Forum of Mathematics, Sigma (2023), **Vol. 11**:e11 1–16
doi:10.1017/fms.2023.9

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RESEARCH ARTICLE

On Duclos–Exner's conjecture about waveguides in strong uniform magnetic fields

Enguerrand Bon-Lavigne[1], Loïc Le Treust[2], Nicolas Raymond[3] and Julien Royer[4]

[1]Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France; E-mail: enguerrand.lavigne-bon@univ-amu.fr.
[2]Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France; E-mail: loic.le-treust@univ-amu.fr.
[3]Univ Angers, CNRS, LAREMA, Institut Universitaire de France, SFR MATHSTIC, F-49000 Angers, France;
E-mail: nicolas.raymond@univ-angers.fr.
[4]Institut de mathématiques de Toulouse, Université Toulouse 3, 118 route de Narbonne, F-31062 Toulouse cedex 9, France;
E-mail: julien.royer@math.univ-toulouse.fr.

Received: 2 June 2022; **Revised:** 4 January 2023; **Accepted:** 4 January 2023

2020 Mathematics Subject Classification: *Primary* – 35P15; *Secondary* – 81Q10

Abstract
We consider the Dirichlet Laplacian with uniform magnetic field on a curved strip in two dimensions. We give a sufficient condition on the width and the curvature of the strip ensuring the existence of the discrete spectrum in the strong magnetic field limit, answering (negatively) a conjecture made by Duclos and Exner.

Contents

1. Introduction and statement of the main results

In this article, we address the question of existence of the discrete spectrum for a magnetic Laplacian with Dirichlet boundary condition on a two-dimensional curved waveguide.

1.1. What is a waveguide?

Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be a smooth and injective curve with $|\boldsymbol{\gamma}'| = 1$. We set $\mathbf{N} = (\boldsymbol{\gamma}')^{\perp}$, where for $(a, b) \in \mathbb{R}^2$ we write $(a, b)^{\perp}$ for $(-b, a)$. We denote by κ the algebraic curvature of $\boldsymbol{\gamma}$. It is defined by

$$\boldsymbol{\gamma}'' = \kappa \mathbf{N}.$$

In this article, we work under the assumption that κ is compactly supported. The function

$$\Theta : \begin{cases} \mathbb{R} \times (-\delta, \delta) & \to \quad \mathbb{R}^2 \\ (s, t) & \mapsto \quad \boldsymbol{\gamma}(s) + t\mathbf{N}(s) \end{cases}$$

is smooth with bounded derivatives, and it is injective for $\delta > 0$ small enough. In this case, we set

$$\Omega = \Omega_{\gamma,\delta} = \Theta(\Omega_0), \qquad \text{with} \qquad \Omega_0 = \Omega_{0,\delta} = \mathbb{R} \times (-\delta, \delta).$$

The open set Ω is what we call a waveguide in this work.

1.2. The magnetic Laplacian with Dirichlet boundary conditions

The waveguide Ω is subject to a perpendicular uniform magnetic field with intensity B. That is why we consider a vector potential $\mathbf{A} = (A_1, A_2)$ that is smooth on $\overline{\Omega}$ and such that

$$\partial_{x_1} A_2 - \partial_{x_2} A_1 = 1. \tag{1.1}$$

A fundamental property related to magnetic problems on simply connected domains is the gauge invariance. It is nothing but the fact that equation (1.1) only defines $\mathbf{A}$ up to adding a gradient vector field. Of course, it is trivial that there is a smooth solution to equation (1.1) since it is sufficient to consider $\mathbf{A} = (0, x_1)$. Actually, one will see that there is a natural choice of vector potential in our setting. Finding a gauge that is adapted to the structure of the waveguide is in fact part of our problem, and it has been tackled in the past; see, for instance, [7] where a curvature-dependent gauge is introduced. We now assume that $\mathbf{A}$ can be chosen smooth on $\overline{\Omega}$ and bounded with bounded derivatives (at any order). It will be explained in Proposition 1.2 that we may indeed assume this.

For $B > 0$, we consider on Ω the magnetic Laplacian corresponding to the uniform field equal to B:

$$(-i\nabla - B\mathbf{A})^2 - B, \tag{1.2}$$

subject to Dirichlet boundary conditions. The subtraction of B is made for the convenience of the analysis and does not change the presence or absence of the discrete spectrum (it is based on relating the Schrödinger operator to the square of a Dirac operator). In order to use semiclassical analysis, we also introduce the positive parameter $h = B^{-1}$ and set

$$\mathscr{P}_h = (-ih\nabla - \mathbf{A})^2 - h.$$

The operator $\mathscr{P}_h$ is well defined and selfadjoint on the domain

$$\mathsf{Dom}(\mathscr{P}_h) = H_0^1(\Omega) \cap H^2(\Omega).$$

1.3. A subtle question and a conjecture by P. Duclos and P. Exner

Our aim is to study the existence of the discrete spectrum of $\mathscr{P}_h$ in the semiclassical limit $h \to 0$ (equivalent to the large magnetic field limit; see formula (1.2)). This question of existence is actually subtle since, when h goes to 0, not only the bottom of the spectrum moves but also the bottom of the essential spectrum. In this limit, it is natural to wonder if the bottom of the spectrum stays away from the

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threshold of the essential spectrum or collides with it. This question is all the more appealing that, when the magnetic field is zero, that is when considering the Dirichlet Laplacian on a strip, one knows that the discrete spectrum always exists as soon as the strip is not straight (see, for instance, [4] or the book [9, Chapter 1]). It is also known that (variable) magnetic fields can play against the existence of the discrete spectrum. Such considerations can be found in [14, Theorem 2.8 & Proposition 2.11] where a magnetic Hardy inequality is proved when the magnetic field has *compact support* and used to establish that the discrete spectrum is empty when the magnetic field is strong enough (see also the original work [5])[1].

In the mid nineties, buoyed by the momentum of their work [4], Pierre Duclos and Pavel Exner conjectured that *the discrete spectrum of operator (1.2) is empty when the magnetic field is strong enough (and uniform)*. This conjecture was explicitly formulated 10 years ago during an "Open Problems" session in Barcelona; see [8]. Our main result disproves the conjecture when the waveguide has a *fixed* width δ assumed to be small enough but *independent* of B.

1.4. Main result

Our main result is the following.

Theorem 1.1. *Assume that κ^2 has a unique maximum, that is nondegenerate. There exist $\delta_0 > 0$ and $h_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and all $h \in (0, h_0)$ we have*

$$\inf \operatorname{sp}(\mathscr{P}_h) < \inf \operatorname{sp}_{\mathsf{ess}}(\mathscr{P}_h).$$

In particular, $\mathscr{P}_h$ has nonempty discrete spectrum.

