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A SYMPLECTIC APPROACH TO YANG MILLS THEORY FOR NON COMMUTATIVE TORI

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ABSTRACT. In this note we give a symplectic approach to Yang Mills theory for non commutative n-tori, inspired by the classical theory of Atiyah and Bott.

Introduction. In this note we present a symplectic approach to Yang Mills theory for a large class of "elementary modules" (in the sense of M. A. Rieffel ([17])) over n-dimensional non commutative tori, inspired by the classical theory of M. F. Atiyah and R. Bott ([1]). Roughly speaking, we prove the following: the moduli space consisting of Yang Mills minima modulo gauge equivalence appears as a symplectic manifold obtained through a Marsden-Weinstein reduction from the space of all hermitian connections, which naturally carries a symplectic structure. Actually, these modules can be directly approached using Heisenberg group theory (as in [6]), as we shall see, but we think that our methods and results are interesting on their own; moreover they can, in principle, be extended to cover more general situations, which will be examined elsewhere (see e.g. [24] for some progress in this direction). In order to lay down our constructions, we shall build on the extension of Sobolev space techniques to non commutative tori given in [23].

The present work is organized as follows: in § 1 we collect some background material on non commutative Yang Mills theory ([6], [18], [20]). In § 2 we recall basic notions on non commutative n-tori and their Sobolev spaces; next (§ 3) we present our results, also giving the Heisenberg group approach previously mentioned; § 4 is devoted to final comments.

1. **Preliminaries.** Here we shall be acting within the framework of general C^*-dynamical systems. For the basic definitions and results of non commutative differential geometry on C^*-dynamical systems we refer to [3], [4], [5], [6], [17], [18] (see also [19], [20], [21]). Let (A, G, τ) be a C^*-dynamical system G a finite dimensional Lie group, equipped with a G-invariant faithful trace τ. Let Ξ^∞ be a hermitian smooth vector bundle over it (i.e. a hermitian finitely generated projective (right) module over A^∞, the smooth subalgebra associated to A, naturally obtained from a unique module Ξ over A). Let $\bar{E} := \mathrm{End}_A \Xi$, $E := \mathrm{End}_{A^\infty} \Xi^\infty$, $\mathcal{G} = \{u \in E \mid u \text{ is invertible and } u^{-1} = u^*\}$, $\mathcal{J} = \{a \in E \mid a^* = -a\}$. Since we shall be always working within the smooth category,

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we omit ∞-superscripts. Ξ becomes a left E-module, $\mathcal{G}$ and $\mathcal{J}$ are called, respectively, the *gauge group* and the *gauge algebra*. We shall focus our attention on the space $\mathcal{A}$ consisting of all connections ∇ compatible with the hermitian structure of Ξ. This is an affine space modelled on $\Lambda^1(\mathcal{J})$ (in general $\Lambda^k(\cdot) = \cdot$-valued k-forms on Lie (G), the Lie algebra of G). $\mathcal{A}$ is acted on by G by means of the following formula:

$$g(\nabla) = g \cdot \nabla \cdot g^{-1} = \nabla + g\nabla(g^{-1}), \quad g \in \mathcal{G}$$

$\big(\nabla(T) := [\nabla, T]$, for $T \in E\big)$, which becomes, at Lie algebra level,

$$\hat{\phi}_{\nabla} := \phi(\nabla) = -\nabla(\phi) = \frac{d}{dt}(e^{t\phi} \cdot \nabla \cdot e^{-t\phi})|_{t=0}, \quad \phi \in \mathcal{J}.$$

$\hat{\phi}$ represents the vector field on $\mathcal{A}$ induced by ϕ. Choosing a base connection ∇^0 $\big($in terms of which any other hermitian connection ∇ can be written as $\nabla = \nabla^0 + a$, $a \in \Lambda^1(\mathcal{J})\big)$, the above formulae read:

$$\begin{aligned} g(\nabla) &= \nabla^0 + g\nabla^0(g^{-1}) + gag^{-1}, \\ \phi(\nabla) &= -\nabla^0(\phi) - [\phi, a]. \end{aligned} \tag{1.1}$$

The curvature of ∇, i.e. $F^{\nabla} := \nabla^2$, always fulfils Bianchi's identity $\nabla(F^{\nabla}) := [\nabla, F^{\nabla}] = 0$ and is gauge covariant:

$$F^{g(\nabla)} = gF^{\nabla}g^{-1}.$$

In terms of the covariant derivative operators, F^{∇} has the expression

$$F^{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$(X, Y \in \mathrm{Lie}(G))$. It will be assumed throughout the paper that $\mathrm{Lie}(G) \cong \mathbb{R}^n$, $n = 2p$. Lie(G) will be equipped with a complex structure inducing a real metric h and a symplectic form ω, reading in a suitable (Darboux) symplectic basis $\{e_i, e_{\bar{\imath}}\}_{i=1,2,\ldots,p}$ as $\omega = \sum_{i=1}^{p} e_i \wedge e_{\bar{\imath}}$. Let $*$ denote the Hodge operator induced by h. Using the natural (normalized) trace on $\bar{E}$ induced by τ and denoted by the same symbol, we can introduce a pre-Hilbert space structure on $\Lambda^k(E)$ as follows:

$$\langle a, b\rangle = \tau\big(*(a^* \wedge *b)\big), \quad a, b \in \Lambda^k(E),\ k = 0, 1, \ldots, n$$

(here a^* is the hermitian conjugate of a). The generalized *de Rham sequence*

$$0 \longrightarrow \Lambda^0(E) \xrightarrow{\nabla} \Lambda^1(E) \xrightarrow{\nabla} \Lambda^2(E) \xrightarrow{\nabla} \cdots \tag{1.2}$$

(here ∇ is the natural extension of ∇ to $\Lambda^k(E)$: $\nabla(T) := [\nabla, (-1)^k T]$, $T \in \Lambda^k(E)$) gives rise to an *adjoint sequence*

$$0 \longleftarrow \Lambda^0(E) \xleftarrow{\nabla^*} \Lambda^1(E) \xleftarrow{\nabla^*} \Lambda^2(E) \longleftarrow \cdots$$

with ∇^* the adjoint to ∇ with respect to $\langle\ ,\ \rangle$, and to corresponding *Laplace operators*

$$\Delta = \nabla^*\nabla + \nabla\nabla^*$$

(at each level; if $k = 0$, $\Delta = \nabla^*\nabla$ and if $k = n$, $\Delta = \nabla\nabla^*$). If F^∇ is *central*, i.e. $F^\nabla(T) = [F^\nabla, T] = 0$, $T \in E$, then (1.2) becomes a *complex* ($\nabla^2 = 0$) and we can form the *Hodge cohomology groups*

$$H^k_\nabla = \{\ker\Delta \mid \Delta\colon \Lambda^k(E) \longrightarrow \Lambda^k(E)\}$$

whose elements are called *harmonic k-forms*, $k = 0, 1, \ldots, n$. Using an explicit orthonormal basis of Lie(G) ($\cong$ Lie$(G)^*$ via h), say $\{e_1, e_2, \ldots, e_n\}$, we gather some useful formulae in

PROPOSITION 1.1.

