ON THE MODULAR REPRESENTATIONS OF THE SYMMETRIC GROUP
PART IV

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1. Introduction. The study of the modular representation theory of the symmetric group has been greatly facilitated lately by the introduction of the *graph* **(9**, III**)**, the *q-graph*[1] **(5)** and the *hook-graph* **(4)** of a Young diagram $[\lambda]$. In the present paper we seek to coordinate these ideas and relate them to the *r-inducing* and *restricting* processes **(9**, II**)**.

If we denote the number of nodes of class r which can be added to or removed from $[\lambda]$ by d and d^* respectively, then the Main Theorem 6.3 expresses the change in weight of $[\lambda]$, which arises as a result of r-inducing or restricting, in terms of d and d^*. Further explicit results connect d and d^* with the corresponding δ, δ^* associated with the q-core of $[\lambda]$, which are illustrated in Tables I and II at the end of the paper.

It is interesting to note that the set of Young diagrams thus associated with a given $[\lambda]$ constitutes a Boolean Algebra of *dimension* $d + d^*$, whose partial ordering is that established by r-inducing. Two diagrams, or elements of the Boolean Algebra, of the same dimension d^* have the same weight w. Moreover, *dual* elements also have the same weight, and this shows itself in the symmetry of Tables I and II.

That these results are so explicit is somewhat surprising. No attempt is made here to apply them to the study of the structure of the indecomposables of the regular representation of S_n, this being left to a subsequent paper.

2. The graph $G[\lambda]$ and the q-graph $\mathbf{G}[\lambda]$. We begin by introducing the notion of the graph of a Young diagram $[\lambda] = [\lambda_1, \lambda_2, \ldots \lambda_m]$ obtained by replacing the (i, j) node of $[\lambda]$ by

2.1 $$g_{i,j} = j - i.$$

We shall denote this *graph* $(g_{i,j})$ by $G[\lambda]$. The quantity $1/\rho$ appearing in Young's semi-normal representation of S_n is given **(9**, III**)** by

2.2 $$\frac{1}{\rho} = g_{i,j} - g_{k,l},$$

where $i < k$ and $j > l$

If we reduce $g_{i,j}$ modulo q and require that the residue be non-negative, i.e. set

2.3 $$g_{i,j} \equiv \mathbf{g}_{i,j} \pmod{q}, \qquad 0 \leqslant \mathbf{g}_{i,j} < q,$$

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[1]As in **(7, 11)** we use q instead of p to indicate that q may be composite.

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we obtain D. E. Littlewood's *q-graph* $(\mathfrak{g}_{i,j})$ which we shall denote by $\mathbf{G}[\lambda]$. An immediate consequence of 2.1 and 2.3 is the relation

2.4 $$\mathfrak{g}_{i,j-1} = \mathfrak{g}_{i+1,j} \equiv \mathfrak{g}_{i,j} - 1 \pmod{q},$$

from which it follows that

2.5 *Any right or skew hook of* $\mathbf{G}[\lambda]$ *of length kq with head node of class r, is made up of a succession of residues*

$$r, r-1, \ldots, 1, 0, q-1, \ldots, 1, 0, q-1, \ldots, r+2, r+1,$$

each residue appearing k times.

Thus we may associate the class of its head node **(10)** with any kq-hook of $[\lambda]$. The significance of this association so far as the *star diagram* **(3, 4)** or *q-quotient* $[\lambda]_q$ of $[\lambda]$ is concerned will appear shortly. The leg length of such a hook will depend on the core.

It follows from 2.5 that the residue content of $\mathbf{G}[\lambda]$ is uniquely determined by that of the core and the *weight* w of $[\lambda]$, which is the number of removable q-hooks. Littlewood proved the following important result **(5**, p. 337**)**:

2.6 *A necessary and sufficient condition that two diagrams* $[\lambda']$ *and* $[\lambda'']$ *have the same weight and the same q-core is that* $\mathbf{G}[\lambda']$ *and* $\mathbf{G}[\lambda'']$ *contain the same set of residues modulo q.*

Another approach to the problem is to consider the hooks with corner nodes in the first column of $[\lambda]$, setting

2.7 $$l_i = \lambda_i + m - i,$$

where m is the number of rows of $[\lambda]$. The following theorem[2] supplements 2.6 and makes it possible to actually construct the core of a diagram, given the residue content of its q-graph.