We can be more precise and provide some bounds for the bottoms of spectrum and essential spectrum. For this, we compare the spectral properties of the magnetic Laplacian on Ω to those on Ω_0. On Ω_0, we set $\mathbf{A}_0(s,t) = (-t, 0)$ and we consider in $L^2(\Omega_0)$ the operator $\mathscr{P}_{h,0} = (-ih\nabla - \mathbf{A}_0)^2 - h$, with Dirichlet boundary conditions. For $h > 0$, we set

$$\lambda_{\text{ess}}(h) = \inf \operatorname{sp}(\mathscr{P}_{h,0}).$$

The following proposition gives a rather naive lower bound of the infimum of the essential spectrum. It is likely not optimal (due to the presence of the h^2 factor), but it will be sufficient for our analysis.

Proposition 1.1. *We have*

$$\operatorname{sp}_{\text{ess}}(\mathscr{P}_h) = \operatorname{sp}_{\text{ess}}(\mathscr{P}_{h,0}) = \operatorname{sp}(\mathscr{P}_{h,0}) = [\lambda_{\text{ess}}(h), +\infty)$$

and

$$\lambda_{\text{ess}}(h) \geqslant \frac{(\pi h)^2}{4\delta^2} e^{-\delta^2/h}.$$

To prove an upper bound on the bottom of the spectrum, we first introduce on Ω_0 the function ϕ_0 defined by

$$\phi_0(s,t) = \frac{t^2 - \delta^2}{2}.$$

Then we define $\hat{\phi}_0 = \phi_0 \circ \Theta^{-1} \in \mathscr{C}^\infty(\overline{\Omega})$. In particular, $\hat{\phi}_0$ vanishes on $\partial\Omega$. In order to perform the analysis of the bottom of the spectrum, we will use a function ϕ, looking like $\hat{\phi}_0$ at infinity, defined thanks to the following proposition. We will use the following notation for the Schwartz space

$$\mathcal{S}(\overline{\Omega}) = \{\psi \in \mathscr{C}^\infty(\overline{\Omega}) : \forall(\alpha,\beta) \in \mathbb{N}^2, \quad \exists C_{\alpha,\beta} > 0 : \|x^\alpha \partial^\beta \psi\|_\infty \leqslant C_{\alpha,\beta}\}.$$

[1] Let us also mention that, in [14], the spectrum is also analyzed (by means of resolvent convergence) in the shrinking limit $\delta \to 0$ with a possibly δ-dependent magnetic field. Deriving effective operators in such regimes can actually be done in a quite general framework; see [10].

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Proposition 1.2. *There exists a unique $\phi \in \mathscr{C}^\infty(\overline{\Omega})$ such that $\Delta\phi = 1$, $\phi_{|\partial\Omega} = 0$, and $\phi - \hat{\phi}_0 \in \mathcal{S}(\overline{\Omega})$. Moreover, there exists $c_0 > 0$ such that $\partial_\nu \phi \geqslant c_0$ on $\partial\Omega$, ν being the outward pointing normal to the boundary.*

Then, by gauge invariance, we can choose $\mathbf{A} = (\nabla\phi)^\perp$ in the definition of $\mathscr{P}_h$. In particular, we may assume that $\mathbf{A}$ is bounded on Ω, as announced in Section 1.2. Here comes our result ensuring the existence of the discrete spectrum.

Theorem 1.2. *Assume that ϕ given by Proposition 1.2 has a unique minimum $\phi_{\min}$ (reached at $x_{\min} \in \Omega$) that is nondegenerate and smaller than $\min \phi_0 = -\delta^2/2$. Then, as $h \to 0$, we have*

$$\inf \mathrm{sp}(\mathscr{P}_h) \leqslant \frac{J}{\pi}\sqrt{\det \mathrm{Hess}_{x_{\min}}\phi}\, e^{2\phi_{\min}/h}\big(1 + o(1)\big),$$

with

$$J = 2 \inf_{f \in \mathscr{E}} \|(\partial_\nu\phi)^{\frac{1}{2}} f\|^2_{\partial\Omega},$$

and

$$\mathscr{E} = \{f \in \mathscr{O}(\Omega) \cap H^1(\Omega) : f(x_{\min}) = 1\},$$

where $\mathscr{O}(\Omega)$ is the set of holomorphic functions on Ω.

Remark 1.3.

1. The set $\mathscr{E}$ is not empty as we can see by considering a function of the form $f : z \mapsto c(z - z_1)^{-2}$ with $z_1 \notin \overline{\Omega}$ and c such that $f(x_{\min}) = 1$.
2. Due to a classical trace theorem and the fact that $\partial_\nu\phi$ is bounded, J is finite.
3. The fact that ϕ has a unique minimum (which is nondegenerate) can be ensured under explicit assumptions on the curvature κ and on the width of the waveguide; see Proposition 1.3 below.
4. By using Proposition 1.1 and under the assumption on ϕ in Theorem 1.2, we have $\inf \mathrm{sp}(\mathscr{P}_h) < \inf \mathrm{sp}_{\mathrm{ess}}(\mathscr{P}_h)$.

Our proof of Theorem 1.2 is based on extensions of strategies used in [1][2], where the asymptotic simplicity of the low-lying eigenvalues is established, under generic assumptions on Ω. Let us emphasize that, in [1], Ω is assumed to be *bounded* and that the assumption on ϕ can be ensured, in the uniform magnetic field case, when Ω is *strictly convex* (thanks to the works by Kawohl [12, 13]). In the present setting, Ω is neither bounded, nor convex. Moreover, in our unbounded setting, one needs to be very careful since the functional spaces (such as the Hardy spaces) involved in [1] are no more obviously well defined. The study of such spaces on strips[3] has an interest of its own, and their use to deduce precise spectral asymptotics will be the object of a future work. Fortunately, we do not need them to disprove Duclos–Exner's conjecture.

To complete our analysis, it remains to give a sufficient condition under which the assumption of Theorem 1.2 is satisfied.

Proposition 1.3. *Assume that $\kappa \in \mathscr{C}_0^\infty(\mathbb{R})$ and that κ^2 has a unique maximum, which is nondegenerate. There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, ϕ has a unique minimum $\phi_{\min}$ in Ω that is nondegenerate. Moreover, $\phi_{\min} < (\phi_0)_{\min}$.*

Theorem 1.1 follows from Proposition 1.1, Proposition 1.2, Theorem 1.2 and Proposition 1.3. Due to our motivation to disprove a conjecture from the nineties, we provide the reader with rather self-contained proofs (and sometimes recall basic arguments). In Section 2, we analyze the essential spectrum and we prove Proposition 1.1. In Section 3, the existence of the function ϕ is established and we prove Propositions 1.2 and 1.3. In Section 4, we prove Theorem 1.2.

[2]Motivated by the seminal works [6] and [11].
[3]Which started a long time ago; see, for instance, [15].

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2. The essential spectrum

In this section, we prove Proposition 1.1, which follows from Propositions 2.1 and 2.2. We first recall a classical result.

