 2. *If (R,S) is a regular topological d.g. near-ring such that $Ann_r e = 0$ for every non-zero idempotent e in S then (R,S) is a division topological d.g. near-ring.*

Proof. Let $t \in S^*$. Since R is regular we have $s \in S^*$ such that $tst = t$. Now $e = ts$ is a non-zero idempotent in S and as $e(e - 1) = e^2 - e = 0$ we have $e - 1 \in \mathrm{Ann}_r e$. Consequently $e = 1$ and $ts = 1$. Thus S^* forms a multiplicative group and the result follows by Theorem 1.

COROLLARY 1. *A regular topological d.g. near-ring with no non-trivial closed right ideals is a division topological d.g. near-ring.*

COROLLARY 2. *If R is a regular topological d.g. near-ring whose only distributive idempotents are 0 and 1 then R is a division topological d.g. near-ring.*

COROLLARY 3. *If R is a regular topological d.g. near-ring with no non-zero distributive left divisors of zero then R is a division topological d.g. near-ring.*

COROLLARY 4. *If (R,S) is a regular topological d.g. near-ring with no non-zero left divisors of zero in S then (R,S) is a division topological d.g. near-ring.*

THEOREM 3. *If (R,S) is a regular topological d.g. near-ring with the property that for each $t \in S^*$ there exists a unique $x \in R$ such that $txt = t$ then (R,S) is a division topological d.g. near-ring.*

Proof. Let $t \in S^*$ and suppose there exists $y \in R$ such that $ty = 0$. Since R is regular, we have $s \in S$ such that $tst = t$. Now $t(s + y)t = tst + tyt = tst + 0 = tst = t$ and so by the uniqueness condition we have $s + y = s$. Hence $y = 0$ and t is not a left divisor of zero. Thus S has no non-zero left divisors of zero and so by Corollary 4 of Theorem 2, R is a division topological d.g. near-ring.

THEOREM 4. *If* (R,S) *is a subdirectly irreducible regular topological d.g. near-ring such that for every idempotent* e *in* S *we have* $Ann_r e \subseteq Ann_\ell e$ *and* $Ann_r e$ *a closed ideal then* R *is a division topological d.g. near-ring.*

Proof. Suppose S has non-zero idempotents e such that $\mathrm{Ann}_r e \neq 0$. Let C be the set of all such idempotents and $I = \bigcap_{e \in C} \mathrm{Ann}_r e$. As (R,S) is regular and subdirectly irreducible we have $I \neq 0$ and $I \cap S^* \neq \Phi$. Take $t \in I \cap S^*$. Then $t \in \mathrm{Ann}_r e \subseteq \mathrm{Ann}_\ell e$ and so $te = 0$ for all $e \in C$. Also since (R,S) is regular there exists $s \in S^*$ such that $tst = t$. Now $e_1 = st$ is a non-zero idempotent in S and since $e_1 e = (st)e = s(te) = 0$ we have $e \in \mathrm{Ann}_r e_1$ and so $e_1 \in C$. Thus $t \in I \subseteq \mathrm{Ann}_r e_1 \subseteq \mathrm{Ann}_\ell e_1$ and $0 = te_1 = tst = t$. This is a contradiction and consequently $C = \Phi$ and by Theorem 2, R is a division topological d.g. near-ring.

COROLLARY 1. *If* R *is a subdirectly irreducible, regular topological d.g. near-ring such that for every distributive idempotent* e *we have* $Ann_r e \subseteq Ann_\ell e$ *and* $Ann_r e$ *a closed ideal then* R *is a division topological d.g. near-ring.*

COROLLARY 2. *If* (R,S) *is a subdirectly irreducible regular topological d.g. near-ring and if all idempotents in* S *are central then* (R,S) *is a division topological d.g. near-ring.*

By Proposition 3.6 we have

COROLLARY 3. *If* (R,S) *is a subdirectly irreducible regular topological d.g. near-ring with no non-zero nilpotent elements in* S *then* (R,S) *is a division topological d.g. near-ring.*

THEOREM 5. *Let* (R,S) *be a regular topological d.g. near-ring. Then* R *is topologically isomorphic to a topological subdirect product of division topological d.g. near-rings if and only if every idempotent in* S *is central.*

Proof. Suppose R is topologically isomorphic to a topological subdirect product of division topological d.g. near-rings $R_\alpha (\alpha \in A)$.

Let e be an idempotent in S . Then $\Pi_\alpha(e)$ is a distributive idempotent of R_α and so $\Pi_\alpha(e) = 0$ or 1 . Thus $\Pi_\alpha(e)$ is central in R_α for all $\alpha \in A$ and consequently e is central in R .