2.8 *A diagram is a q-core if and only if each class of congruent l_i's contains all smaller non-negative integers congruent to the largest one in the class, the* 0*-class being empty.*

The details of this construction are being given elsewhere.

3. r-inducing and r-restricting. The reciprocity theorem of Frobenius is of deep significance in the representation theory of finite groups over a field of characteristic zero. The relation between *inducing* and *restricting* thus provided is particularly simple in the case of the symmetric group S_{n+1} if the subgroup under consideration is taken to be S_n.

Consider first the *inducing* process, taking the irreducible representation $[\lambda]$ of S_n, to yield the reducible representation **(7; 8)**

$$[\lambda] \, . \, [1]$$

[2]As stated in **(11)**, the theorem was not quite correct.

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of S_{n+1} whose irreducible components are obtained by adding a node to $[\lambda]$ in all possible ways. For example:

3.1 $$[3, 2, 1] \uparrow [4, 2, 1] + [3^2, 1] + [3, 2^2] + [3, 2, 1^2].$$

Conversely, if we take an irreducible representation $[\lambda]$ of S_n and *restrict* it to the operations of S_{n-1}, the irreducible components of the resulting representation of S_{n-1} will be obtained by removing a node from $[\lambda]$ in all possible ways. For example:

3.2 $$[4, 2, 1] \downarrow [3, 2, 1] + [4, 1^2] + [4, 2].$$

The two symbols $\uparrow \downarrow$ are convenient to indicate inducing and restricting, respectively, particularly in the modular case to which we now proceed.

If we think of the processes as operating on $\mathbf{G}[\lambda]$ instead of on $[\lambda]$, we may distinguish the residue class of the added node by inserting an r above or below the arrow. Thus we may add a node of class r *only* and designate the process as *r-inducing*. For example, taking $q = 3$, $r = 0$,

3.3 $$[3, 2, 1] \overset{0}{\uparrow} [4, 2, 1] + [3, 2, 1^2].$$

Similarly, we may limit the restricting process to *r-restricting*, so that

3.4 $$[4, 2, 1] \underset{0}{\downarrow} [3, 2, 1] + [4, 1^2].$$

What is the significance of these limited processes as regards the modular representation theory of S_n? We state the following modification of 2.6:

3.5 *The necessary and sufficient condition that two diagrams* $[\lambda']$ *and* $[\lambda'']$ *obtained by adding (removing) a node to (from) a given diagram* $[\lambda]$ *should have the same q-core is that the added (removed) nodes should be of the same residue class in* $\mathbf{G}[\lambda]$.

While 3.5 is not essential in the application of the inducing or restricting processes, since one may readily determine the q-core of a diagram **(1, 2, 6)**, nevertheless the simplification thus introduced makes it possible to keep track of changes in the star diagram or the q-quotient $[\lambda]_q$ and consequently in the weight of $[\lambda]$. We shall study these changes in detail with the aid of the hook-graph described in the following section. We prove here an important preliminary result, after making the following

DEFINITIONS. We shall call[3]

(i) the number d of nodes of class r which can be added to $[\lambda]$ the *r-defect* of $[\lambda]$;

(ii) the number d^* of nodes of class r which can be removed from $[\lambda]$ the *r-affect* of $[\lambda]$;

denoting by δ, δ^* the *r-defect* and *r-affect* of the q-core of $[\lambda]$.

[3]The r-defect must not be confused with Brauer's *defect* group or *defect* of a block **(1)**.

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3.6 *Neither adding nor removing a kq-hook of class different from r or $r - 1$ changes d, d^*, δ, δ^*.*

Proof. Since the class of a kq-hook is defined to be the class of its head node, a hook of class different from r or $r - 1$ cannot begin or end in a node of class r. Thus a node of class r must be *internal* to such a skew hook, and if it could have been removed from $[\lambda]$, the addition or removal of such a hook does not affect this possibility. Similarly, if a node of class r could be added to $[\lambda]$, the addition or removal of a kq-hook of class different from r or $r - 1$ does not affect the possibility of such an addition. Moreover, the core remains the same so d, d^*, δ, δ^* remain unchanged.