Lemma 2.1. *Let $\phi \in \mathscr{C}^\infty(\overline{\Omega})$ be bounded with bounded derivatives and $\mathbf{A} = (\nabla\phi)^\perp$. For all $\psi \in H^1_0(\Omega)$, we have*

$$\|(-ih\nabla - \mathbf{A})\psi\|^2_{L^2(\Omega)} - h\|\psi\|^2_{L^2(\Omega)} = 4h^2\int_\Omega e^{-2\phi/h}|\partial_{\bar z}u|^2\mathrm{d}x,$$

where $u := e^{\phi/h}\psi \in H^1_0(\Omega)$ and $z = x_1 + ix_2$, $x = (x_1, x_2)$.

Proof. We have

$$\begin{aligned}4h^2\int_\Omega e^{-2\phi/h}|\partial_{\bar z}u|^2\mathrm{d}x &= \int_\Omega |e^{-\phi/h}(h\partial_1 + ih\partial_2)u|^2\mathrm{d}x\\ &= \int_\Omega |(h\partial_1 + ih\partial_2)e^{-\phi/h}u - [h\partial_1 + ih\partial_2, e^{-\phi/h}]u|^2\mathrm{d}x\\ &= \int_\Omega |(h\partial_1 + i\partial_2\phi + ih\partial_2 + \partial_1\phi)\psi|^2\mathrm{d}x\\ &= \int_\Omega |(h\partial_1 - iA_1 + ih\partial_2 + A_2)\psi|^2\mathrm{d}x\\ &= \int_\Omega |(L_1 + iL_2)\psi|^2\mathrm{d}x\,, \quad L_j = -ih\partial_j - A_j.\end{aligned}$$

Then, we get

$$\begin{aligned}4h^2\int_\Omega e^{-2\phi/h}|\partial_{\bar z}u|^2\mathrm{d}x &= \|(-ih\nabla - \mathbf{A})\psi\|^2 + 2\mathrm{Re}\,\langle L_1\psi, iL_2\psi\rangle\\ &= \|(-ih\nabla - \mathbf{A})\psi\|^2 + 2\mathrm{Im}\,\langle L_1\psi, L_2\psi\rangle.\end{aligned}$$

Note that

$$\begin{aligned}2\mathrm{Im}\,\langle L_1\psi, L_2\psi\rangle &= 2\mathrm{Im}\,\langle \psi, L_1L_2\psi\rangle\\ &= 2\mathrm{Im}\,\langle \psi, L_2L_1\psi + [L_1, L_2]\psi\rangle\\ &= 2\mathrm{Im}\,\langle L_2\psi, L_1\psi\rangle - 2h\|\psi\|^2.\end{aligned}$$

The conclusion follows. □

Proposition 2.1. *For all $h > 0$, we have*

$$\mathrm{sp}(\mathscr{P}_{h,0}) = [\lambda_{\mathrm{ess}}(h), +\infty),$$

and

$$\lambda_{\mathrm{ess}}(h) \geqslant \frac{(\pi h)^2}{4\delta^2}e^{-\delta^2/h}.$$

Proof. By using the Fourier transform with respect to s, we have

$$\mathscr{P}_{h,0} = \int^\oplus \mathscr{P}_{h,0,\xi}\mathrm{d}\xi,$$

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where the operator

$$\mathscr{P}_{h,0,\xi} = -h^2\partial_t^2 + (\xi + t)^2 - h$$

is equipped with the Dirichlet conditions at $t = \pm\delta$. Let us denote by $(\gamma_n(\xi, h))_{n\geqslant 1}$ the increasing sequence of its eigenvalues. A straightforward application of the min-max theorem shows that, for all $h > 0$,

$$\lim_{\xi\to\pm\infty} \gamma_n(\xi, h) = +\infty.$$

We get

$$\mathrm{sp}(\mathscr{P}_{h,0}) = [\min_{\xi\in\mathbb{R}} \gamma_1(\xi, h), +\infty) = \mathrm{sp}_{\mathrm{ess}}(\mathscr{P}_{h,0}).$$

By the min-max principle, we have

$$\inf \mathrm{sp}(\mathscr{P}_{h,0}) = \inf_{\psi\in H_0^1(\Omega_0)\setminus\{0\}} \frac{\|(-ih\nabla - \mathbf{A}_0)\psi\|^2 - h\|\psi\|^2}{\|\psi\|^2},$$

and, by letting $\psi = e^{-\phi_0/h}u$, we get

$$\inf \mathrm{sp}(\mathscr{P}_{h,0}) = \inf_{u\in H_0^1(\Omega_0)\setminus\{0\}} \frac{4h^2\|e^{-\phi_0/h}\partial_{\bar z}u\|^2}{\|e^{-\phi_0/h}u\|^2}.$$

This allows to get the rough lower bound

$$\begin{aligned}\inf \mathrm{sp}(\mathscr{P}_{h,0}) &\geqslant e^{-\delta^2/h} \inf_{u\in H_0^1(\Omega_0)\setminus\{0\}} \frac{4h^2\|\partial_{\bar z}u\|^2}{\|u\|^2}\\ &\geqslant h^2 e^{-\delta^2/h}\lambda_1^{\mathrm{Dir}}((-\delta,\delta))\\ &\geqslant \frac{(\pi h)^2}{4\delta^2} e^{-\delta^2/h}.\end{aligned}$$

This last argument already appeared in [11, Theorem 3.1]. □

Let us recall the following classical result.

Lemma 2.2. *Consider* $(T_1, \mathrm{Dom}(T_1))$ *and* $(T_2, \mathrm{Dom}(T_2))$ *two closed operators on a Banach space* E *and having the same domain. Assume that there exists* $z_0 \in \rho(T_1) \cap \rho(T_2)$ *such that the operator* $K : (T_1 - z_0)^{-1} - (T_2 - z_0)^{-1} : E \longrightarrow E$ *is compact. Then,*

$$\mathrm{sp}_{\mathrm{ess}}(T_1) = \mathrm{sp}_{\mathrm{ess}}(T_2).$$

Proof. Let us recall the proof and note that it does not require the selfadjointness of T_1 or T_2. We recall that $\lambda \in \mathrm{sp}_{\mathrm{ess}}(T_1)$ if and only if $T_1 - \lambda$ is not a Fredholm operator with index 0.

Consider $\lambda \notin \mathrm{sp}_{\mathrm{ess}}(T_1)$, and write

$$\begin{aligned}T_2 - \lambda = T_2 - z_0 + (z_0 - \lambda) &= \left(\mathrm{Id} + (z_0 - \lambda)(T_2 - z_0)^{-1}\right)(T_2 - z_0)\\ &= \left(\mathrm{Id} + (\lambda - z_0)K + (z_0 - \lambda)(T_1 - z_0)^{-1}\right)(T_2 - z_0)\\ &= \left((\lambda - z_0)K + (T_1 - \lambda)(T_1 - z_0)^{-1}\right)(T_2 - z_0).\end{aligned}$$

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Now, notice that $T_2 - z_0 : \mathrm{Dom}(T_2) \to E$ is Fredholm with index 0 (since it is bijective). The operator $(T_1 - z_0)^{-1} : E \to \mathrm{Dom}(T_1)$ is also bijective and thus Fredholm with index 0. Therefore, $(T_1 - \lambda)(T_1 - z_0)^{-1} : E \to E$ is also Fredholm with index 0 (see [3, Corollary 5.7]). Since K is compact,

$$(\lambda - z_0)K + (T_1 - \lambda)(T_1 - z_0)^{-1}$$

is still Fredholm with index 0 (see [3, Corollary 5.9]). Thus, $T_2 - \lambda$ is Fredholm with index 0 (again by [3, Corollary 5.7]). □

Thanks to Lemma 2.2, it is rather easy to get the following.