Conversely, suppose every idempotent in S is central. Now by Proposition 2.3 R is topologically isomorphic to a topological subdirect product of subdirectly irreducible topological d.g. near-rings $R_\alpha (\alpha \in A)$. Then by Proposition 3.9 and Corollary 2 of Theorem 4 we have the R_α as division topological d.g. near-rings for $\alpha \in A$ and the result follows.

COROLLARY 1. *Let (R,S) be a regular topological d.g. near-ring. Then R is topologically isomorphic to a topological subdirect product of division topological d.g. near-rings if and only if S has no non-zero nilpotent elements.*

COROLLARY 2. *Let (R,S) be a regular topological d.g. near-ring. Then*

(i) every idempotent in S is central if and only if every distributive idempotent in R is central;

(ii) S has no non-zero nilpotent elements if and only if R has no non-zero nilpotent distributive elements;

(iii) every distributive idempotent in R is central if and only if R has no non-zero nilpotent distributive elements.

4. Endomorphism near-ring of a relatively free group.

Let (R_1,S_1) be a division discrete d.g. near-ring, F a free group of the variety $v(R_1^+)$ of left (R_1,S_1)-groups generated by the left (R_1,S_1)-group R_1^+ , Λ a basis of F and R the endomorphism d.g. near-ring of F with the finite topology induced by Λ (see [5]).

Let $\Lambda_0 = S_1\Lambda$ and $S = \{x \in R: \Lambda x \subseteq \Lambda_0\}$. Then by Proposition 2.2 of [5], (R,S) is a topological d.g. near-ring.

PROPOSITION 4.1. *Every closed right ideal of R is a d.g. right (R,S)-module.*

Proof. Let I be a non-zero closed right ideal of R . Since

$e_\lambda R \cap \{x \in R: \lambda x = 0\} = 0$ we have $e_\lambda R$ to be a discrete d.g. right (R,S)-module for all $\lambda \in \Lambda$.

Now by the Corollary of Theorem 2 of [5] we have $R \cong \coprod_{\lambda \in \Lambda} e_\lambda R$ and by Proposition 5.4 of [5], R is a simple topological d.g. near-ring. Further, by Theorem 1 of [7], $e_\lambda R\, e_\lambda \cong R_1$ and so $e_\lambda R\, e_\lambda$ is a division discrete d.g. near-ring. Thus by Theorem 4 of [7], $e_\lambda R$ is an irreducible d.g. right(R,S)-module and consequently $I \cap e_\lambda R = 0$ or $e_\lambda R$ for all $\lambda \in \Lambda$. Hence $I \cong \coprod_{\lambda \in \Lambda_1} e_\lambda R$ where $\Lambda_1 = \{\lambda \in \Lambda : I \cap e_\lambda R = e_\lambda R\}$ and so I is a d.g. right (R,S)-module.

THEOREM 6. *(R,S) is a regular topological d.g. near-ring.*

Proof. Let $t \in S^*$ and $[\Lambda t] = \{\lambda \in \Lambda$: there exist $s_1 \in S_1^*$ and $\lambda_1 \in \Lambda$ such that $\lambda_1 t = s_1 \lambda\}$. For each $\lambda \in [\Lambda t]$ choose a λ_1 such that $\lambda_1 t = s_1 \lambda$ for some $s_1 \in S_1^*$ and $s \in S$ by $\lambda s = s_1^{-1}\lambda_1$ if $\lambda \in [\Lambda t]$ and $\lambda s = 0$ otherwise. Let $\lambda^1 \in \Lambda$. If $\lambda^1 t = 0$ then $\lambda^1 tst = 0 = \lambda^1 t$. Suppose $\lambda^1 t \neq 0$. Then $\lambda^1 t = s_1 \lambda$ where $s_1 \in S_1^*$ and $\lambda \in [\Lambda t]$. If now $\lambda s = s_2^{-1}\lambda_2$ then $\lambda_2 t = s_2 \lambda$ and so $\lambda^1 tst = s_1 \lambda\, st = s_1 s_2^{-1} \lambda_2 t = s_1 s_2^{-1} s_2 \lambda = s_1 \lambda = \lambda t$. Thus we have $tst = t$ and so by Proposition 4.1, (R,S) is a regular topological d.g. near-ring.

PROPOSITION 4.2. *If R_1 is not a ring and $S_1 = T_0(R_1)$ then $S = T_0(R)$.*

Proof. Suppose $t \in T(R)$. Then given $\lambda \in \Lambda$,

$$\lambda t = \sum_1^n x_i \lambda_i + \sum_1^m c_j \qquad\qquad (x_i \neq 0)$$

where the λ_i are distinct and the c_j are commutators of the form $[----[[y_4 y_4^1, [y_1 y_1^1, y_2 y_2^1]], y_3 y_3^1]----]$ and the representation chosen so that m is minimal. Now

$$t(e_{\lambda_i\lambda^1} + e_{\lambda_i\lambda^{11}}) = te_{\lambda_i\lambda^1} + te_{\lambda_i\lambda^{11}} \quad \text{for all } \lambda^1, \lambda^{11} \in \Lambda$$

and so $\lambda t(e_{\lambda_i\lambda^1} + e_{\lambda_i\lambda^{11}}) = \lambda te_{\lambda_i\lambda^1} + \lambda te_{\lambda_i\lambda^{11}}$ for all $\lambda^1, \lambda^{11} \in \Lambda$.