(i)

$$\nabla : \Lambda^0(\mathcal{J}) \longrightarrow \Lambda^1(\mathcal{J})$$

$$\nabla b = \sum_{i=1}^n \nabla_i(b)e_i, \quad b \in \Lambda^0(\mathcal{J})\ (\nabla_i := \nabla_{e_i})$$

$$\nabla^*\colon \Lambda^1(\mathcal{J}) \longrightarrow \Lambda^0(\mathcal{J})$$

$$\nabla^* a = -\sum_{i=1}^n \nabla_i(a_i), \quad a = \sum_{i=1}^n a_i e_i \in \Lambda^1(\mathcal{J}).$$

(ii)

$$\nabla : \Lambda^1(\mathcal{J}) \longrightarrow \Lambda^2(\mathcal{J})$$

$$\nabla a = \sum_{i<j}[\nabla_i(a_j) - \nabla_j(a_i)]e_i \wedge e_j, \quad a = \sum_{i=1}^n a_i e_i \in \Lambda^1(\mathcal{J}),$$

$$\nabla^*\colon \Lambda^2(\mathcal{J}) \longrightarrow \Lambda^1(\mathcal{J})$$

$$\nabla^* b = -\sum_{j=1}^n [\sum_{i=1}^n \nabla_j(b_{ij})]e_j, \quad b = \sum_{i<j} b_{ij} e_i \wedge e_j \in \Lambda^2(\mathcal{J}).$$

(iii)

$$\Delta\colon \Lambda^0(\mathcal{J}) \longrightarrow \Lambda^0(\mathcal{J})$$

$$\Delta b = \nabla^*\nabla(b) = -\sum_{i=1}^n \nabla_i^2(b), \quad b \in \Lambda^0(\mathcal{J}),$$

$$\Delta : \Lambda^1(\mathcal{J}) \longrightarrow \Lambda^1(\mathcal{J})$$

$$\begin{aligned}\Delta a &= (\nabla^*\nabla + \nabla\nabla^*)(a)\\ &= \sum_{j=1}^n [\Delta(a_j) + \sum_{i=1}^n F^\nabla_{ij}(a_i)]e_j, \quad a = \sum_{i=1}^n a_i e_i \in \Lambda^1(\mathcal{J})\end{aligned}$$

(Weitzenböck's formula).

The proof of Proposition 1.1 is straightforward. Corresponding formulae for any Lie group can be found in [20]. Moreover, *Weitzenböck* type formulae can be easily given for any $k = 1, 2, \ldots, n$.

The *Yang-Mills functional* $\mathcal{YM}\colon \mathcal{A} \longrightarrow \mathbb{R}^+$ is defined as follows ([6]):

$$\mathcal{YM}(\nabla) := \langle F^{\nabla}, F^{\nabla} \rangle = \|F^{\nabla}\|^2, \quad \nabla \in \mathcal{A}.$$

It is gauge invariant, i.e., $\mathcal{YM}\big(g(\nabla)\big) = \mathcal{YM}(\nabla)\ \forall g \in \mathcal{G}$.

The *Yang-Mills equations* are the Euler-Lagrange equations pertaining to $\mathcal{YM}$; their solutions are called *critical* (or *Yang-Mills*) connections. In view of gauge invariance, ∇ critical entails $g(\nabla)$ critical $\forall g \in \mathcal{G}$. The Yang-Mills equations read precisely as in the classical case ([18], [20]):

PROPOSITION 1.2. *(i) Let Ξ be a hermitian smooth vector bundle on a C^*-dynamical system (A, G, τ). The Yang-Mills equations read:*

$$\nabla^*(F^{\nabla}) = 0$$
$$\nabla(F^{\nabla}) = 0. \qquad \textit{(Bianchi identity)}$$

(ii) ([20], [18]) Explicitly, if G is abelian, the first equation becomes:

$$\sum_{i=1}^{n} \nabla_i(F_{ij}^{\nabla}) = 0, \quad j = 1, 2, \ldots, n.$$

(iii) ([20]) The Hessian $\mathcal{H}_{\nabla}$ at a critical connection reads:

$$\mathcal{H}_{\nabla}(a, a) = \|\nabla a\|^2 + 2\langle F^{\nabla}, a \wedge a\rangle, \quad a \in \Lambda^1(\mathcal{J}).$$

The proof of Proposition 1.2 mimics the classical one. We recall the following result of A. Connes and M. A. Rieffel ([6]):

THEOREM ([6]). *Let* Lie(G) *be abelian, and assume Ξ admits connections in with constant curvature. Then the set in $\mathcal{A}$ on which $\mathcal{YM}$ attains a minimum consists exactly of those connections in $\mathcal{A}$ with constant curvature. Furthermore, the curvatures of all these minimizing connections will be the same.*

PROOF. If F^{∇} is constant, for some $\nabla \in \mathcal{A}$, then clearly ∇ satisfies the Yang-Mills equations. Moreover, a straightforward computation shows that, for any $a \in \Lambda^1(\mathcal{J})$.

$$\mathcal{YM}(\nabla + a) = \mathcal{YM}(\nabla) + \|\nabla a + a \wedge a\|^2,$$

whence ∇ is a minimum of $\mathcal{YM}$ and all other minima are given by the connections $\nabla + a$ fulfilling

$$\nabla a + a \wedge a = 0, \tag{1.3}$$

i.e., precisely the connections having the same (constant) curvature as ∇, since, in general,

$$F^{\nabla+a} = F^{\nabla} + \nabla a + a \wedge a. \qquad \blacksquare$$

Observe that if F^{∇} is constant, then $\mathcal{H}_{\nabla}(a,a) = \|\nabla a\|^2 \geq 0$ and $\mathcal{H}_{\nabla}(a,a) = 0 \Leftrightarrow \nabla a = 0$; this equation is the linearization of (1.3) and formally represents the "tangent space" to the "manifold" of constant curvature connections in ∇. In order to achieve complete rigour, we must resort to Sobolev space techniques, which will be done starting from the next section, concentrating on the non commutative torus case.

2. **Non commutative tori and their Sobolev spaces.** For background on non commutative tori we refer to [6], [17], [18] and references therein. The *non commutative torus* A_θ is by definition the *cocycle* C^**-algebra* $C^*(D,\sigma)$, where $D \cong \mathbb{Z}^n$ (we confine ourselves to the case $n = 2p$), the standard lattice in $L^* = \mathbb{R}^{*^n}$, and $\sigma(x,y) = e^{-i\pi\sigma(x,y)}$, $x, y \in L^*$, with $\theta \in \Lambda^2 L$. For our purposes we require θ to be non degenerate, i.e., $\theta(x,y) = 0\ \forall y \in L^* \Rightarrow x = 0$. (L^*, θ) is then a symplectic vector space.

There is a natural action of the (commutative) torus T^n on A_θ (see below) which allows one to identify the ensuing smooth algebra as $\mathcal{S}(D,\sigma)$ ([17]). Let us denote this by $\mathcal{S}_\theta$. A_θ is generated by unitaries $\{u_x\}_{x\in\mathbb{Z}^n}$ (belonging to $\mathcal{S}_\theta$) fulfilling the *commutation relations*

$$u_y u_x = e^{2\pi i\theta(x,y)} u_x u_y.$$

Any element $\Phi \in \mathcal{S}_\theta$ can be uniquely expressed as

$$\Phi = \sum_{\mathbb{Z}^n} \Phi_{\mathbf{m}} u_1^{m_1} u_2^{m_2} \cdots u_n^{m_n},$$

$\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, $\{\Phi_{\mathbf{m}}\} \in \mathcal{S}(\mathbb{Z}^n)$, $\{u_k\}$ corresponding to the standard basis of $\mathcal{L}^*$. The action of T^n is given by $\big($if $(z_1, z_2, \ldots, z_n) \in T^n$, i.e., $z_k \in \mathbb{C}, |z_k| = 1\big)$

$$\begin{aligned} a(I) &= I, \\ a_{z_j}(u_j) &= z_j u_j, \\ a_{z_j}(u_k) &= u_k, \quad k \neq j, \end{aligned}$$

(where I denotes the identity of A_θ). The (T^n-invariant) normalized trace τ on A_θ is given by

$$\tau(\Phi) = \Phi_{\mathbf{0}}, \quad \mathbf{0} = (0,0,\ldots,0), \quad \Phi \in A_\theta.$$

We shall concentrate on particular "elementary" modules $\big($in M. A. Rieffel's terminology ([17])$\big)$

$$V^T := \mathcal{S}(M), \quad M = \mathbb{R}^p,$$

which are acted on by $\mathcal{S}_\theta$ via:

$$(fu_x)(m) = e\Big(\big\langle m - \frac{T'(x)}{2}, T''(x)\big\rangle\Big) f\big(m - T'(x)\big)$$

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$m \in M, f \in \mathcal{S}(M)$, $e(a) := e^{2\pi i a}$, $a \in \mathbb{R}$. Here $\langle \ , \ \rangle$ denotes the standard duality between $\mathbb{R}^p$ and $\mathbb{R}^{*p}$, and $T = (T', T'')$ is an *invertible* map $T: \mathbb{R}^{*2p} \longrightarrow \mathbb{R}^p \times \mathbb{R}^{*p}$ such that:

$$\langle T'(y), T''(x) \rangle - \langle T'(x), T''(y) \rangle = \theta(x, y),$$

$x, y \in \mathbb{Z}^n$. u_x can of course be defined for any $x \in \mathbb{R}^{*n}$ (see [17] for the general definition of T). It is known from the general theory that $E := \mathrm{End}_{\mathcal{S}_\theta} V^T$ is, in this case, of type $\mathcal{S}_\psi$ as well and is based on the lattice $D^\perp \cong \mathbb{Z}^n$ determined by the requirement $y \in D^\perp$ if and only if $\theta(x, y) = 0 \pmod 1$ $\forall x \in D$. More precisely, we have:

PROPOSITION 2.1. $\mathrm{End}_{\mathcal{S}_\theta} V^T \cong \mathcal{S}_{-\theta^{-1}}$.