Proposition 2.2. *For all $h > 0$, we have* $\mathrm{sp}_{\mathrm{ess}}(\mathscr{P}_h) = \mathrm{sp}_{\mathrm{ess}}(\mathscr{P}_{h,0})$.

Proof. The operator $\mathscr{P}_h$ is unitarily equivalent to the selfadjoint operator $\widetilde{\mathscr{P}}_h$ (on $L^2(\Omega_0, \mathrm{d}s\mathrm{d}t)$ with domain $\mathsf{Dom}(\widetilde{\mathscr{P}}_h) = H^2(\Omega) \cap H_0^1(\Omega) = \mathsf{Dom}(\mathscr{P}_{h,0})$) given by

$$\widetilde{\mathscr{P}}_h = -\partial_t^2 + (a^{-\frac{1}{2}}(D_s - \tilde{A}(s,t))a^{-\frac{1}{2}})^2 - \frac{\kappa^2}{4a^2} - h\,, \quad a(s,t) = 1 - t\kappa(s),$$

where $\tilde{A}(s,t) = t - \kappa(s)\frac{t^2}{2}$. The unitary transformation is made of a changing of coordinates via Θ, a flattening of the metric induced by the Jacobian of Θ and a change of magnetic gauge. Since κ is compactly supported, we see that $\widetilde{\mathscr{P}}_h$ acts as $\mathscr{P}_{h,0}$ away from a compact set.

Let us now apply Lemma 2.2 with $T_1 = \mathscr{P}_{h,0}$, $T_2 = \widetilde{\mathscr{P}}_h$ and $z_0 = i$. The resolvent formula gives

$$K = (T_1 - z_0)^{-1}(T_2 - T_1)(T_2 - z_0)^{-1}.$$

In our case, we have

$$T_2 - T_1 = a^{-\frac{1}{2}}\big[(D_s - \tilde{A})a^{-1}(D_s - \tilde{A})\big]a^{-\frac{1}{2}} - (D_s - t)^2 - \frac{\kappa^2}{4a^2}.$$

Computing some commutators shows that we can find three smooth functions W_1, W_2 and W_3 on $\overline{\Omega_0}$, compactly supported with respect to s, such that

$$T_2 - T_1 = W_1(s,t)D_s^2 + W_2(s,t)D_s + W_3(s,t).$$

Then, by elliptic regularity and the Kolmogorov–Riesz theorem (see [3, Theorem 4.14 & Remark 4.15]), we notice that $W(\widetilde{\mathscr{P}}_h - i)^{-1} : L^2(\Omega_0) \to H^1(\Omega_0)$ is compact for all $W \in \mathscr{C}_0^\infty(\overline{\Omega_0})$. This shows that the terms involving W_2 and W_3 in K are compact operators on $L^2(\Omega_0)$ (by using that the set of compact operators forms an ideal). Concerning the term involving W_1, we notice, on the one hand, that $D_s^2(\widetilde{\mathscr{P}}_h - i)^{-1}$ is bounded on $L^2(\Omega_0)$ and, on the other hand, that $(\mathscr{P}_{h,0} - i)^{-1}W_1 : L^2(\Omega_0) \to L^2(\Omega_0)$ is compact since the operators

$$[(\mathscr{P}_{h,0} - i)^{-1}, W_1] = -(\mathscr{P}_{h,0} - i)^{-1}[\mathscr{P}_{h,0}, W_1](\mathscr{P}_{h,0} - i)^{-1}$$

and $W_1(\mathscr{P}_{h,0} - i)^{-1} : L^2(\Omega_0) \to L^2(\Omega_0)$ are compact.

Applying Lemma 2.2, the conclusion follows. □

3. On the function ϕ

In this section, we prove Propositions 1.2 and 1.3. We recall that ϕ_0 and $\hat{\phi}_0$ were defined before Proposition 1.2.

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3.1. Proof of Proposition 1.2

Assume that two functions ϕ_1 and ϕ_2 satisfy the conclusions of the proposition. Then $\phi_1 - \phi_2$ is harmonic in Ω and belongs to $H_0^1(\Omega)$. This implies that $\phi_1 = \phi_2$ and gives uniqueness.

Since the tube Ω is straight at infinity, we have $\Delta\hat{\phi}_0 = 1$ outside a compact set. In particular, $1 - \Delta\hat{\phi}_0 \in L^2(\Omega)$. By the Poincaré inequality (see, for instance, [4] for the case of a waveguide) and the Riesz representation theorem, there exists a unique $f_0 \in H_0^1(\Omega)$ such that

$$\forall\varphi \in H_0^1(\Omega), \quad \int_\Omega \nabla f_0 \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega (1 - \Delta\hat{\phi}_0)\varphi \,\mathrm{d}x.$$

Then $-\Delta f_0 = 1 - \Delta\hat{\phi}_0$ in the sense of distributions, and f_0 belongs to $\mathscr{C}^\infty(\overline{\Omega})$ by elliptic regularity (notice that elliptic regularity on the straight waveguide Ω_0 is proved as in the classical case of the half-space and that we can then deduce elliptic regularity on Ω via the diffeomorphism Θ with the same proof as for bounded domains; see, for instance, [2, §9.6]).

Let $V = 1 - \Delta\hat{\phi}_0$, and consider a bounded Lipschitzian function Φ on Ω. We have

$$\langle -\Delta f_0, e^{2\Phi} f_0\rangle = \int_\Omega V e^{2\Phi} f_0 \mathrm{d}x.$$

Taking the real part and integrating by parts in the left-hand side, we get the "Agmon formula"

$$\|\nabla(e^\Phi f_0)\|^2_{L^2(\Omega)} - \|f_0 e^\Phi \nabla\Phi\|^2_{L^2(\Omega)} = \mathrm{Re} \int_\Omega V e^{2\Phi} f_0 \mathrm{d}x.$$

Since V has compact support, it follows that

$$\|\nabla(e^\Phi f_0)\|^2_{L^2(\Omega)} - \|f_0 e^\Phi \nabla\Phi\|^2_{L^2(\Omega)} \leqslant \|V\|_{L^2(\Omega)} \|f_0\|_{L^2(\Omega)} \max_{\mathrm{suppV}} e^{2\Phi}.$$