For convenience, we shall abbreviate "a node of class r" to an *r-node.* Similarly we shall describe the position such a node may occupy in $\mathbf{G}[\lambda]$ as an *r-position.*

4. The hook-graph $H[\lambda]$. Since the hook structure of $[\lambda]$ is different for every q, it is not only convenient but also of general significance, to make all such computations once and for all **(4)**. To this end we set in place of the (i, j) node of $[\lambda]$ the quantity

4.1 $$h_{i,j} = (\lambda_i - j) + (\lambda'_j - i) + 1,$$

where $[\lambda']$ is the transpose of $[\lambda]$. Clearly, $h_{i,j}$ is the length of the right hook having its corner at the (i,j) node of $[\lambda]$. We denote the *hook-graph* $(h_{i,j})$ by $H[\lambda]$. Note that $l_i = h_{i,1}$ if $m = \lambda'_1$ in 2.7.

We have immediately from 2.1 and 4.1 that

4.2 $$\begin{aligned} h_{i,k} - h_{j,k} &= (\lambda_i - i) - (\lambda_j - j) \\ &= g_{i,j} + \lambda_i - \lambda_j, \end{aligned}$$

so that this difference is independent of k, which provides a useful check on the construction of $H[\lambda]$. The following relation between $\mathbf{G}[\lambda]$ and $H[\lambda]$ is fundamental in all that follows:

4.3 *If in* $\mathbf{G}[\lambda]$ *the ith row ends in s and the jth column in t, then in* $H[\lambda]$

$$h_{i,j} \equiv s - t + 1 \pmod{q}.$$

Proof. From 4.1 we have

$$\begin{aligned} h_{i,j} &= (\lambda_i - j) + (\lambda'_j - i) + 1 \\ &= (\lambda_i - i) - (j - \lambda'_j) + 1 \\ &= g_{i,\lambda_i} - g_{\lambda'_j,j} + 1, \end{aligned}$$

so that by 2.3 we have the desired result:

$$h_{i,j} \equiv \mathbf{g}_{i,\lambda_i} - \mathbf{g}_{\lambda'_j,j} + 1 \pmod{q}.$$

Clearly an r-node can be added at the end of a row of $\mathbf{G}[\lambda]$ whose final node is of class $r - 1$, provided such an r-position is also at the foot of a column whose final node is of class $r + 1$, and only in such places. But the $h \equiv 0 \pmod{q}$ which yield the constituent of $[\lambda]_q$ of class $r - 1$ lie in rows which end in $(r - 1)$

-nodes and consequently in columns which end in r-nodes, and those which yield the constituent of class r lie in rows which end in r-nodes and in columns which end in $(r+1)$-nodes. Thus the addition to $\mathbf{G}[\lambda]$ of an r-node modifies one or both of these constituents of $[\lambda]_q$. On the other hand, the addition of an r-node cannot affect the other constituents of $[\lambda]_q$. A similar argument applies to the removal of an r-node, proving the following analogue of 3.6:

4.4 *Neither adding nor removing an r-node changes the constituents of* $[\lambda]_q$ *of class different from r or r* $-$ 1, *but does modify one or both of these constituents.*

In the following sections we shall study the effects of r-inducing and restricting so far as the *weight* is concerned. To simplify matters we might assume that $[\lambda]_q$ has only constituents of class r and $r-1$, in view of 3.6 and 4.4. However, for the considerations of this paper such an assumption is unnecessary.

5. The change in weight ∇. Consider any diagram $[\lambda]$ with r-defect $d > 0$ so that we may add an r-node at some r-position P at the intersection of the ith row and jth column of $\mathbf{G}[\lambda]$. The effect on $[\lambda]_q$ will be two-fold.

