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KEPLER–POINSOT-TYPE REALIZATIONS OF REGULAR MAPS OF KLEIN, FRICKE, GORDAN AND SHERK

BY
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ABSTRACT. The paper describes polyhedral realizations for Felix Klein's map $\{3, 7\}_8$ of genus 3, for Gordan's map $\{4, 5\}_6$ of genus 4, and for two maps of genus 5, the Klein-Fricke map of type $\{3, 8\}$ and Sherk's map of type $\{4, 6\}$. The polyhedra have self-intersections but high symmetry and thus are close analogues to the Kepler–Poinsot-polyhedra.

1. **Introduction**. In his famous work on elliptic functions Felix Klein constructed a regular map on a Riemann surface of genus 3 that plays the key role for the transformation of the equation of degree 7, just as the icosahedron does for the transformation of the quintic equation (cf. [12, 13, 4]). This map has 168 orientation preserving automorphisms which is the maximum number on a Riemann surface of genus 3 (cf. Hurwitz [11]). Another representation of this surface is given by Klein's quartic

$$x^3y + y^3z + z^3x = 0,$$

which is a plane algebraic curve (of order 4) in homogeneous complex variables (cf. [12], p. 446).

About the same time Gordan found a regular map on a Riemann surface of genus 4 with exactly 120 orientation preserving automorphisms which is the maximum number on a Riemann surface of genus 4 (cf. [8], compare also [7, 23]). Gordan's map was rediscovered later by Brahana and Coxeter (cf. Coxeter–Moser [4], p. 139).

Gordan also gave a representation of his map as a sextic ([8], p. 379) which, by a suitable transformation (A. Duma, private communication), can be reduced to the quintic

$$x^2y^3 + y^2z^3 + x^3yz - xz^4 = 0,$$

which is a plane algebraic curve in homogeneous complex variables.

These maps are particular instances of Coxeter's regular maps $\{p, q\}_r$ with groups PGL(2, 7) of order 336 and $C_2 \times S_5$ of order 240, respectively. The rotation group of the former is the simple group PSL(2, 7) (cf. [4]).

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Bearing in mind that the (reflexible) regular maps are the most natural topological generalizations of the Platonic solids, sharing with them the flag-transitivity of the automorphism group, it is of interest to look for polyhedral realizations of regular maps in Euclidean 3-space E^3. The recent discovery of polyhedral models for Klein's map $\{3, 7\}_8$, for 6 of Coxeter's 4-dimensional regular skew polyhedra and for Dyck's regular map $\{3, 8\}_6$ were first steps in this direction (cf. [16, 17, 18], see also McMullen–Schulz–Wills [15]). However, the realization of Dyck's $\{3, 8\}_6$ could only be obtained by allowing self-intersections as in the case of the Kepler–Poinsot-polyhedra (cf. Coxeter [3]). This reflects that the realizability of a regular map as a polyhedron without self-intersections seems to be a very rare property.

The purpose of this short note is to describe polyhedral realizations of Kepler–Poinsot-type for Klein's map $\{3, 7\}_8$, Gordan's map $\{4, 5\}_6$, and two maps of genus 5, the Klein–Fricke map of type $\{3, 8\}$ and Sherk's map of type $\{4, 6\}$ (cf. [14], Ch. 7, [20], p. 17, and [7], p. 54). By a polyhedron of Kepler–Poinsot-type (KP-type) we mean a finite family P of plane (Jordan-)polygons in E^3 that fit together like the faces of a map on a surface but has self-intersections, and whose union has the rotation group or the full symmetry group of some Platonic solid. In fact, in all known cases only convex polygonal faces occur. We require 'minimal self-intersections', that is: if F and G are vertices, edges or faces of P with $F \cap G \neq \emptyset$, then $F \cap G$ shall either be a vertex, an edge or a face of P, or be a point set of dimension less than $\dim(F)$ and $\dim(G)$.

Our model of Klein's $\{3, 7\}_8$ has the octahedral rotation group of order 24 as its symmetry group, analogous to the infinite non-polyhedral model which Klein described in [13], p. 466–469.

Comparing our new model of $\{3, 7\}_8$ with that in [16] (where the tetrahedral rotation group is the symmetry group) it seems that higher symmetry can only be obtained by allowing self-intersections. Both models of Klein's $\{3, 7\}_8$ are chiral in the sense that they occur in a right hand version and in a left hand version (cf. [3]).