By the Poincaré inequality, we have

$$\|\nabla(e^\Phi f_0)\|^2 \geqslant \lambda_1(\Omega)\|e^\Phi f_0\|^2,$$

where $\lambda_1(\Omega) > 0$ is the infimum of the spectrum of the Dirichlet Laplacian on Ω. This shows that

$$\left(\lambda_1(\Omega) - \|\nabla\Phi\|^2_\infty\right) \int_\Omega e^{2\Phi}|f_0|^2 \mathrm{d}x \leqslant \|V\|_{L^2(\Omega)} \|f_0\|_{L^2(\Omega)} \max_{\mathrm{suppV}} e^{2\Phi}.$$

Choosing $\Phi(x) = \Phi_m(x) = \alpha \min(\langle x\rangle, m)$ (with $\alpha > 0$ fixed small enough) and letting $m \to +\infty$, we see by the Fatou lemma that there exists $C > 0$ such that

$$\int_\Omega e^{2\alpha\langle x\rangle}|f_0|^2 \mathrm{d}x \leqslant C\|f_0\|_{L^2(\Omega)}.$$

Coming back to the Agmon formula, we deduce that

$$\int_\Omega e^{2\alpha\langle x\rangle}|\nabla f_0|^2 \mathrm{d}x < +\infty.$$

Then we adapt the classical strategy used for elliptic regularity. Let $R > 0$ be so large that κ is supported in $(-R, R)$. Let $\chi_0 \in C^\infty(\Omega_0; [0, 1])$ equal to 0 on $(-R, R) \times (-\delta, \delta)$ and equal to 1 on $\Omega \setminus (-R - 1, R + 1) \times (-\delta, \delta)$. Let $\chi = \chi_0 \circ \Theta^{-1}$. Then the above arguments apply with f_0 replaced

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by $\partial_s(\chi f_0) = \boldsymbol{\gamma}' \cdot \nabla(\chi f_0)$. Since $\Delta(\chi f_0)$ is compactly supported, we deduce that for any $\beta \in \mathbb{N}^2$ with $|\beta| = 2$ we have

$$\int_\Omega e^{2\alpha\langle x\rangle} |\partial^\beta f_0|^2 \mathrm{d}x < +\infty.$$

We proceed by induction on $|\beta|$ to get the same estimate for any $\beta \in \mathbb{N}^2$, and we deduce in particular that f_0 belongs to the Schwartz class $\mathscr{S}(\overline{\Omega})$.

We set $\phi = \hat{\phi}_0 - f_0$. It is smooth, it satisfies the Dirichlet condition, $\phi - \hat{\phi}_0$ belongs to $\mathscr{S}(\overline{\Omega})$ and $\Delta\phi = 1$. It remains to discuss the uniform positivity of the normal derivative. By the Hopf lemma, we already know that $\partial_\nu \phi > 0$ on $\partial\Omega$, so it is enough to show that this estimate is uniform at infinity.

We have

$$\partial_\nu \phi = \partial_\nu \hat{\phi}_0 - \partial_\nu f_0.$$

Since Θ is a rotation at infinity, we see by the explicit expression of ϕ_0 that there exists $c_1 > 0$ such that, for all $x \in \partial\Omega$ with a sufficiently large curvilinear abscissa,

$$\partial_\nu \hat{\phi}_0 \geqslant 2c_1.$$

On the other hand, since $f_0 \in \mathscr{S}(\overline{\Omega})$ we have

$$\lim_{\substack{|x|\to+\infty \\ x\in\partial\Omega}} \partial_\nu f_0(x) = 0.$$

Then $\partial_\nu \phi(x) \geqslant c_1$ for $x \in \partial\Omega$ large enough, and we deduce the uniform positivity of $\partial_\nu \phi$ on $\partial\Omega$.

3.2. Proof of Proposition 1.3

For $(s,t) \in \Omega_0$, we set

$$a(s,t) = \det\big(\mathrm{Jac}(\Theta)(s,t)\big) = 1 - t\kappa(s)$$

and notice that $a(s,t) \geqslant \frac{1}{2}$ as soon as δ is small enough.

Let $\tilde{\phi} = \phi \circ \Theta$. For $s \in \mathbb{R}$ and $\tau \in (-1,1)$, we set

$$a_\delta(s,\tau) = a(s,\delta\tau) \quad \text{and} \quad \psi(s,\tau) = \delta^{-2} a_\delta(s,\tau)^{\frac{1}{2}} \tilde{\phi}(s,\delta\tau).$$

Finally, we define on $\mathbb{R} \times (-1,1)$ the differential operator

$$\mathscr{M}_\delta = \partial_\tau^2 + \delta^2\big(a_\delta^{-\frac{1}{2}} \partial_s a_\delta^{-\frac{1}{2}}\big)^2 + \frac{\delta^2\kappa^2}{4a_\delta^2},$$

where $a_\delta^{-\frac{1}{2}} \partial_s a_\delta^{-\frac{1}{2}} : u \mapsto a_\delta^{-\frac{1}{2}} \partial_s (a_\delta^{-\frac{1}{2}} u)$.

Lemma 3.1. *We have $\mathscr{M}_\delta \psi = a_\delta^{\frac{1}{2}}$ and $\psi(\cdot, \pm 1) = 0$.*

Proof. Since $\tilde{\phi}(s, \pm\delta) = 0$ we have $\psi(\cdot, \pm 1) = 0$ for all $s \in \mathbb{R}$. In the tubular coordinates, the equality $\Delta\phi = 1$ reads

$$(a^{-1}\partial_s a^{-1} \partial_s + a^{-1}\partial_t a \partial_t)\tilde{\phi} = 1.$$

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Setting $\check{\phi} = a^{\frac{1}{2}}\tilde{\phi}$, we get

$$\left[\left(a^{-\frac{1}{2}}\partial_s a^{-\frac{1}{2}}\right)^2 + \left(a^{-\frac{1}{2}}\partial_t a^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}}\partial_t a^{-\frac{1}{2}}\right)\right]\check{\phi} = a^{\frac{1}{2}},$$

or

$$\left[\left(a^{-\frac{1}{2}}\partial_s a^{-\frac{1}{2}}\right)^2 + \left(\partial_t - \frac{\kappa}{2a}\right)\left(\partial_t + \frac{\kappa}{2a}\right)\right]\check{\phi} = a^{\frac{1}{2}},$$

which gives

$$\left[\left(a^{-\frac{1}{2}}\partial_s a^{-\frac{1}{2}}\right)^2 + \partial_t^2 + \frac{\kappa^2}{4a^2}\right]\check{\phi} = a^{\frac{1}{2}}.$$

Since $\psi(s,\tau) = \delta^{-2}\check{\phi}(s,\delta\tau)$, the conclusion follows. □

Proof of Proposition 1.3. We look for an approximation Ψ_5 of ψ, in the sense that

$$\mathscr{M}_\delta(\psi - \Psi_5) = \mathscr{O}_{H^2(\mathbb{R}\times(-1,1))}(\delta^5)\,, \quad \psi - \Psi_5 \in H^2 \cap H_0^1(\mathbb{R}\times[-1,1]). \tag{3.1}$$

By elliptic regularity, this will give

$$\|\psi - \Psi_5\|_{H^4(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3),$$

and then, by Sobolev embeddings,

$$\|\psi - \Psi_5\|_{\mathscr{C}^2(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3). \tag{3.2}$$

We look for Ψ_5 of the form $\psi_0 + \delta\psi_1 + \delta^2\psi_2 + \delta^3\psi_3 + \delta^4\psi_4$. Note that we could proceed similarly to get a rest of order $\mathscr{O}(\delta^N)$ in $\mathscr{C}^k(\mathbb{R}\times[-1,1])$ for any N and k.