Thus $x_i(\lambda^1 + \lambda^{11}) = x_i\lambda^1 + x_i\lambda^{11}$ for all $\lambda^1, \lambda^{11} \in \Lambda$ as

$c_je_{\lambda_i\lambda^1} = 0 = c_je_{\lambda_i\lambda^{11}}$ for all c_j .

But F is a free group of the variety $v(R_1^+)$ and so $x_i(y_1 + z_1)$

$= x_iy_1 + x_iz_1$ for all $y_1, z_1 \in R_1$ and so $x_i \in T(R)$.

Thus $\lambda t = \sum_1^n s_i\lambda_i + \sum_1^n c_j$ where $s_i \in T(R_1)$. Now suppose at least two of the s_i , say s_1 , s_2 are non-zero. Then since $\lambda t(e_{\lambda_2\lambda^{11}} + e_{\lambda_1\lambda^1})$

$= \lambda te_{\lambda_2\lambda^{11}} + \lambda te_{\lambda_1\lambda^1}$ we have $s_1\lambda^1 + s_2\lambda^{11} + (\sum_1^m c_j)(e_{\lambda_2\lambda^{11}} + e_{\lambda_1\lambda^1})$

$= s_2\lambda^{11} + s_1\lambda^1$.

But $e_{\lambda_2\lambda^{11}} + e_{\lambda_1\lambda^1} = e_{\lambda_1\lambda^1} + e_{\lambda_2\lambda^{11}}$ for all $\lambda^1, \lambda^{11} \in \Lambda$.

Therefore $s_1\lambda^1 + s_2\lambda^{11} + (\sum_1^m c_j)(e_{\lambda_2\lambda^{11}} + e_{\lambda_1\lambda^1}) = s_1\lambda^1 + s_2\lambda^{11}$

and so $(\sum_1^m c_j)(e_{\lambda_2\lambda^{11}} + e_{\lambda_1\lambda^1}) = 0$.

Hence $- s_2\lambda^{11} - s_1\lambda^1 + s_2\lambda^{11} + s_1\lambda^1 = 0$ for all $\lambda^1, \lambda^{11} \in \Lambda$.

Now as F is a $v(R_1^+)$-free group we have $-s_2y_1 - s_1x_1 + s_2y_1 + s_1x_1 = 0$ for all $x_1, y_1 \in R_1$. Thus $s_1^{-1} s_2y_1$ and so y_1 belongs to the centre of R_1^+ for all $y_1 \in R$ and consequently R_1^+ is abelian. This is a contradiction since R_1 is not a ring. Thus at most one of the s_i is non-zero and so

$$\lambda t = s_1\lambda_1 + \sum_1^m c_j \ .$$

Suppose $m \geq 1$. We choose λ_1^1 occuring in c_1 such that $\lambda_1^1 \neq \lambda_1$ and let $\lambda_2^1 \ldots \lambda_2^1$ be the other elements of Λ occurring in the $c_j^{\prime}s$, $j = 1,\ldots m$. Let $x = e_{\lambda_1^1}$ and $y = e_{\lambda_2^1} + \ldots + e_{\lambda_2^1}$. Then $\lambda t(x + y) = \lambda tx + \lambda ty$ and so

$$s_1\lambda_1(x + y) + (\sum_1^m c_j)(x + y) = s_1\lambda_1 x + (\sum_1^m c_j)x + s_1\lambda_1 y + (\sum_1^m c_j)y$$

That is, $s_1\lambda_1 y + \sum_1^m c_j = s_1\lambda_1 y + (\sum_2^m c\ )y$ and so $\sum_1^m c_j = \sum_2^m (c_j y)$.

This is a contradiction of the minimality of m and so $m = 0$. Hence $\lambda t = s_1\lambda_1 \in \Lambda$ for all $\lambda \in \Lambda$ and $t \in S$. Consequently $S = T_0(R)$.

THEOREM 7. *If* $(R_1, T_0(R_1))$ *is a division discrete d.g. near-ring then* $(R, T_0(R))$ *is a regular topological d.g. near-ring.*

Proof. The result is known to be true when R_1 is a ring and follows from Theorem 6 and Proposition 4.2 when R_1 is not a ring.

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