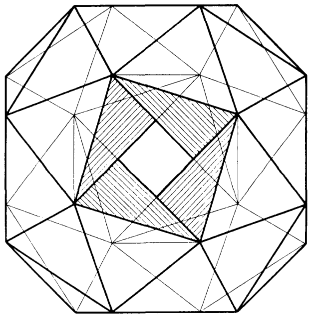
The model of the Klein–Fricke map of type $\{3, 8\}$ has the tetrahedral rotation group as its symmetry group. Identifying antipodal vertices turns this polyhedron into the model for Dyck's map $\{3, 8\}$ on a surface of genus 3 described in [18], thereby gaining full tetrahedral symmetry. This gives a geometric proof of the fact that the Klein–Fricke map is a 2-fold covering of Dyck's map.

In the other two cases the symmetry group is the full octahedral symmetry group, so these models have the maximal possible symmetry.

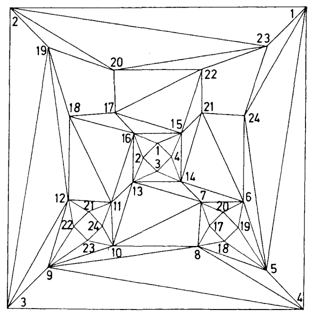
Finally we mention that for our realizations of Klein's and Gordan's regular maps the symmetry group acts transitively on the vertices, whereas our realization of Sherk's map is face transitive under its symmetry group.

For an introduction to the theory of regular maps the reader is referred to [4]; for the connections to Riemann surfaces compare also the survey article of Duma [5].

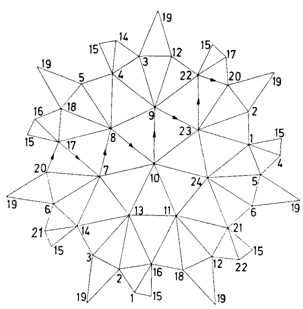
2. **Construction of the polyhedra**. The construction of the four polyhedra follows essentially the same line. In each case the polyhedron is derived from some Archimedean solid or a pair of Archimedean solids by removing certain faces of the solids



$\{3\ 7\}_8$ Fig. 1 $f = 4(6, 21, 14)$



$\{3, 7\}_8$ Fig. 2



$\{3, 7\}_8$ Fig. 3

and suitably joining these holes by tunnels. Then the polyhedron has self-intersections but within each tunnel any two faces do not intersect except for vertices and edges.

The proof of isomorphism with the map is then similar to those of [16, 18]. Starting from a topological diagram of the polyhedron the regularity is checked by proving the existence of certain automorphisms ρ and σ, where ρ cyclicly permutes the edges bounding one face and σ cyclicly permutes the edges surrounding one vertex of that face (cf. [4]). As a map on a surface of genus 3, 4 or 5 cannot be chiral (that is, irreflexible in the sense of [4]) the automorphism group must then be flag-transitive (cf. Garbe [7]), and thus the polyhedron is regular. Once the regularity is known the isomorphism with the map in question is easily obtained from the complete list of regular maps on a surface of genus 3, 4 or 5, respectively (cf. [19], [7]).

Figure 1 shows how a polyhedral model of Klein's map $\{3, 7\}_8$ can be derived from the snub cube by suitably joining any two antipodal square faces by a 'tunnel', which is combinatorially the mantle of a quadrangular antiprism. (In Figure 1 only one tunnel is drawn.) Considering polygons on the boundary of the snub cube that become part of a Petrie polygon (cf. [4]) for our polyhedron we see there is a unique way of constructing the polyhedron such that these Petrie polygons have length 8. This is also the length of the antiprismatic Petrie polygons given by the three tunnels. The symmetry group of the polyhedron is the octahedral rotation group (isomorphic to S_4) and is transitive on the vertices of the polyhedron. In particular, the automorphism ρ above can be realized by a Euclidean rotation in one of the equilateral triangles.

Starting from the Schlegel diagram of the snub cube (cf. Grünbaum [9]) in Figure 2 a diagram of the polyhedron is obtained by representing within three suitable quadrangular faces of the Schlegel diagram the eight faces of the respective tunnel. Now, labelling the vertices of our diagram by the numbers $1, \ldots, 24$ as in Figure 2, and rearranging the diagram in the way maps are usually drawn leads to the map of Figure 3. Here, the automorphism σ (defined with respect to the vertex 10) is given by

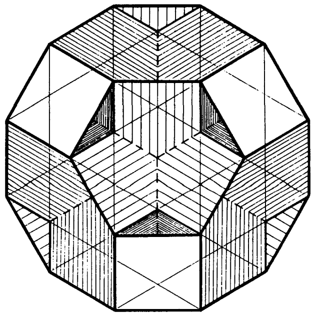
$$\begin{cases} \sigma = (7\ 13\ 11\ 24\ 23\ \ 9\ 8)(1\ 22\ 4\ 17\ 14\ 16\ 21) \\ \qquad (2\ 12\ \ 5\ 20\ \ 3\ 18\ 6) \end{cases}$$

Thus the polyhedron is regular and, being a regular polyhedron of type $\{3, 7\}$ and genus 3, necessarily isomorphic to Klein's $\{3, 7\}_8$. Another proof of isomorphism is obtained by directly comparing our Figure 3 with Figure 4 in [16].