PROOF. In terms of the standard basis of $\mathbb{Z}^n$, $\{e_1, e_2, \ldots, e_n\}$, $y \in D^\perp$ if and only if

$$\theta(y, e_k) = z_k \in \mathbb{Z}, \quad k = 1, 2, \ldots, n.$$

Let $y^k = \theta^{-1} e_k$; then any $y \in D^\perp$ can be written as $y = \sum_{k=1}^n m_k y^k$, $m_k \in \mathbb{Z}$. An immediate computation shows:

$$\theta(y^i, y^j) = -(\theta^{-1})_{ji},$$

whence, by setting $V_i := u_{y^i}$, we find the commutation relations

$$V_j V_i = e\big(-(\theta^{-1})_{ji}\big) V_i V_j,$$

and our assertion follows. ∎

An $\mathbb{R}^n$-family of constant curvature connections compatible with a suitable metric on V^T is given by the formula (in terms of covariant derivatives operators, see [17])

$$\nabla_X = Q_{SX} + \langle a, SX \rangle I, \quad \alpha \in \mathbb{R}^p \times \mathbb{R}^{*p}, \tag{2.1}$$

where $S := (S_1, S_3) = (T^{-1})^*: L \longrightarrow \mathbb{R}^{*p} \times \mathbb{R}^p$, and if $z = (s, u) \in \mathbb{R}^{*p} \times \mathbb{R}^p$ we have $Q_z = Q_s^1 + Q_u^3$ with

$$\begin{aligned} (Q_s^1 f)(m) &= 2\pi i \langle m, s \rangle f(m), \\ (Q_u^3 f)(m) &= \sum_{i=1}^p u_j \frac{\partial f}{\partial m_j}(m), \quad f \in \mathcal{S}(M), \end{aligned}$$

and I is here the identity operator on V^T, i.e., the identity of E.

The curvature form F^∇ reads:

$$\begin{aligned} F^\nabla(X, Y) &= [\nabla_X, \nabla_Y] \\ &= 2\pi i \big(\langle S_3(X), S_1(Y) \rangle - \langle S_3(Y), S_1(X) \rangle \big) I \\ &=: 2\pi i \omega(X, Y) I, \quad X, Y \in L. \end{aligned} \tag{2.2}$$

ω is a symplectic form on L, in this particular case (see [17], pages 290–293): it is essentially θ, upon identification of L with L^*. In particular, it does not vanish.

Let now ∇ denote any connection of the family (2.1). We have the following

PROPOSITION 2.2. *∇ acts on E via the formula:*

$$\nabla_X(V_y) = 2\pi i\langle X, y\rangle V_y, \quad x \in L,\ y \in D^{\perp}. \tag{2.3}$$

PROOF. Using the general formula (cf. [17])

$$(\Phi f)(m) = \int_{\tilde{D}^{\perp}} \Phi(w) e(\langle m, w''\rangle) f(m + w')\, e(\langle \frac{1}{2}w', w''\rangle)\, dw$$

($\Phi \in E$), with dw the counting measure on

$$\tilde{D}^{\perp} = \{ w = (w', w'') = \big(-T'(y), T''(y)\big), y \in D^{\perp}\},$$

a straightforward computation shows that

$$\begin{aligned}[(\nabla_X \Phi) f](m) &:= \{ [\nabla_X, \Phi] f\} (m) \\ &= \int_{\tilde{D}^{\perp}} 2\pi i \langle X, y\rangle \Phi(w)\, e(\langle m, w''\rangle) f(m + w') e(\langle \frac{1}{2}w', w''\rangle)\, dw,\end{aligned}$$

whence (2.3) follows. ∎

Upon fixing a metric on $\mathbb{R}^n$ in such a way that $\{y^k\} =: \{e_k\}$, $k = 1, 2, \ldots, n$ becomes an orthonormal basis, we have for the 0th-order Laplace operator the formula (by Proposition (1.1))

$$\begin{aligned}[(\Delta\Phi) f](m) &= -\sum_{i=1}^{n} [(\nabla_i^2 \Phi) f](m) \\ &= 4\pi^2 \int_{\tilde{D}^{\perp}} \sum_{i=1}^{n} |\langle y, e_i\rangle|^2 \Phi(w) e(\langle m, w''\rangle) f(m + w') e(\langle \frac{1}{2}w', w''\rangle)\, dw,\end{aligned}$$

$f \in \mathcal{S}(M)$. Let $y_i := \langle y, e_i\rangle$; we have, in particular

$$\Delta V_y = 4\pi^2 \cdot \sum_{i=1}^{n} y_i^2 \cdot V_y.$$

Moreover writing

$$\Phi = \sum_{\mathbb{Z}^n} \Phi_{\mathbf{m}} V_1^{m_1} V_2^{m_2} \cdots V_n^{m_n},$$

$\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, we find:

$$\|\Phi\|_0^2 = \langle \Phi, \Phi\rangle = \sum_{\mathbb{Z}^n} |\Phi_{\mathbf{m}}|^2 = \|\{\Phi_{\mathbf{m}}\}\|_{l^2(\mathbb{Z}^n)}^2$$

(here we use the normalized trace on E) and

$$\langle \Phi, \Delta\Phi\rangle = 4\pi^2 \sum_{\mathbb{Z}^n} m^2 |\Phi_{\mathbf{m}}|^2,$$

where $m^2 := \sum_{i=1}^{n} m_i^2$.

Let $\Lambda_0^k(E)$ denote the Hilbert space completion of $\Lambda^k(E)$ with respect to the norm induced from $\langle\ ,\ \rangle$, which will be labelled as $\|\ \|_0$. *Sobolev norms* are introduced in $\Lambda^k(E)$, $k = 0, 1, \ldots, n$ by the definition (compare with [23])

$$\begin{aligned}\|b\|_s^2 &:= \|b\|_0^2 + \langle b, \Delta^s b\rangle \\ &= \langle b, (I + \Delta^s)b\rangle, \quad b \in \Lambda^k(E),\end{aligned}$$

$s = 1, 2, \ldots$, where Δ is the (kth) Laplace operator attached to ∇. An equivalent norm is, for instance, the following:

$$\|b\|_s^2 := \langle b, (I + \Delta)^s b\rangle .$$

Then we complete accordingly, obtaining *Sobolev spaces* $\Lambda_s^k(E)$.

The following assertions are part of, or immediately follow from the theory developed in [23] and Proposition 2.2.

THEOREM 2.3. *(i)* $\Lambda_s^k(E) \cong \Lambda_s^k(T^n)$, $k = 0, 1, \ldots, n$; $s = 0, 1, 2 \ldots$.
(ii) $\Lambda_s^k(E) \hookrightarrow \Lambda^k(\bar{E})$ *if* $s > p = \frac{n}{2}$ *(Sobolev Embedding Theorem).*
(iii) The Hodge-de Rham complex pertaining to ∇ *is elliptic and has the same cohomology groups as the de Rham complex on* T^n.
(iv) The inclusion $\Lambda_s^0(E) \hookrightarrow \Lambda_{s'}^0(E)$, $s > s'$, *is compact (Rellich's Theorem) and Hilbert-Schmidt for* $s - s' > \frac{n}{2}$ *(Maurin's Theorem).*
(v) $\Lambda_s^0(E)$ *is a Banach* $*$*-algebra for* $s > n$.
(vi) $\Lambda_{s'}^0(E)$ *is a topological* $\Lambda_s^0(E)$*-module for* $s > n$, $0 \leq s' \leq s$.