(i) Consider first those $h \equiv -1 \pmod q$ in $H[\lambda]$ which are thereby changed into $h \equiv 0 \pmod q$. Setting $s = r - 1$ in 4.3 it follows that the number of $h_{i,k} \equiv -1 \pmod q$ for $k < j$ is equal to the number of foot-nodes in $\mathbf{G}[\lambda]$ of class $r+1$ *below* P; denote this number by $(r+1)_{fB}$. On the other hand, the number of $h_{l,j} \equiv -1 \pmod q$ for $l < i$, which lie in the jth column, is similarly obtained by setting $t = r+1$ in 4.3, and is the number of head-nodes of class $r-1$ lying *above* P, which we may denote by $(r-1)_{hA}$. Thus adding an r-node at P leads to an *increase* in the weight of $[\lambda]$ by an amount

5.1 $$\Delta = (r+1)_{fB} + (r-1)_{hA}.$$

(ii) The second effect of adding an r-node at P is to change those $h \equiv 0 \pmod q$ which appear in the ith row and the jth column of $H[\lambda]$ into $h \equiv 1 \pmod q$. As before, it follows that the number of $h_{i,k} \equiv 0 \pmod q$ for $k < j$ is equal to the number of foot-nodes of class r *below* P, which number we denote by $(r)_{fB}$. On the other hand, the number of $h_{l,j} \equiv 0 \pmod q$ for $l < i$, which lie in the jth column is equal to the number of head-nodes of class r which lie *above* P, which number we denote by $(r)_{hA}$. Thus adding, an r-node at P leads to a *decrease* in the weight of $[\lambda]$ by an amount

5.2 $$\bar{\Delta} = (r)_{fB} + (r)_{hA}.$$

Combining the two effects we see that the total change in the weight of $[\lambda]$ caused by adding an r-node in the r-position P is given by

5.3 $$\nabla = \Delta - \bar{\Delta} = \{(r+1)_{fB} + (r-1)_{hA}\} - \{(r)_{fB} + (r)_{hA}\}.$$

If we compare the result of adding an r-node in two different r-positions P' and P'' to yield two different q-graphs $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$, then we know by 3.5

that $[\lambda']$ and $[\lambda'']$ have the same q-core and the same weight, and ∇ has the same value in each case. Effectively, 3.5 states that adding an r-node to $\mathbf{G}[\lambda]$ can be passed back through the removable hooks to the core, and that any change in weight is due to the effect of such an addition on the core. It is not without interest to follow through the changes in the terms on the right-hand side of 5.3 which arise when the r-node is added at different r-positions P, or when a kq-hook is removed from $\mathbf{G}[\lambda]$ which does not begin or end at P, but we leave this to the reader.

We now examine briefly the effect of *removing* an r-node, and there is no loss of generality if we consider it to be the one previously added at P on the rim of $\mathbf{G}[\lambda]$ to yield $\mathbf{G}[\lambda']$; of course we shall obtain $\mathbf{G}[\lambda]$ again. As before, there are two effects to consider.

(iii) Those $h \equiv 0 \pmod q$ in $H[\lambda]$ which were changed into $h \equiv 1 \pmod q$ in $H[\lambda']$ by adding an r-node at P in (ii) are precisely those which now yield $h \equiv 0 \pmod q$ in the reverse process.

(iv) Similarly, those $h \equiv 0 \pmod q$ in $H[\lambda']$ are now changed into $h \equiv -1 \pmod q$ in $H[\lambda]$.

Thus r-restricting *interchanges* the roles of Δ and $\bar{\Delta}$ and so changes the sign of the difference ∇.

5.4 *The change in weight arising from the addition of an r-node to* $[\lambda]$ *is given by*

$$\nabla = \Delta - \bar{\Delta}$$

where

$$\Delta = (r+1)_{fB} + (r-1)_{hA}, \quad \bar{\Delta} = (r)_{fB} + (r)_{hA},$$

and ∇ *may be positive or negative. Similarly, the change in weight arising from removing an r-node from* $[\lambda]$ *is given by* $-\nabla$.

6. An explicit formula for ∇. While the results of the preceding section are complete they do not express ∇ explicitly in terms of $[\lambda]$. To do this we study the functions Δ and $\bar{\Delta}$ in greater detail.