There exist $M_0,\ldots,M_4 \in \mathcal{L}(H^4(\mathbb{R}\times[-1,1]), H^2(\mathbb{R}\times[-1,1]))$ (we denote by $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ the space of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$) such that in $\mathcal{L}(H^4(\mathbb{R}\times[-1,1]), H^2(\mathbb{R}\times[-1,1]))$ we have

$$\mathscr{M}_\delta = \sum_{k=0}^{4}\delta^k M_k + \mathscr{O}(\delta^5).$$

In particular,

$$M_0 = \partial_\tau^2\,, \quad M_1 = 0\,, \quad M_2 = \partial_s^2 + \frac{\kappa^2}{4}.$$

Similarly, in $H^2(\mathbb{R}\times[-1,1])$ we have by Lemma 3.1

$$\mathscr{M}_\delta\psi = \sum_{k=0}^{4}\delta^k\alpha_k + \mathscr{O}(\delta^5),$$

with

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{\kappa\tau}{2}, \quad \alpha_2 = -\frac{\tau^2\kappa^2}{8},$$

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and $\alpha_3, \alpha_4 \in \mathscr{C}^\infty(\overline{\Omega})$. We compute ψ_k by induction on k. It satisfies

$$M_0\psi_0 = \alpha_0, \quad M_0\psi_1 = -M_1\psi_0 + \alpha_1, \quad M_0\psi_k = -\sum_{j=2}^{k} M_j\psi_{k-j} + \alpha_k,$$

with the Dirichlet condition $\psi_k(\cdot, \pm 1) = 0$. This gives in particular

$$\psi_0(s,\tau) = \frac{\tau^2-1}{2}, \quad \psi_1(s,\tau) = \frac{\kappa(s)}{12}(\tau - \tau^3).$$

Then ψ_2 has to be a solution of

$$M_0\psi_2 = -M_2\psi_0 - \frac{\kappa^2\tau^2}{8} = \frac{\kappa^2}{4}\left(-\frac{\tau^2-1}{2} - \frac{\tau^2}{2}\right) = \frac{\kappa^2}{4}\left(\frac{1}{2} - \tau^2\right).$$

This leads to take

$$\psi_2(s,\tau) = \frac{\kappa^2}{4}\left(\frac{\tau^2-1}{4} - \frac{\tau^4-1}{12}\right) = \frac{\kappa^2}{4}\left(\frac{\tau^2}{4} - \frac{\tau^4}{12} - \frac{1}{6}\right).$$

Due to the asymptotic behavior of ϕ given in Proposition 1.2 and the fact that $a = 1$ at infinity, $a^{\frac{1}{2}}\tilde{\phi} - \phi_0$ and hence $\psi - \psi_0$ belong to the Schwartz class. Thus, Ψ_5 satisfies equation (3.1) and hence equation (3.2). Now setting $\Psi = \psi_0 + \delta\psi_1 + \delta^2\psi_2$ we deduce

$$\|\psi - \Psi\|_{\mathscr{C}^2(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3).$$

This gives

$$\|\delta^{-2}\tilde{\phi}(s,\delta\tau) - a(s,\delta\tau)^{-\frac{1}{2}}\Psi\|_{\mathscr{C}^2(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3)$$

or

$$\left\|\delta^{-2}\tilde{\phi}(s,\delta\tau) - \left(1 + \delta\tau\frac{\kappa}{2} + \delta^2\frac{3}{8}\tau^2\kappa^2\right)\Psi\right\|_{\mathscr{C}^2(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3).$$

Then

$$\left\|\delta^{-2}\tilde{\phi}(s,\delta\tau) - f_\delta(s,\tau)\right\|_{\mathscr{C}^2(\mathbb{R}\times[-1,1])} = \mathscr{O}(\delta^3)\,, \tag{3.3}$$

where we have set

$$f_\delta(s,\tau) = \psi_0 + \delta\left(\psi_1 + \frac{\tau\kappa}{2}\psi_0\right) + \delta^2\left(\frac{3\tau^2\kappa^2}{8}\psi_0 + \frac{\tau\kappa}{2}\psi_1 + \psi_2\right).$$

We have

$$f_\delta(s,\tau) = \frac{\tau^2-1}{2} - \frac{\delta\kappa(s)}{6}\left(\tau - \tau^3\right) + \delta^2\kappa^2 P_2(\tau)\,,$$

where

$$P_2(\tau) = \frac{3\tau^2(\tau^2-1)}{16} + \frac{\tau(\tau-\tau^3)}{24} + \frac{\tau^2}{16} - \frac{\tau^4}{48} - \frac{1}{24}.$$

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Let us explain why f_δ has a unique minimum, nonattained at infinity and which is nondegenerate. Firstly, when $s \notin \operatorname{supp}\kappa$, we have

$$f_\delta(s,\tau) = \frac{\tau^2-1}{2} \geqslant -\frac{1}{2} = f_\delta(s,0).$$

This shows that f_δ has a minimum in Ω_0. This minimum is in fact strictly less than $-\frac{1}{2}$ and thus attained at points where the curvature is not 0. Indeed, consider s_0 the maximum of κ^2. We have $\kappa(s_0) \neq 0$, $\kappa'(s_0) = 0$, and $\kappa(s_0)\kappa''(s_0) < 0$. Let us notice that

$$\begin{aligned} f_\delta\left(s_0, \frac{\delta\kappa(s_0)}{6}\right) &= -\frac{1}{2} + \delta^2\kappa^2(s_0)\left(\frac{1}{72} - \frac{1}{36} - \frac{1}{24}\right) + \mathscr{O}(\delta^3) \\ &= -\frac{1}{2} - \frac{\delta^2\kappa^2(s_0)}{18} + \mathscr{O}(\delta^3). \end{aligned}$$

This shows that, for δ small enough,

$$\inf_{(s,\tau)\in\mathbb{R}\times(-1,1)} f_\delta(s,\tau) \leqslant -\frac{1}{2} - \frac{\delta^2 \max\kappa^2}{18} + C\delta^3 < -\frac{1}{2}$$

and that the infimum is a minimum (which is not attained at infinity).