From (iii), we get, in particular:

$$H_\nabla^k \cong H^k(T^n) \cong \Lambda^k(\mathbb{R}^n),$$

whence $h_\nabla^k = \binom{n}{k}$, $k = 0, 1, \ldots, n$. Here, as usual, we set $h_\nabla^k = \dim H_\nabla^k$.

REMARK. The Sobolev spaces introduced above depend, a priori, on ∇. But this is not the case, according to

PROPOSITION 2.4. *The Sobolev norms pertaing to any two connections* ∇_0, $\nabla = \nabla_0 + a$, $a \in \Lambda^1(\mathcal{J})$, *are equivalent (we definitely set* $s > 0$*).*

PROOF. It is enough to prove the result for $\Lambda_s^0(E)$; its validity for $\Lambda_s^k(E)$, $k = 1, 2, \ldots, n$, will be then obtained by using similar arguments in conjunction with Weitzenböck type formulae (see Proposition 1.1, (iii), and the remark following it).

Let s be even, $s = 2p$, and $b \in H_s := \Lambda_s^0(E)$ (defined via ∇_0, any connection of the family (2.1). Then $\langle b, \Delta^s b\rangle = \|\Delta^p b\|_0^2$. But $\Delta^p = \Delta_0^p + K$, where $K: H_s \longrightarrow H_0$ is a compact operator, by Rellich's theorem, and hence continuous. So:

$$\|\Delta^p b\|_0^2 \leq 2(\|\Delta_0^p b\|_0^2 + \|Kb\|_0^2) \leq C\|b\|_s^2$$

(for some $C > 0$). Exchanging the role of ∇_0 and ∇ yields the result for $s = 2p$.

Let now s be odd: $s = 2p + 1$. Then

$$\begin{aligned}\langle b, \Delta^s b\rangle &= \langle \Delta^p b, \Delta \cdot \Delta^p b\rangle = \|\nabla \Delta^p b\|_0^2 \\ &\leq 2(\|\nabla_0 \Delta^p b\|_0^2 + \|a\Delta^p b\|_0^2) \\ &\leq 4(\|\nabla_0 \Delta_0^p b\|_0^2 + \|\nabla_0 K b\|_0^2) + D\|\Delta^p b\|_0^2\end{aligned}$$

(for suitable $D > 0$).

Now $b \in H_s$ implies $Kb \in H_3$ and $\nabla_0 Kb \in H_2$. But $K: H_s \longrightarrow H_2$ is continuous (for $s > 2$) and so

$$\|\nabla_0 Kb\|_0^2 \leq C_1 \|b\|_s^2$$

(for some $C_1 > 0$), whence, on proceeding now as above, the desired result follows. The remaining case $s = 1$ is immediately settled. ∎

We also have the following

PROPOSITION 2.5. *$\nabla : \Lambda_{s+1}^0(E) \longrightarrow \Lambda_s^1(E)$ has closed range for any $\nabla = \nabla_0 + a$, $a \in \Lambda^1(\mathcal{J})$.*

PROOF. If $\xi \in \overline{\mathrm{Im}(\nabla)}^s \cap \Lambda_s^1(E)$, then $\xi \in \overline{\mathrm{Im}(\nabla)}^0$ — (obvious notation). It is easy to see that $\langle \xi, \eta \rangle = 0 \; \forall \eta \in \mathrm{Ker}\, \nabla^*$. In particular, if ϕ is harmonic, then $\nabla \phi \in \mathrm{Ker}\, \nabla^*$. But if $\xi \in \Lambda_s^1(E)$, then $\langle \xi, \Delta\phi \rangle = 0$, whence $\nabla^* \xi$ is orthogonal to $\mathrm{Ker}\, \Delta$ and this entails, by the Fredholm alternative, $\nabla^* \xi = \Delta\lambda$, for some $\lambda \in \Lambda_{s+1}^0(E)$, which in turn implies $\nabla^*(\xi - \nabla \lambda) = 0$, whence our assertion follows easily. Recall that the Laplacian $\Delta = \nabla^* \nabla : \Lambda_s^0(E) \longrightarrow \Lambda_{s-2}^0(E)$ differs from Δ_0 by a compact operator, and this entails the Fredholm alternative for it. ∎

Let $\mathcal{A}_s$, $\mathcal{G}_s$, $\mathcal{J}_s$ denote the Sobolev completions of $\mathcal{A}$, $\mathcal{G}$, $\mathcal{J}$, respectively. $\mathcal{G}_s$ becomes a Lie group modelled on the Hilbert space $\mathcal{J}_s$ (local charts for $\mathcal{G}_s$ being produced through the exponential map), provided we act in the "good range" $s > n$, which henceforth will be tacitly assumed. The action of $\mathcal{G}_{s+1}$ on $\mathcal{A}_s$ given by (1.1) is then smooth.

3. **Symplectic description of Yang-Mills minima.** To begin with, let us proceed in some generality, without referring explicitly to non commutative tori. The following description may, in fact, work in general, provided the relevant concepts involved retain their meaning.

For the notions of symplectic geometry discussed here we refer, for instance to [2], [11], [1], [13]. We have the following

THEOREM 3.1. *(i) $\mathcal{A}_s$ carries a weak symplectic structure.*

(ii) $\mathcal{G}_{s+1}$ acts on $\mathcal{A}_s$ in a hamiltonian manner.

(iii) The map $\mu: \mathcal{A}_s \longrightarrow \mathcal{J}_{s-1}$ defined as

$$\mu(\nabla) := \hat{F}^{\nabla}$$

(with ^ denoting the component of F^{∇} "parallel" to ω, using the $\| \;\|_0$ metric on $\Lambda_{s-1}^0(E)$) is a $\mathcal{G}_{s+1}$-equivariant moment map.

PROOF. (i) Let us define, for $\nabla \in \mathcal{A}_s$, $a, b \in \Lambda_s^1(\mathcal{J})$,

$$\hat{\omega}_{\nabla}(a, b) := \langle a \wedge b, \omega \rangle = \tau\big(*(a \wedge b \wedge *\omega)\big).$$

$\hat{\omega}_\nabla$ is a weak symplectic structure on $\Lambda^1_s(\mathcal{J}) \cong T_\nabla \mathcal{A}_s$ (tangent space to $\mathcal{A}_s$ in ∇) since the map from $\Lambda^1_s(\mathcal{J})$ to $\Lambda^1_s(\mathcal{J})^*$ given by $a \longrightarrow a \lrcorner \omega$ ($\lrcorner$ denotes contraction) is injective but not surjective. In fact, if $\hat{\omega}_\nabla(a,b) = 0 \; \forall b \in \Lambda^1_s(\mathcal{J})$, then

$$0 = \hat{\omega}_\nabla(a,b) = \sum_{i=1}^{p} \tau(a_i b_{\bar{\imath}} - a_{\bar{\imath}} b_i), \quad \forall b \in \Lambda^1_s(\mathcal{J}),$$

in terms of the symplectic basis $\{e_i, e_{\bar{\imath}}\}_{i=1,2,\ldots,p}$ referred to in § 1, and this clearly entails $a = 0$. The latter assertion follows immediately once we observe that (as sets) $\mathcal{J}_s \underset{\neq}{\subset} \mathcal{J}_0 \subseteq \mathcal{J}_s^*$. Thus we naturally obtain a closed, non degenerate smooth 2-form on $\mathcal{A}_s$, i.e. a (weak) symplectic form $\hat{\omega}$ defined at each ∇ by the above formula.
(ii) The action of $\mathcal{G}_{s+1}$ on $\mathcal{A}_s$ is symplectic, i.e., if $g \in \mathcal{G}_{s+1}$, then

$$g * \hat{\omega} = \hat{\omega}.$$

Here $*$ denotes pull-back. We check this explicitly:

$$\begin{aligned}(g * \hat{\omega})_\nabla \; &:= \hat{\omega}_{g(\nabla)}\big(g(a), g(b)\big) \\ &= \hat{\omega}_{\nabla + g\nabla(g^{-1})}(gag^{-1}, gbg^{-1}) = \langle gag^{-1} \wedge gbg^{-1}, \omega \rangle \\ &= \hat{\omega}_\nabla(a,b).\end{aligned}$$

Now define, for $\phi \in \mathcal{J}_{s+1}$, the smooth function on $\mathcal{A}_s$:

$$\lambda_\phi(\nabla) := \langle F^\nabla, \phi\,\omega \rangle.$$

Then

$$\begin{aligned}(d\lambda_\phi)_\nabla(a) = \langle \nabla a, \phi\,\omega \rangle &= \sum_{i=1}^{p} \tau[(\nabla_i a_{\bar{\imath}} - \nabla_{\bar{\imath}} a_i)\phi\,] \\ &= \sum_{i=1}^{p} \tau(-a_{\bar{\imath}} \nabla_i \phi + a_i \nabla_{\bar{\imath}} \phi) \\ &= -\langle \nabla \phi \wedge a, \omega \rangle = (\hat{\phi} \lrcorner \hat{\omega})_\nabla(a),\end{aligned}$$

whence the action of $\mathcal{G}_{s+1}$ on $\mathcal{A}_s$ is indeed hamiltonian.