Consider first the function

5.2 $$\bar{\Delta} = (r)_{fB} + (r)_{hA}.$$

Certainly all removable r-nodes of $\mathbf{G}[\lambda]$ contribute to $\bar{\Delta}$, since each one is a possible head-node and(or) foot-node of a kq-hook. But other r-nodes contribute as well.

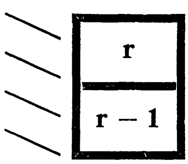

FIG. 1

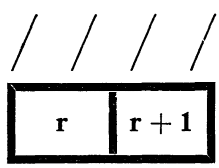

FIG. 2

We have illustrated in Figs. 1 and 2 parts of the rim of $\mathbf{G}[\lambda]$ in which no r-node is removable, and yet the arrangement in Fig. 1, appearing say ϵ_A times *above* P contributes ϵ_A to $(r)_{hA}$. Similarly, the arrangement in Fig. 2 appearing ϵ_B times *below* P contributes ϵ_B to $(r)_{fB}$. Thus

6.1 $$\bar{\Delta} = (r)_{hA} + (r)_{fB} = d^* + \epsilon_A + \epsilon_B.$$

On the other hand, no r-node can be added to the right of $r - 1$ in Fig. 1 and below $r + 1$ in Fig. 2. But the quantity

5.1 $$\Delta = (r + 1)_{fB} + (r - 1)_{hA}$$

enumerates not only $(r - 1)$-nodes in configurations such as Fig. 1 appearing ϵ_A times above P and $(r + 1)$-nodes in configurations such as Fig. 2 appearing ϵ_B times below P, but also all places where an r-node *can* be added, excluding the position P itself, so that

6.2 $$\Delta = (r + 1)_{fB} + (r - 1)_{hA} = (d - 1) + \epsilon_A + \epsilon_B.$$

It is to be noted that the epsilons depend on the choice of P on the rim of $[\lambda]$. Subtracting 6.1 from 6.2, these variable terms disappear and we have the desired explicit expression for ∇.

In the restricting process we must interchange the roles of Δ and $\bar{\Delta}$. An exactly analogous argument leads to the equations

$$\bar{\Delta}' = d^{*\prime} - 1 \ + \epsilon'_A + \epsilon'_B,$$
$$\Delta' = d' \ + \epsilon'_A + \epsilon'_B,$$

which, when subtracted, yield the change in weight $\nabla' = \bar{\Delta}' - \Delta'$. If we are considering the same r-position, first inducing and then restricting as at the end of §5, then

$$\begin{aligned}\nabla' &= \bar{\Delta}' - \Delta' = d^{*\prime} - d' - 1\\ &= (d^* + 1) - (d - 1) - 1\\ &= -(d - d^* - 1)\\ &= -(\Delta - \bar{\Delta}) = -\nabla,\end{aligned}$$

as in 5.4. We collect together these results in our

6.3 MAIN THEOREM. *The change in weight of* $[\lambda]$ *arising by adding an r-node is given by*

$$\text{(a)} \quad d - d^* - 1,$$

and by removing an r-node is given by

$$\text{(b)} \quad d^* - d - 1,$$

where d and d^* *are, respectively, the r-defect and r-affect of* $[\lambda]$.

The assumption that $[\lambda]$ is a p-core rules out the appearance of configurations such as Fig. 1 above any r-position and such as Fig. 2 below any r-position, since otherwise a kq-hook beginning or ending to the left of, or above, P would be removable and $[\lambda]$ would not be a core. For a similar reason $\delta^* = 0$ if $\delta \neq 0$. So that 6.3 (a) becomes in this case

6.4 $$\nabla = \delta - 1,$$

for the addition of an r-node. If we restrict a core for which $\delta = 0$ with $\delta^* \neq 0$, a corresponding change takes place in 6.3 (b). We prove the following interesting result:

6.5 *If the r-defect of a q-core* $[\lambda]$ *is* δ, *then the addition of* δ *r-nodes to* $[\lambda]$ *yields a q-core* $[\lambda']$.