Now, we prove that for δ small enough all the possible minima are nondegenerate. Consider a minimum (s_1,τ_1) of f_δ. We have $\tau_1 \in (-1,1)$ and $\kappa(s_1) \neq 0$. Moreover, we must have

$$\partial_\tau f_\delta(s_1,\tau_1) = 0,$$

which implies that

$$\tau_1 = \frac{\delta\kappa(s_1)}{6} + \mathscr{O}(\delta^2). \tag{3.4}$$

Then,

$$f_\delta(s_1,\tau_1) = -\frac{1}{2} - \frac{\delta^2\kappa^2(s_1)}{18} + \mathscr{O}(\delta^3).$$

With the upper bound on the minimum, we deduce that

$$0 \leqslant \kappa^2(s_0) - \kappa^2(s_1) \leqslant C\delta.$$

By using the uniqueness and nondegeneracy of the minimum, this implies that

$$s_1 = s_0 + \mathscr{O}(\delta^{\frac{1}{2}}), \quad \tau_1 = \frac{\delta\kappa(s_0)}{6} + \mathscr{O}(\delta^2), \tag{3.5}$$

where we used equation (3.4) and that $\kappa'(s_0) = 0$.

Let us now estimate the second derivative of f_δ at (s_1,τ_1). We have

$$\partial_s^2 f_\delta(s_1,\tau_1) = -\frac{\kappa(s_0)\kappa''(s_0)}{9}\delta^2 + o(\delta^2), \quad \partial_s\partial_\tau f_\delta(s_1,\tau_1) = \mathscr{O}(\delta^{\frac{3}{2}}),$$

and

$$\partial_\tau^2 f_\delta(s_1,\tau_1) = 1 + \mathscr{O}(\delta^2).$$

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We infer that there exist $\delta_0, c > 0$ such that for all $\delta \in (0, \delta_0)$ and all minimum (s_1, τ_1),

$$\mathrm{Hess}_{(s_1,\tau_1)} f_\delta \geqslant c\delta^2.$$

By definition, this means that the minima are nondegenerate.

Let us finally prove that there is only one minimum. Consider two minima $X_1 = (s_1, \tau_1)$ and $X_2 = (s_2, \tau_2)$. From equation (3.5), we have, uniformly in $\lambda \in [0, 1]$,

$$X_1 + \lambda(X_2 - X_1) = \left(s_0, \frac{\delta\kappa(s_0)}{6}\right) + (\mathscr{O}(\delta^{\frac{1}{2}}), \mathscr{O}(\delta^2)). \tag{3.6}$$

Since the differential of f_δ vanishes at X_1, the Taylor formula gives

$$f_\delta(X_2) - f_\delta(X_1) = \int_0^1 (1-\lambda)\mathrm{Hess}_{X_1+\lambda(X_2-X_1)} f_\delta(X_2 - X_1, X_2 - X_1)\mathrm{d}\lambda.$$

By using equation (3.6), we deduce as before that there exist $\delta_0, c > 0$ such that for all $\delta \in (0, \delta_0)$ and all $\lambda \in [0, 1]$,

$$\mathrm{Hess}_{X_1+\lambda(X_2-X_1)} f_\delta \geqslant c\delta^2.$$

This shows that

$$0 = f_\delta(X_2) - f_\delta(X_1) \geqslant \frac{c\delta^2}{2}|X_1 - X_2|^2.$$

Therefore, for δ small enough, f_δ has a unique minimum $X(\delta)$, which is not attained at infinity and nondegenerate, and

$$\mathrm{Hess}_{X(\delta)} f_\delta \geqslant c\delta^2.$$

By a perturbative argument using equation (3.3), this shows that $\delta^{-2}\tilde{\phi}(s, \delta\tau)$ has also a unique minimum, which is not attained at infinity and nondegenerate. The same conclusion follows for ϕ. □

4. Upper bound for the bottom of the spectrum

This last section is devoted to the proof of Theorem 1.2. From the min-max principle, we have

$$\inf \mathrm{sp}(\mathscr{P}_h) = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|(-ih\nabla - \mathbf{A})\psi\|^2 - h\|\psi\|^2}{\|\psi\|^2}.$$

From Lemma 2.1, we have

$$\inf \mathrm{sp}(\mathscr{P}_h) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{4h^2 \int_\Omega e^{-2\phi/h}|\partial_{\bar{z}} u|^2 \mathrm{d}x}{\int_\Omega e^{-2\phi/h}|u|^2 \mathrm{d}x}. \tag{4.1}$$

Let us construct a convenient test function. It is natural to consider a test function in the form

$$u(x) = f(x)\chi(x),$$

where $f \in \mathscr{O}(\Omega) \cap H^1(\Omega)$ is such that $f(x_{\min}) \neq 0$ and the cut-off function χ ensures that u satisfies the boundary condition. It is chosen of the form $\chi = \rho \circ \Theta^{-1}$ with $\rho(s, \pm\delta) = 0$ and $\rho(s, t) = 1$ for all $s \in \mathbb{R}$ and $t \in (-\delta + \epsilon, \delta - \epsilon)$. This function ρ will be determined below to optimize an upper bound; see equation (4.5).

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4.1. Estimate of the numerator

By using the change of variable Θ, we have

$$4h^2 \int_{\Omega} e^{-2\phi/h} |\partial_{\bar{z}} u|^2 \mathrm{d}x = h^2 \int_{\Omega_0} e^{-2\tilde{\phi}(s,t)/h} |\tilde{f}(s,t)|^2 \Big(a^{-2} |\partial_s \rho|^2 + |\partial_t \rho|^2 \Big) a(s,t) \mathrm{d}s\mathrm{d}t\,, \tag{4.2}$$

with $\tilde{\phi} = \phi \circ \Theta$ and $\tilde{f} = f \circ \Theta$. Since ρ is constant on $\mathbb{R} \times [-\delta + \epsilon, \delta - \epsilon]$, the right-hand side is actually an integral over $\mathbb{R} \times \big((-\delta, -\delta + \epsilon) \cup (\delta - \epsilon, \delta) \big)$. We begin with the contribution of the integral over $\mathbb{R} \times (\delta - \epsilon, \delta)$. We will choose ϵ smaller that h. Then, using the Taylor formula to expand $\tilde{\phi}(s,t)$ and $a(s,t)$ near $t = \delta$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\delta-\epsilon}^{\delta} e^{-2\tilde{\phi}(s,t)/h} |\partial_t \rho|^2 |\tilde{f}(s,t)|^2 a(s,t) \mathrm{d}t\mathrm{d}s \\
&\qquad\leqslant (1 + C\epsilon + C\epsilon^2/h) \int_{\mathbb{R}} \int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 |\tilde{f}(s,t)|^2 a(s,\delta) \mathrm{d}t\mathrm{d}s.
\end{aligned}$$