We now check that, upon defining *Poisson brackets* $\{\;,\;\}$ by

$$\{\lambda_\phi, \lambda_\psi\}(\nabla) := \hat{\omega}_\nabla(\hat{\phi}_\nabla, \hat{\psi}_\nabla),$$

$\phi, \psi \in \mathcal{G}_{s+1}$, we have

$$\{\lambda_\phi, \lambda_\psi\} = \lambda_{[\phi,\psi]}.$$

Indeed:

$$\begin{aligned}\{\lambda_\phi, \lambda_\psi\}(\Delta) &= \hat{\omega}_\nabla(\nabla\phi, \nabla\psi) = -\hat{\omega}_\nabla(\nabla\psi, \nabla\phi) \\ &= (d\lambda_\phi)_\nabla(\hat{\psi}_\nabla) = \hat{\psi}(\lambda_\phi)(\nabla) = \frac{d}{dt}\langle F^{(e^{t\psi})(\nabla)}, \phi\,\omega \rangle|_{t=0} \\ &= \langle F^\nabla, [\phi, \psi]\omega \rangle = \lambda_{[\phi,\psi]}(\nabla).\end{aligned}$$

(iii) The map $\mu: \mathcal{A}_s \longrightarrow \mathcal{J}_{s-1}$ is smooth and $\mathcal{G}_{s+1}$-equivariant since

$$F^{g(\nabla)} = gF^{\nabla} g^{-1}$$

and, moreover, if $\phi \in \mathcal{J}_{s+1}$, then

$$\big(\mu(\nabla), \phi\big) = (\hat{F}^{\nabla}, \phi) := \langle F^{\nabla}, \phi\,\omega \rangle = \lambda_{\phi}(\nabla),$$

whence it is an equivariant moment map:

$$\begin{aligned} \mu\big(g(\nabla)\big) &= \lambda_{\phi}\big(g(\nabla)\big) = \langle F^{g(\nabla)}, \phi\,\omega \rangle \\ &= \lambda_{g^{-1}(\phi)}(\nabla), \end{aligned}$$

with $g^{-1}(\phi) := g^{-1}\phi\, g$ (*adjoint* action of $\mathcal{G}_{s+1}$ on $\mathcal{J}_{s+1}$).

If we work at the pre-Hilbert space level, we can identify $\mathcal{J}$ with $\mathcal{J}^*$ by means of τ, and transform the adjoint action of $\mathcal{G}$ on $\mathcal{J}$ into the *coadjoint* action of $\mathcal{G}$ on $\mathcal{J}^*$, adhering to the *Kirillov-Kostant-Souriau* theory ([11], [13]). Theorem 3.1 generalizes corresponding classical results ([1]). ∎

Now suppose that F^{∇} is constant (and of the form $F^{\nabla} = \hat{F}^{\nabla}\,\omega$); the *isotropy group*

$$\mathcal{G}_{F^{\nabla}} = \{ g \in \mathcal{G}_{s+1} \mid gF^{\nabla} g^{-1} = F^{\nabla} \}$$

then coincides with $\mathcal{G}_{s+1}$. If we consider the *moduli space* $\mathcal{M}$ consisting, by definition, of the Yang-Mills minima up to gauge equivalence, we see that, taking into due account the Connes-Rieffel theorem discussed in § 1, it appears formally as a *Marsden-Weinstein symplectic quotient*

$$\mathcal{M} \cong \mu^{-1}(\hat{F}^{\nabla}) / \mathcal{G}_{F^{\nabla}},$$

since $gF^{\nabla} g^{-1} = F^{\nabla}$ if and only if $g\hat{F}^{\nabla} g^{-1} = \hat{F}^{\nabla}$. Under suitable conditions, to be described below, this picture can be implemented rigorously, i.e., $\mathcal{M}$ becomes a genuine Hausdorff symplectic manifold.

Let us now give some definitions. A connection ∇ is called *irreducible* if $\nabla\phi = 0$, $\phi \in E$, entails $\phi = \lambda I$, $\lambda \in \mathbb{C}$. Otherwise it will be called *reducible*. Let $\tilde{A}_s$ denote the manifold of irreducible connections (in $\mathcal{A}_s$). Let $\hat{\mathcal{G}}_{(s)}$ denote the *effective* gauge group $\mathcal{G}_{(s)}/T$ (T being the circle group) and set $\hat{\mathcal{J}}_{(s)} := \mathrm{Lie}(\hat{\mathcal{G}}_{(s)})$. Thus, if $a \in \mathcal{J}_{(s)}$, we have an orthogonal decomposition $a = \tau(a)I + \hat{a}$, with $\tau(\hat{a}) = 0$, $\hat{a} \in \hat{\mathcal{J}}_{(s)}$. If $g = e^{a} = e^{\tau(a)}\hat{g}$ ($\hat{g} = e^{\hat{a}}$), then $g(\nabla) = \hat{g}(\nabla)$ and this justifies our terminology. If ∇ is irreducible, then $\nabla\phi = 0$, $\phi \in \hat{\mathcal{J}}_{s+1}$, entails $\phi = 0$. This last fact descends from *elliptic regularity*, which holds in view of Theorem 2.3 and Propositions 2.4 and 2.5. Indeed, if $\nabla\phi = 0$, for $\phi \in \hat{\mathcal{J}}_{s+1}$, then $\Delta\phi = 0$, whence $\phi \in \hat{\mathcal{J}}$, and $\phi = 0$ since $\Delta\phi = 0 \Rightarrow \nabla\phi = 0 \Rightarrow \phi = 0$ (by the irreducibility of ∇ and the very definition of $\hat{\mathcal{J}}$).

Let us now prove the following provisional theorem, which will be later partially replaced by a more complete result.

THEOREM 3.2. *Let $s > n$, and let all the objects involved in the statements below pertain to a module V^T over A_θ subject to the conditions discussed in the previous section. In particular, let ∇ be a constant curvature connection on V^T (one can choose ∇ to belong to the family (2.1)). Then*

(i) $\tilde{\mathcal{A}}_s/\hat{\mathcal{G}}_{s+1}$ is a Hausdorff manifold.

(ii) $\tilde{\mathcal{A}}_s \cap \mu^{-1}(\hat{F}^\nabla)$ is a submanifold of $\tilde{\mathcal{A}}_s$.

(iii) $\mathcal{M} \equiv \tilde{\mathcal{A}}_s \cap \mu^{-1}(\hat{F}^\nabla)/\hat{\mathcal{G}}_{s+1}$ is a submanifold of $\tilde{\mathcal{A}}_s/\hat{\mathcal{G}}_{s+1}$ and appears as a Marsden-Weinstein reduction of $\tilde{\mathcal{A}}_s$.

(iv) Around ∇, $\mathcal{M}$ looks like

$$\{\nabla + a \mid a = \sum_{j=1}^{n} a_j e_j,\ a_j \in i\mathbb{R},\ \|a\|_s = \|a\|_0 < \epsilon, \textit{ for some } \epsilon > 0\}.$$

Before giving the proof of Theorem 3.2, we make the following

REMARKS. (i) We shall see that moduli spaces defined in Theorem 3.2 (iii), and during the discussion preceding it, and denoted by the same symbol $\mathcal{M}$, will indeed coincide, since they will consist of irreducible (and smooth) connections.
(ii) The arguments presented here have been adapted from [14] (see also [10]). We again proceed in some generality, in order to make clear that the arguments given here extend beyond the concrete examples discussed, provided the relevant objects are meaningful and the Sobolev theory holds.