Proof. We need only consider the $h \equiv -1 \pmod q$ which appear at the intersections of rows and columns ending in r-positions, the number of these positions being δ. Adding an r-node at each position changes each such $h \equiv -1 \pmod q$ of $H[\lambda]$ into an $h \equiv +1 \pmod q$ of $H[\lambda']$. No new $h \equiv 0 \pmod q$ appear, by 4.4. Thus $[\lambda']$ must be a q-core as required.

We state the corresponding theorem for r-restricting without proof.

6.6 *If the r-affect of a q-core* $[\lambda]$ *is* δ^*, *then the removal of* δ^* *r-nodes from* $[\lambda]$ *yields a q-core* $[\lambda']$.

Taking 6.5 and 6.6 together we have:

6.7 *Every q-core is obtainable by adding to the zero core first* δ *nodes of class r, then* δ' *nodes of class* r', *and so on, two successive values of r being necessarily distinct.*

It should be noted that the sequence of such additions for different r is not uniquely determined, so that 6.7 does not lead to a generating function for cores. Consider for example the 3-core $[4, 2^2, 1^2]$. The sequence of additions of δ nodes may be any one of the following:

6.8 $$I_0\, I_1\, I_2^2\, I_1\, I_0^3\, I_2^2, \qquad I_0\, I_2\, I_1^2\, I_2\, I_0^3\, I_2^2, \qquad I_0\, I_2\, I_1^2\, I_0^2\, I_2^3\, I_0,$$

where I_r^n indicates the addition of n nodes of class r.

There is a restriction on the choice of r for r-inducing on a q-core:

6.9 *The r-defect* δ *(r-affect* δ^**) of a given core* $[\lambda]$ *must vanish for at least one value of r.*

Proof. If the class of the end node in the first row of $\mathbf{G}[\lambda]$ is r, then no node of class r can be added to $\mathbf{G}[\lambda]$, since this would imply that a kq-hook could be removed from $[\lambda]$ beginning in the first row and ending above the supposed r-position. Thus $\delta = 0$ for at least one value of r. By a similar argument $\delta^* = 0$ for at least one value of r; this is also implied by 2.8.

7. The r-Boolean Algebra. The totality of diagrams obtained from a given diagram $[\lambda]$ by r-inducing and r-restricting at every stage constitutes a *Boolean*

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Algebra which we shall denote by rBA. To see this it is convenient to introduce the r-affect d^* as a label, writing

$$[\lambda] \equiv [\lambda^{d^*}],$$

and setting

7.1 $$d + d^* = l.$$

The diagram $[\lambda^0]$ for which $d^* = 0$ is the 0-element of rBA and $[\lambda^l]$ for which $d^* = l$ is the I-element of rBA. The *dimension* of rBA is l, while that of any given diagram is d^*. The operations $\cup$ and $\cap$ are defined in a natural manner. Clearly

$$[\lambda'] \cup [\lambda'']$$

is that diagram $[\lambda]$ of smallest dimension such that $\mathbf{G}[\lambda]$ contains $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$. Similarly

$$[\lambda'] \cap [\lambda'']$$

is that diagram $[\lambda]$ of largest dimension such that $\mathbf{G}[\lambda]$ is contained in both $\mathbf{G}[\lambda']$ and $\mathbf{G}[\lambda'']$. The existence and uniqueness of the diagram $[\lambda]$ follows in each case from the nature of our construction.

Since we are concerned here with the *weight* w and not with the linkage properties **(2)** of the diagrams of rBA, it is unnecessary to distinguish diagrams having the same dimension d^*, since all these have the same weight w, d, d^*, δ, δ^*. Tables I and II give the values of these parameters in two typical cases.

If we denote the r-defect and affect of $[\lambda^i]$ by d_i and d_i^* respectively, then the weight w_i of $[\lambda^i]$ can be obtained by repeated application of 6.3 and is readily seen to be given by one or other of the following expressions:

7.2 $$w_i = \sum_{j=0}^{i-1} (d_j - d_j^* - 1) = \sum_{j=l}^{i+1} (d_j^* - d_j - 1),$$

according as we induce from $[\lambda^0]$ upwards or restrict from $[\lambda^l]$ downwards.