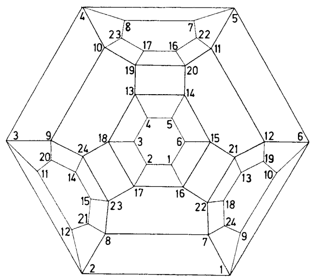
The polyhedral model for Gordan's map $\{4, 5\}_6$ is constructed in a similar fashion from the truncated octahedron (cf. Figure 4). Here, the hexagonal faces are removed, and any pair of antipodal hexagonal holes is then joined by a prismatic tunnel, giving four tunnels with six rectangular faces each. The symmetry group of the polyhedron is the octahedral group and is transitive on the vertices. Again the automorphism ρ is given by a Euclidean rotation in one of the square faces.

The existence of σ is checked by means of a diagram of the polyhedron derived from the Schlegel diagram of the truncated octahedron by representing within four hexagonal faces the six faces of the respective tunnel (see Figure 5). Labelling the vertices by $1, \ldots, 24$ and turning this diagram into the map of Figure 6 proves the existence of σ (defined with respect to the vertex 1), in particular

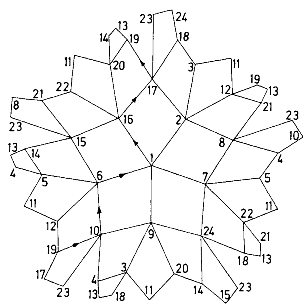
$$\begin{cases} \sigma = (2\ 16\ \ 6\ \ 9\ 7)(8\ 17\ 15\ 10\ 24) \\ \qquad (3\ 22\ 12\ 20\ 5)(4\ 18\ 21\ 19\ 14). \end{cases}$$



$\{4, 5\}_6$ Fig. 4 $f = 6(4, 10, 5)$



$\{4, 5\}_6$ Fig. 5



$\{4, 5\}_6$ Fig. 6

Being regular, of type $\{4, 5\}$, and genus 4, our polyhedron must be isomorphic to Gordan's $\{4, 5\}_6$ (cf. [7]).

A polyhedral model for Sherk's regular map of type $\{4, 6\}$ and genus 5 is obtained from a pair of homothetic cubes by removing all their faces and then joining each hole of the outer cube to the antipodal hole of the inner cube by a prismatic tunnel, giving six tunnels with trapezoidal faces each (see Figure 7 which shows an orthogonal

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projection along a common diagonal of the two cubes, which is a 3-fold rotation axis of the polyhedron). Again, the symmetry group is the octahedral group but this time is transitive on the faces. Here, neither ρ nor σ is realizable by a symmetry.

A diagram for the polyhedron can be derived from a pair of Schlegel diagrams of cubes (see Figure 8). Here, the second Schlegel diagram representing the inner cube is not drawn in the usual way. Its faces are 'separated', and vertices with the same label have to be identified. Now, turning this diagram into the map of Figure 9 shows that

$$\sigma = (4\ 5'\ 7\ 4'\ 5\ 7')(1'\ 3'\ 8')(1\ 3\ 8)(2\ 2')$$

has the required properties with respect to the vertex 6. Clearly, σ^2 is just the Euclidean 120°-rotation about the diagonal $\overline{6\ 2}$ of the outer cube. On the other hand, choosing the face F with vertices 6, 7, 1′ and 4′ the automorphism ρ defined with respect to F turns out to be

$$\rho = (6\ 7\ 1'\ 4')(2\ 3'\ 5'\ 8)(3\ 5\ 8'\ 2')(1\ 4\ 6'\ 7').$$

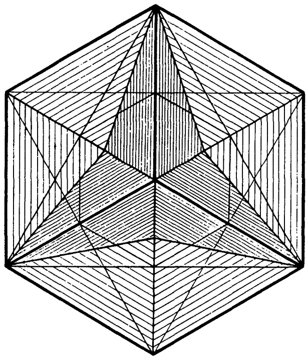
This can be checked by a new figure of the map, in which F becomes the central face. Finally, as there is only one regular map of type {4, 6} on a surface of genus 5 (cf. [7]), our polyhedron must be isomorphic to Sherk's map.

Our model of the Klein–Fricke map of type {3, 8} is constructed from a pair of homothetic icosahedra by removing all their faces and joining suitable hexagonal 'paths' by 'tunnels'. These hexagonal 'paths' are obtained as follows.