PROOF OF THEOREM 3.2. (i) A natural candidate for an atlas of $\tilde{\mathcal{A}}_s/\hat{\mathcal{G}}_{s+1}$ is provided by local charts of the form

$$\mathcal{O}_{\nabla,\epsilon} = \{\nabla + a, a \in \Lambda^1_s(\mathcal{J}) \mid \nabla^* a = 0,\ \|a\|_s < \epsilon\},$$

for $\epsilon > 0$ small enough. The condition $\nabla^* a = 0$ (*Coulomb (or Hodge) gauge condition*) means that a is orthogonal to the tangent space to the orbit of $\mathcal{G}_{s+1}$ through ∇. Here ∇ is a general hermitian irreducible connection; in order to simplify the following calculations, we assume it to have constant curvature ($\nabla^2 = 0$), but the argument works in general, for a dense set of connections $\nabla + a$, $a \in \Lambda^1_s(\mathcal{J})$, $\nabla + a$ irreducible: it is easy to check that if ∇ is irreducible, then $\nabla + a$ is such, for a small enough, upon comparison of their associated (0th-order) Laplacians; the argument is completed by resorting to Proposition 2.5.

Let

$$\hat{\mathcal{G}}_{s+1,I,\epsilon} := \{g \in \hat{\mathcal{G}}_{s+1} \mid \|g - I\|_{s+1} < \epsilon\}.$$

Consider the map Φ: $\hat{\mathcal{G}}_{s+1,I,\epsilon} \times \mathcal{O}_{\nabla,\epsilon} \longrightarrow \mathcal{A}_s$ given by, for $\nabla' = \nabla + a$,

$$\Phi(g, \nabla') = \nabla + g\nabla(g^{-1}) + gag^{-1}.$$

The differential of Φ at (I, ∇) reads:

$$(d\Phi)_{(I,\nabla)}(\gamma, b) = -\nabla\gamma + b,$$

$\gamma \in \hat{\mathcal{G}}_{s+1}$, $b \in \Lambda^1_s(\mathcal{J})$, $\nabla^* b = 0$; thus, it is injective since ∇ is irreducible, and surjective since $\operatorname{Im}\nabla$ is closed (Proposition 2.5), so $\mathcal{A}_s = \operatorname{Im}\nabla \oplus \operatorname{Ker}\nabla*$ (Sobolev indices omitted). $(d\Phi)_{(I,\nabla)}$ is then an isomorphism by the open mapping theorem; hence $\hat{\mathcal{G}}_{s+1,I,\epsilon} \times \mathcal{O}_{\nabla,\epsilon}$ is diffeomorphic to a neighbourhood of ∇ in $\mathcal{A}_s$, for some small $\epsilon > 0$. We shall prove that $\mathcal{O}_{\nabla,\epsilon}$ injects into the quotient, i.e., for $\epsilon > 0$ sufficiently small, if $g(\nabla_2) = \nabla_1$, then $g \in \hat{\mathcal{G}}_{s+1,I,\epsilon}$. This will be achieved by *bootstrapping*. If $\nabla_j = \nabla + a_j$ and $\|a_j\|_s < \delta$, $\delta > 0, j = 1, 2$, then

$$\nabla g = g a_2 - a_1 g$$

(by (1.1)). Moreover,

$$\begin{aligned}\|\nabla g\|_0 &\leq \|g a_2\|_0 + \|a_1 g\|_0\\ &\leq C\|g\|_0(\|a_1\|_s + \|a_2\|_s) < \infty,\end{aligned}$$

for some $C > 0$. Actually $\|g\|_0 = 1$ if we use the normalized trace τ on E to define $\|\ \|_0$; g does not necessarily belong to E, *a priori*, but it is a unitary operator on the GNS-Hilbert space H_τ induced by τ and its norm as a vector in H_τ is precisely $\|g\|_0$. Retrieving our calculation, we get

$$\|g\|_1^2 = \|g\|_0^2 + \langle g, \Delta g\rangle < \infty,$$

i.e., $g \in \hat{\mathcal{G}}_1$. Similarly,

$$\begin{aligned}\|\nabla g\|_1 &\leq \|g a_2\|_1 + \|a_1 g\|_1\\ &\leq C_1\|g\|_1(\|a_1\|_s + \|a_2\|_s), \quad C_1 > 0.\end{aligned}$$

But

$$\begin{aligned}\|\nabla g\|_1^2 &= \|\nabla g\|_0^2 + \langle \nabla g, \Delta \nabla g\rangle\\ &= \langle g, \Delta g\rangle + \langle g, \nabla^*(\nabla^*\nabla + \nabla\nabla^*)\nabla g\rangle\\ &= \langle g, \Delta g\rangle + \langle g, \Delta^2 g\rangle,\end{aligned}$$

(as $\nabla^2 = 0$), and so $g \in \hat{\mathcal{G}}_2$. Similarly, we get $g \in \hat{\mathcal{G}}_{s+1}$.

An analogous computation also shows that $g \in \hat{\mathcal{G}}_{s+1,I,\epsilon}$ for ϵ small enough. In particular one also obtains as a corollary that if $a_j \in \Lambda^1(\mathcal{J})$, $j = 1, 2$, then $g \in \mathcal{G}$. This shows that the topology of $\tilde{\mathcal{A}}/\mathcal{G}$ does *not* depend on the Sobolev structure. It remains to prove the Hausdorff property. Let ∇_1, ∇_2 be two inequivalent connections in $\mathcal{O}_{\nabla,\epsilon}$; suppose that there exist sequences $\{\nabla_j^n\} \longrightarrow \nabla_j$, $j = 1, 2$, in $\|\ \|_s$ and $g_n(\nabla_s^n) = \nabla_1^n$. Setting $\nabla_j^n = \nabla + a_j^n$, we have, as before:

$$\nabla g_n = g_n a_2^n - a_1^n g_n.$$

Proceeding now as above, we have $\|g_n\|_{s+1}$ uniformly bounded by a constant $C > 0$; upon passing to a subsequence, still denoted by $\{g_n\}$, we have, by Rellich's theorem (Theorem 2.3, (iv)), $g_n \longrightarrow g$ (say) in $\|\ \|_s$. This is easily seen to entail $g \in \hat{\mathcal{G}}_{s+1}$ and $g(\nabla_2) = \nabla_1$, so that ∇_1 and ∇_2 are gauge equivalent, contrary to our initial assumption.

The various charts $\mathcal{O}_{\nabla,\epsilon}$ overlap smoothly (direct check), and hence $\tilde{\mathcal{A}}_s$ is a Hausdorff manifold as claimed. This proves (i).

(ii) We must show that the differential $d\mu$ is surjective at any irreducible ∇. If $d\mu_\nabla$ is not surjective, there exists $\phi \in \hat{\mathcal{J}}_{s+1}$, $\phi \neq 0$, such that

$$\big(d\mu_\nabla(a), \phi\big) = 0, \quad a \in \Lambda^1_s(\mathcal{J}).$$

But then

$$\begin{aligned} 0 &= \big(d\mu_\nabla(a), \phi\big) = \langle \widehat{\nabla a}, \phi \rangle \\ &= (d\lambda_\phi)_\nabla(a) = \hat{\omega}_\nabla(\nabla\phi, a), \end{aligned}$$

whence, $\hat{\omega}$ being non degenerate, $\nabla\phi = 0$ with $\phi \neq 0$, which contradicts our hypothesis.