From our definitions of d and d^* it follows immediately that

7.3 $$d_i - d_i^* = (d_{i+1} - d_{i+1}^*) + 2,$$

so that second differences of w are constant. Thus:

7.4 *If from the 2^l diagrams belonging to an r-Boolean Algebra a typical one be chosen of each dimension d^*, then these diagrams can be located on a line*

$$d + d^* = l,$$

when d is plotted against d^*, or on a parabola

$$w = w_l + d^*(l - d^*),$$

when w is plotted against d^*.

It should be noted that successive r-inducing(restricting) applied in *all* possible ways yields diagrams of dimension d^*, each with a multiplicity

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$d^*!(d!)$. Counting each distinct diagram once only, the number of diagrams of dimension d^* is

$$\binom{l}{d^*}$$

so that the total number of elements of rBA is 2^l, as stated in the theorem.

That the addition or removal of an r-node *commutes* with the addition or removal of any kq-hook which does not begin or end at P leads to the relation

7.5 $$d - d^* = \delta - \delta^*.$$

Proof. Since the change in weight of $[\lambda]$ for r-inducing is given by $d - d^* - 1$, this change must be accounted for by a corresponding change in weight of the core of $[\lambda]$, which, by the same argument, amounts to $\delta - \delta^* - 1$. Thus the quantities in 7.5 must be equal.

We have noted the special properties of δ, δ^* in §6, namely that if $\delta \neq 0$, then $\delta^* = 0$ and conversely. From 7.5 we have[4]

7.6 $$\begin{aligned} \delta &= \tfrac{1}{2}\{d - d^* + |d - d^*|\}, \\ \delta^* &= \tfrac{1}{2}\{d^* - d + |d - d^*|\}. \end{aligned}$$

With each diagram $[\lambda^i]$ of dimension i is associated a unique *complement* $[\lambda^{l-i}]$ of dimension $l - i$, *dual* to it in rBA. The following relations express the fundamental property of this duality relation and explain the symmetry of the tables.

7.7 $$d_i^* = d_{l-i}, \quad \delta_i^* = \delta_{l-i}.$$

Proof. The first relation is immediate. Using this and 7.6 we have:

$$\begin{aligned} \delta_i^* &= \tfrac{1}{2}\{d_i^* - d_i + |d_i - d_i^*|\} \\ &= \tfrac{1}{2}\{d_{l-i} - d_{l-i}^* + |d_{l-i} - d_{l-i}^*|\} \\ &= \delta_{l-i}. \end{aligned}$$

The examples used to illustrate these ideas in Tables I and II, have been chosen to bring out two things. In the first place, the oddness or evenness of l determines whether there is or is not a level of r-inducing in rBA where the weight remains constant. In the second place:

7.8 $$d = \delta, \qquad d^* = \delta^*,$$

for the 0 and I-elements of rBA, and one of these equalities implies the other by 7.5 or 7.7. In Table I these elements are cores. When this is not the case, as in Table II, these elements have special properties which we shall not consider here. Thus

[4]Drawn to my attention by J. S. Frame.

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7.9 *If* $d = \delta \neq 0$ *for a diagram* $[\lambda]$ *then* $d^* = \delta^* = 0$ *and* $[\lambda]$ *is the* 0*-element of an r-Boolean Algebra. Conversely, if* $d^* = \delta^* \neq 0$ *then* $d = \delta = 0$ *and* $[\lambda]$ *is the I-element of an r-Boolean Algebra. These conditions are necessary as well as sufficient.*

TABLE I

w	0	4	6	6	4	0
d^*	0	1	2	3	4	5
d	5	4	3	2	1	0
δ^*	0	0	0	1	3	5
δ	5	3	1	0	0	0

In Table I $[\lambda^0] = [8, 6, 4, 2]$, $[\lambda^5] = [9, 7, 5, 3, 1]$, $q = 3$, $r = 2$.

TABLE II

w	7	10	11	10	7
d^*	0	1	2	3	4
d	4	3	2	1	0
δ^*	0	0	0	2	4
δ	4	2	0	0	0

In Table II $[\lambda^0] = [5, 4^3, 3]$, $[\lambda^4] = [6, 5, 4^3, 1]$, $q = 2$, $r = 1$.

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