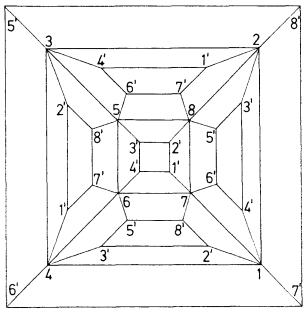
First observe that on the boundary of a 3-cube a hyperplane passing through the centroid and orthogonal to a diagonal cuts out (the boundary of) a regular hexagon. This way the four diagonals give four hexagonal paths covering all the vertices and edges of the cuboctahedron. Inserting suitable diagonals into the six square faces of the cuboctahedron gives a dissection of its boundary that is combinatorially isomorphic to the boundary complex of the icosahedron. Now turning this dissection into the complex of the regular icosahedron provides four hexagonal paths in the graph of the icosahedron. These four paths are linked by a net of eight triangular faces of the icosahedron corresponding to the eight triangular faces of the cuboctahedron (see Figure 10).

Assume now that a pair of homothetic icosahedra is given with the vertices labelled 1, . . . , 12 and 1′, . . . , 12′, respectively (where i' corresponds to i, for $i = 1, \ldots, 12$). Choosing on the two icosahedra the same sets of four hexagonal paths together with the respective nets of eight triangular faces, and joining corresponding paths by twelve triangles that fit together like the faces in the mantle of a hexagonal antiprism gives in fact a model of the map in question. However, the twelve triangles have to be chosen as shown in the diagram of Figure 11: first the hexagonal 'bases' have to be twisted and then joined antiprismatically. Altogether the polyhedron has 24 8-valent vertices and 64 ($= 2\cdot 8 + 4\cdot 12$) triangular faces.

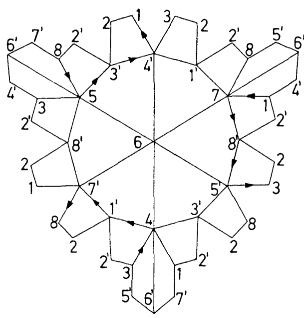
For the proof of isomorphism with the Klein–Fricke map we only have to prove regularity, since there is only one regular map of type {3, 8} and genus 5 (cf. [7]). Again the automorphism ρ is realizable by a Euclidean rotation in one of the eight faces on



{4, 6; 5} Fig. 7 $f = 4(4, 12, 6)$



{4, 6; 5} Fig. 8



{4, 6; 5} Fig. 9

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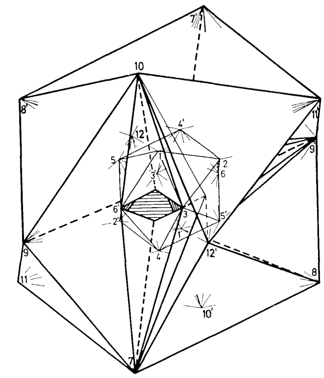
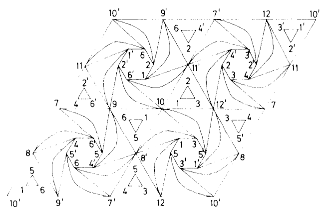
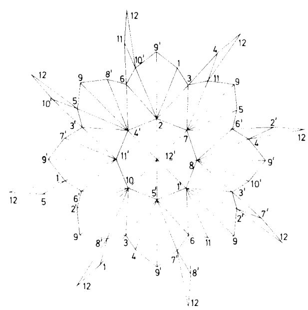


{3, 8; 5} Fig. 10



{3, 8; 5} Fig. 11



{3, 8; 5} Fig. 12

the outer icosahedron. To check the existence of σ we turn the diagram of Figure 11 into the map of Figure 12. Then

$$\begin{aligned}\sigma = {} & (2\ 4'\ 11'\ 10\ 5'\ 1'\ 8\ 7)(3\ 6\ 3'\ 6')\\ & (9\ 9')(1\ 8'\ 7'\ 2'\ 4\ 11\ 10'\ 5)\end{aligned}$$

has the required properties with respect to the vertex 12′. Note that the symmetry group of the polyhedron is the tetrahedral rotation group consisting of rotations in 'outer' triangles and half-turns about the midpoints of the six missing edges of the outer icosahedron.

One of the most interesting features of this construction is that it reveals the fact that the Klein–Fricke map is a 2-fold covering of Dyck's map $\{3, 8\}_6$. In fact, identifying antipodal vertices in our model gives exactly the model for $\{3, 8\}_6$ described in [18]). Note that then the Petrie-polygons of lengths 12 (within the 'tunnels') collapse to Petrie-polygons of lengths 6.

Concluding we remark that polyhedra with certain transitivity properties of the symmetry group have also been studied in [10] and [22].

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