(iii) In view of (i) and (ii), we have to seek a solution to the system

$$\text{(3.1)} \qquad \begin{cases} \widehat{\nabla a} = f \\ \nabla^* a = 0, \qquad f \in \hat{\mathcal{J}}_{s-1}, a \in \Lambda^1_s(\mathcal{J}). \end{cases}$$

Let $b \in \Lambda^1_s(\hat{\mathcal{J}})$ be a solution to the equation

$$\widehat{\nabla b} = f,$$

which exists according to (ii); if we find $\xi \in \Lambda^1_s(\mathcal{J})$ such that

$$\text{(3.2)} \qquad \begin{cases} \nabla\xi = 0 \\ \nabla^*\xi = -\nabla^* b. \end{cases}$$

then we see that $a = b + \xi$ is a solution of (3.1). Solving (3.2) is equivalent (using the irreducibility of ∇ in conjunction with $\nabla^2 = 0$) to solving the Poisson equation

$$\Delta\xi = -\nabla\nabla^* b,$$

which is possible by the Fredholm alternative, and in view of Weitzenböck's formula (Proposition 1.1, (iii)), with $F^\nabla(\cdot) = 0$.

(iv) $\mathcal{M}$ is locally described through the equations

$$\text{(3.3)} \qquad \begin{cases} \nabla a + a \wedge a = 0, \\ \nabla^* a = 0, \end{cases}$$

which entail

$$\Delta a + \nabla^*(a \wedge a) = 0.$$

The tangent space $T_\nabla \mathcal{M}$ at ∇ is depicted by the equations

$$\text{(3.4)} \qquad \begin{cases} \nabla a = 0 \\ \nabla^* a = 0 \quad (\Leftrightarrow \Delta a = 0), \end{cases}$$

i.e., by H^1_∇. Owing to the irreducibility of ∇, the only solutions to (3.4) are provided by scalar 1-forms, which solve the full equations (3.3). $\mathcal{M}$ is independent of the Sobolev structure, since scalar forms are automatically smooth.

Alternatively, one can use the implicit function theorem in the following guise: the map $P\colon \Lambda^1_s(\hat{\mathcal{J}}) \longrightarrow \Lambda^1_{s-2}(\hat{\mathcal{J}})$ defined by

$$P(\hat{a}) = \Delta\hat{a} + \nabla^*(\hat{a} \wedge \hat{a})$$

has differential (at $\hat{a} = 0$) equal to Δ, which is an isomorphism; hence the equation $P(\hat{a}) = 0$ has no solution different from $\hat{a} = 0$ in a small neighbourhood of ∇. ∎

The following theorem is essentially a reformulation of Theorem 3.2, (iv) above, but we give it since we shall need a slight variation of it in the sequel.

THEOREM 3.3. *(i) Let I be an open interval in $\mathbb{R}$ containing 0 and let $I \ni t \longrightarrow \nabla(t) = \nabla + a(t)$ be a smooth curve of solutions to (3.3), with $a(t) = ta_1 + b(t)$, where $t^{-1}b(t) \longrightarrow 0$ if $t \longrightarrow 0$. Then its tangent vector in ∇, namely a_1, fulfils the equation $\Delta a_1 = 0$, and so it is a scalar 1-form.*

(ii) If $a(t)$ is real analytic in t, then $a(t)$ is a scalar 1-form.

PROOF. (i) If $\nabla + a(t)$ fulfils (3.3), then

$$t\nabla a_1 + G(t) = 0,$$
$$t\nabla^* a_1 + \nabla^* b(t) = 0,$$

with an obvious definition of $G(t)$, and $t^{-1}G(t) \longrightarrow 0$ if $t \longrightarrow 0$. Dividing by $t \neq 0$ and letting $t \longrightarrow 0$ we get the assertion, since ∇ is irreducible.

(ii) This follows easily from power series expansion (upon setting $a(t) = \sum_{k=1}^{\infty} t^k a_k$), which yields equations

$$\begin{cases} \nabla a_k + \sum_{h=1}^{k-1} a_h \wedge a_{k-h} = 0, & \\ \nabla^* a_k = 0, & k = 1, 2, \dots, \end{cases}$$

which are solved recursively and give scalar 1-forms a_k, $k = 1, 2, \dots$.

This result shows once more that, in the irreducible case, H^1_∇ is the genuine tangent space to the moduli space. In general it is not true that any element solving the linearized equations (3.4) appears as the tangent vector in ∇ to a curve of connections solving the full equations (3.3). We shall see an instance of this phenomenon later on (compare, e.g. [10], [14],).

The neighbourhood $\mathcal{O}_{\nabla,\epsilon} \cap \mathcal{M} \subset \mathcal{M}$ cannot be indefinitely extended in general: $\nabla_1 = \nabla + a_1$, $\nabla_2 = \nabla + a_2$, where $a_j, j = 1, 2$, are scalar 1-forms are gauge equivalent if and only if there exists $g \in \hat{\mathcal{G}}$ such that

$$\nabla g = (a_2 - a_1)g \tag{3.5}$$

and ∇ may possess eigenvalues.

If ∇ is not irreducible, then it has a non trivial *stabilizer*

$$\Gamma^\nabla := \{ g \in \hat{\mathcal{G}}_{s+1} \mid g(\nabla) = \nabla \}$$

which measures the lack of injectivity of ∇. Proceeding again as in [14], we easily obtain the following

COROLLARY 3.4. *If ∇ is reducible, a chart around it is given by*

$$\mathcal{O}_{\nabla,\epsilon}/\Gamma^{\nabla}.$$

(ii) If ∇ is reducible and has constant curvature, the formal tangent space to $\mathcal{M}$ in ∇ is $H^1_{\nabla}/\Gamma^{\nabla}$. ■

From now on we definitely deal with non commutative tori: the calculations below just refer to this case.

Let $\mathcal{M}_{\nabla}$ denote the connected component of the moduli space containg ∇. We are now in a position to state the following

THEOREM 3.5. *$\mathcal{M}_{\nabla}$ is a reduced symplectic manifold isomorphic with T^n.*

PROOF. Theorem 3.5 is a consequence of Theorem 3.2. The only thing which remains to be proven is the global character of $\mathcal{M}_{\nabla}$. We must retrieve the calculation in Theorem 3.2, (iv) and reconsider equation 3.5, which reads explicitly, if $g = \sum_{\mathbb{Z}^n} g_{\mathbf{m}} V_1^{m_1} V_2^{m_2} \cdots V_n^{m_n}$:

$$2\pi i m_j g_{\mathbf{m}} = (a_2 - a_1)_j g_{\mathbf{m}}. \tag{3.6}$$

This implies that

$$(2\pi i)^{-1}(a_2 - a_1)_j \in \mathbb{Z}, \quad j = 1, 2, \ldots, n,$$

and this condition is clearly sufficient to get a solution to (3.6), since it is then enough to choose $g = V_1^{m_1} V_2^{m_2} \cdots V_n^{m_n}$, $m_j = (2\pi i)^{-1}(a_2 - a_1)_j$. ■

Actually, we shall see in a minute that it is possible to go beyond Theorem 3.5 by showing that $\mathcal{M}$ is connected, so $\mathcal{M} = \mathcal{M}_{\nabla} \cong T^n$. We are not able to prove this with our geometric tools. Nevertheless, it is possible to achieve this by resorting "ad hoc" to Heisenberg group theory as in [6]; this approach renders part of our previous discussion redundant (we determine $\mathcal{M}$ directly), but causes some loss in geometric insight. Each approach has its own merits, and sheds light on the other.

THEOREM 3.6. $\mathcal{M} \cong T^n$.

PROOF. Let ∇ be a fixed connection of the family (2.1). The formula (2.2) giving its curvature naturally defines a representation of the *Heisenberg Commutation Relations* for p degrees of freedom, upon considering the covariant derivative operators induced by ∇ (suitably rescaled by $-i(2\pi)^{-\frac{1}{2}}$, in order to get symmetric operators, and the right commutation relations) with respect to a Darboux basis for ω. This representation is integrable to a *Weyl* form π_{∇} (or, equivalently, to a unitary representation of the $2p+1$-dimensional *Heisenberg group*, sending the centre into T) on $\mathcal{L}^2(M, d\mu)$ ($d\mu$ denotes the Lebesgue measure on M), for which $\mathcal{S}(M)$ provides the set of C^∞-*vectors*; this can be ascertained by applying Dixmier's ([7]) and Nelson's ([15]) theorems to $\mathcal{S}(M)$ (together with Stone's theorem ([15])), and then by recalling the notion of C^∞-vector space for a unitary representation of a Lie group (see, e.g. [13]), and the concrete form of the Heisenberg operators (see below).