We also want to replace $|\tilde{f}(s,t)|^2$ by $|\tilde{f}(s,\delta)|^2$. To do so, we remark that, for all $(s,t) \in \mathbb{R} \times (\delta - \epsilon, \delta)$,

$$\begin{aligned}
\Big| |\tilde{f}(s,t)|^2 - |\tilde{f}(s,\delta)|^2 \Big| &\leqslant 2 \int_t^{\delta} |\tilde{f}(s,\tau)| |\partial_t \tilde{f}(s,\tau)| \mathrm{d}\tau \\
&\leqslant \Big(\|\tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])} + \|\partial_t \tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])} \Big)
\end{aligned}$$

so that

$$\int_{\mathbb{R}} \Big(\int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 \mathrm{d}t \Big) a(s,\delta) \Big| |\tilde{f}(s,t)|^2 - |\tilde{f}(s,\delta)|^2 \Big| \mathrm{d}s \leqslant \int_{\mathbb{R}} a(s,\delta) R(s,\epsilon,h) \mathrm{d}s,$$

with

$$R(s,\epsilon,h) = \Big(\|\tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])} + \|\partial_t \tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])} \Big) \int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 \mathrm{d}t. \tag{4.3}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\delta-\epsilon}^{\delta} e^{-2\tilde{\phi}(s,t)/h} |\partial_t \rho|^2 |\tilde{f}(s,t)|^2 a(s,t) \mathrm{d}t\mathrm{d}s \\
&\leqslant (1 + C\epsilon + C\epsilon^2/h) \int_{\mathbb{R}} \int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 |\tilde{f}(s,t)|^2 a(s,\delta) \mathrm{d}t\mathrm{d}s \\
&\leqslant (1 + C\epsilon + C\epsilon^2/h) \Big(\int_{\mathbb{R}} \int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 |\tilde{f}(s,\delta)|^2 a(s,\delta) \mathrm{d}t\mathrm{d}s \\
&\qquad\qquad + \int_{\mathbb{R}} a(s,\delta) R(s,\epsilon,h) \mathrm{d}s \Big).
\end{aligned} \tag{4.4}$$

Looking at the right-hand side suggests to consider a function ρ that minimizes $\int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h} |\partial_t \rho|^2 \mathrm{d}t$ among the H^1-functions equal to 1 in $\delta - \epsilon$ and 0 in δ. This leads to the explicit choice

$$\rho(s,t) = \frac{1 - e^{2(t-\delta)\partial_t \tilde{\phi}(s,\delta)/h}}{1 - e^{-2\epsilon \partial_t \tilde{\phi}(s,\delta)/h}}\,, \quad \forall (s,t) \in \mathbb{R} \times (\delta - \epsilon, \delta). \tag{4.5}$$

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The minimum satisfies

$$\int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t\tilde{\phi}(s,\delta)/h}|\partial_t\rho|^2\mathrm{d}t = \frac{2\partial_t\tilde{\phi}(s,\delta)}{h(1-e^{-2\epsilon\partial_t\tilde{\phi}(s,\delta)/h})}.$$

We recall from Proposition 1.2 that $\partial_t\tilde{\phi}(s,\delta) = \partial_\nu\phi(\Theta(s,\delta))$ is uniformly positive. Choosing $\epsilon = h|\ln h|$, we get, uniformly with respect to s,

$$\int_{\delta-\epsilon}^{\delta} e^{-2(t-\delta)\partial_t\tilde{\phi}(s,\delta)/h}|\partial_t\rho|^2\mathrm{d}t = \frac{2\partial_t\tilde{\phi}(s,\delta)}{h} + o(h^{-1}) = \mathscr{O}(h^{-1}), \tag{4.6}$$

where we used that Θ and Θ^{-1} have uniformly bounded Jacobians.

Using that $f \in H^1(\Omega)$, we get

$$\int_{\mathbb{R}}\left(\|\tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])} + \|\partial_t\tilde{f}(s,\cdot)\|^2_{L^2([\delta-\epsilon,\delta])}\right)\mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} 0$$

so that, with equations (4.3) and (4.6), it follows that

$$\int_{\mathbb{R}} a(s,\delta)R(s,\epsilon,h)\mathrm{d}s = o_{h\to 0}(h^{-1}).$$

With equation (4.4), this gives

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\delta-\epsilon}^{\delta} e^{-2\tilde{\phi}(s,t)/h}|\partial_t\rho|^2|\tilde{f}(s,t)|^2 a(s,t)\mathrm{d}t\mathrm{d}s\\ &\quad\leqslant 2h^{-1}\int_{\mathbb{R}}\partial_\nu\phi(\Theta(s,\delta))|\tilde{f}(s,\delta)|^2 a(s,\delta)\mathrm{d}s + o_{h\to 0}(h^{-1}).\end{aligned}$$

Let us now come back to equation (4.2). Considering the term with the tangential derivative, we get with similar computations

$$\int_{\mathbb{R}}\int_{\delta-\epsilon}^{\delta} e^{-2\tilde{\phi}(s,t)/h}|\partial_s\rho|^2|\tilde{f}(s,t)|^2 a(s,t)^{-1}\mathrm{d}t\mathrm{d}s = o_{h\to 0}(h^{-1}).$$

We play the same game with the contribution of the integral over $\mathbb{R}\times(-\delta,-\delta+\epsilon)$ in equation (4.2) (notice that $\partial_t\tilde{\phi}(s,-\delta) = -\partial_\nu\phi(\Theta(s,-\delta))$ is now uniformly negative). We get

$$4h^2\int_\Omega e^{-2\phi/h}|\partial_{\bar{z}}u|^2\mathrm{d}x \leqslant 2h\|(\partial_\nu\phi)^{\frac{1}{2}}f\|^2_{\partial\Omega} + o_{h\to 0}(h).$$

4.2. *Estimate of the denominator and conclusion*

We have

$$\begin{aligned}\int_\Omega e^{-2\phi/h}|u|^2\mathrm{d}x &= \int_\Omega e^{-2\phi/h}|f(x)\chi(x)|^2\mathrm{d}x\\ &= e^{-2\phi_{\min}/h}\int_\Omega e^{-2(\phi-\phi_{\min})/h}|f(x)\chi(x)|^2\mathrm{d}x.\end{aligned}$$

The Laplace method yields (notice that $\chi(x_{\min}) = 1$ for h small enough)

$$\int_\Omega e^{-2\phi/h}|u|^2\mathrm{d}x = he^{-2\phi_{\min}/h}\left(|f(x_{\min})|^2\frac{\pi}{\sqrt{\det \mathrm{Hess}_{x_{\min}}\phi}} + o_{h\to 0}(1)\right).$$

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With equation (4.1), this shows that

$$\inf \mathrm{sp}(\mathscr{P}_h) \leqslant 2\sqrt{\det \mathrm{Hess}_{x_{\min}}\phi}\,\frac{\|(\partial_\nu \phi)^{\frac{1}{2}} f\|^2_{\partial\Omega}}{\pi |f(x_{\min})|^2}(1 + o_{h\to 0}(1))e^{2\phi_{\min}/h},$$

and Theorem 1.2 since this estimate holds for all the functions f in $\mathscr{E}$.

Acknowledgments. We are grateful to David Krejčiřík for sharing the reference [8]. May also the CIRM (and its staff), where this work was initiated, be warmly thanked.

Conflict of Interest. The authors have no conflict of interest to declare.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

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