π_∇ is irreducible, in view of the Stone-von Neumann uniqueness theorem for the Weyl commutation relations ([25], [15], [12], [7]), which states that any such representation is unitarily equivalent to a sum of copies of the *Schrödinger* representation, say π_0, which is also defined on $\mathcal{L}^2(M, d\mu)$ and is irreducible. Recall that this irreducibility is tied to the 1-dimensionality of the ground state space of the quantum harmonic oscillator hamiltonian. π_0 has $\mathcal{S}(M)$ as its set of C^∞-vectors and, as remarked above, the same is true for π_∇ since the covariant derivative operators involved are, by construction, linear combinations of the position and momentum operators attached to π_0 (see § 2).

Any other connection $\nabla' = \nabla + a$, $a \in \Lambda^1(\mathcal{J})$, having the same curvature as ∇, gives rise to another representation of the Heisenberg commutation relations; we have $a\,\mathcal{S}(M) \subset \Lambda^1\big(\mathcal{S}(M)\big)$, and a is clearly a *bounded* perturbation of ∇; so, the use of the Kato-Rellich theorem ([15]), in conjunction with the previous arguments, yields that ∇' also gives rise to an irreducible representation of the Weyl commutation relations, $\pi_{\nabla'}$, on $\mathcal{L}^2(M, d\mu)$ whose set of C^∞-vectors is again $\mathcal{S}(M)$, and which is, of course, unitarily equivalent to π_∇.

Explicitly, we write

$$Q \cdot \nabla \cdot Q^{-1} = \nabla + a,$$

with Q a unitary operator on $\mathcal{L}^2(M, d\mu)$. Proceeding now as in [6], we find that, for any $j = 1, 2, \ldots, n$, the unitary operator $QW_jQW_j^*$ (where W_j is U_j acting on the left) intertwines π_∇ with itself and so, owing to irreducibility, $Q^* W_j Q W_j^* = u_j I$, $|u_j| = 1$, i.e. $W_j Q = u_j Q W_j$. Now let $N = W_x$, $x \in \mathbb{R}^n$, with

$$W_j N = u_j^{-1} N W_j.$$

N exists in view of the commutation relations for A_θ and because θ is assumed to be non degenerate. Let $U = QN$. Then it is immediately checked that $W_j U = U W_j$, $j = 1, 2, \ldots, n$, i.e., $U \in E$. Now $N^{-1} \nabla N = \nabla + \sigma$, with σ a scalar 1-form, and $U(\nabla + \sigma)U^{-1} = \nabla + a$, so U intertwines $\nabla + \sigma$ and $\nabla + a$. This shows that any ∇' having the same constant curvature as ∇ is gauge equivalent to a connection of the family (2.1).

The proof is completed by resorting to the calculation in the preceding theorem. ∎

We now deduce some corollaries of the previous theorem.

COROLLARY 3.7. *The moduli space $\mathcal{M}_{(V^T)^d}$, $d \geq 1$, consisting of the Yang-Mills minima up to gauge equivalence on $(V^T)^d$ (d copies of V^T) is precisely*

$$\mathcal{M}_{(V^T)^d} \cong (T^n)^d / (\Sigma^d)^{n-1},$$

where Σ^d denotes the permutation group over d elements.

PROOF. Let $\nabla = \oplus_{i=1}^d \nabla_i$ on $(V^T)^d$, with ∇_i a connection of the family (2.1). Clearly ∇ has constant curvature and is reducible, with

$$\Gamma^\nabla \cong U(d, \mathbb{C}).$$

The formal tangent space to the moduli space in ∇ is given by $H^1_\nabla / \Gamma^\nabla$ (Corollary 3.4). In this case, it is immediately checked that

$$H^1_\nabla = \{ a = \sum_{i=1}^n a_i e_i \mid a_i \in M(d, \mathbb{C}), a_i^* = -a_i \}.$$

Now, using the same notation as above, let $\nabla(t) = \nabla + a(t)$ by an analytic curve of connections, $t \in I$, $\nabla(0) = \nabla$ fulfilling (3.3). Then, proceeding as in Theorem 3.3, one finds that $a(t)$ is a scalar matrix 1-form and, moreover, $a(t) \wedge a(t) = 0$, i.e. $[a_i(t), a_j(t)] = 0$, $i, j = 1, 2, \ldots, n$.

Conversely, any antihermitian scalar matrix 1-form a fulfilling $a \wedge a = 0$ provides a solution to the full equations (3.3).

The components a_j of a can be simultaneously diagonalized, $a_j = \mathrm{diag}(a_j^1, a_j^2, \ldots a_j^d)$, and if a_1, say, is fixed, then the other components are determined up to a permutation (namely, an element of Σ^d). Moreover, $a_j = \mathrm{diag}(a_j^1, a_j^2, \ldots, a_j^d)$ is gauge equivalent to $a_j' = \mathrm{diag}(a_j'^1, a_j'^2, \ldots, a_j'^d)$ whenever $a_j^k - a_j'^k \in 2\pi i \mathbb{Z}$, $k = 1, 2, \ldots, d$. Hence, in view of Theorems 3.5, 3.6, the moduli space is $(T^n)^d$, modulo the action of $(\Sigma^d)^{n-1}$. The proof is complete. The appearance of Σ^d is natural since it is the Weyl group of $U(d, \mathbb{C})$. In this case the formal tangent space is not the "true" tangent space. ∎

COROLLARY 3.8 ([6]). *Let $\Xi_{p,q}$, $p, q \in \mathbb{Z}$, $(p, q) = 1$, $p + \theta q > 0$, be the basic Heisenberg modules on the non commutative 2-torus A_θ, $\theta \in \mathbb{R} - \mathbb{Q}$. Then (with the obvious notation):*

$$\mathcal{M}_{(\Xi_{p,q})^d} \cong (T^2)^d / \Sigma^d.$$

PROOF. Using the explicit formulae in [6] and reasoning as above yields the desired conclusion. ∎

Let us now drop the condition $(p, q) = 1$. According to Rieffel's results ([18]), the higher critical points of $\mathcal{YM}$ on $\Xi_{p,q}$ are described by fixing a finite partition

$$\{ m_k, p_k, q_k \}, m_k > 0, (p_k, q_k) = 1, p_k + \theta q_k > 0,$$
$$p + \theta q = \sum m_k (p_k + \theta q_k),$$

decomposing $\Xi_{p,q}$ as $\oplus_k (\Xi_{p_k, q_k})^{m_k}$, and setting $\nabla = \oplus_k m_k \nabla_k$, ∇_k a constant curvature connection on Ξ_{p_k, q_k}. The ensuing moduli space $\tilde{\mathcal{M}}$ is then isomorphic to $\prod_k (T^2)^{m_k} / \Sigma^{m_k}$. This result admits an immediate symplectic reinterpretation:

COROLLARY 3.9. $\tilde{\mathcal{M}} \cong \mu^{-1}(\hat{F}^\nabla) / \mathcal{G}_{F^\nabla}$.

PROOF. $F^\nabla = \oplus_k F^{m_k \nabla_k}$, whence $\mathcal{G}_{F^\nabla} \cong \prod_k \mathcal{G}_{F^{m_k \nabla_k}}$, which easily implies the desired result. ∎

4. **Concluding remarks.** (i) It is possible to extend the notion of *holomorphic structure* in our non commutative context ([20], [21], [22]) by taking (with respect to the complex structure of $\mathbb{R}^n \cong \mathbb{C}^p$) the anti-holomorphic part ∇'' of an *integrable* compatible connection (i.e., a ∇ with $\nabla''^2 = 0$) and declaring two such structures *equivalent* if they can be obtained from each other by the action of the *complex gauge group* $\mathcal{G}^c$, i.e. the group of all invertible endomorphisms of Ξ. We plan to use this notion elsewhere to discuss the holomorphic geometry on non commutative vector bundles along the lines of the beautiful classical portrait ([1], [8], [9]).

(ii) As mentioned in the introduction, we discussed, in [24], $\mathcal{YM}$-moduli spaces for a certain class of Rieffel's modules, finding, in this case, highly reducible connections and infinite dimensional moduli spaces.

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