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A SIEVE ON ANALYTIC FUNCTIONS

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ABSTRACT. A sieve lemma is found, applicable to certain families of analytic functions on the unit disk, analogous to a large sieve lemma of P. X. Gallagher for sets of integers.

The following version of the large sieve is due to P. X. Gallagher [2]:

LEMMA 1. *Let A be a subset of $\{0,\ldots,N\}$ such that for each prime power q in a finite set $\mathcal{S}$, we have that A lies in at most r_q residue classes modulo q. Then*

$$|A| \le \frac{\sum_{q\in\mathcal{S}} \Lambda(q) - \log N}{\sum_{q\in\mathcal{S}} \Lambda(q)/r_q - \log N},$$

provided the denominator of the right-hand side is positive. Here $\Lambda(q)$ is von Mangoldt's function, which is $\log p$ where q is a power of p, and 0 for non-prime-powers.

Lemma 1 has various applications, several of which can be found in [3]. A careful examination of the (short) proof reveals that it uses little beyond the product formula, $\|n\|_\infty \prod_p \|n\|_p = 1$.

Consider Jensen's formula for a holomorphic function f in the disk $D(r)$ of radius r, where $f(0) \neq 0$ ([4], p. 15 or [1], p. 206):

$$0 = \log|f(0)| - \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{z\in D(r)} \operatorname{ord}_z f \cdot \log\left|\frac{R}{z}\right|.$$

Such a formula yields, by reasoning parallel to that used to prove Gallagher's lemma, the following "sieve lemma":

LEMMA 2. *Let $\mathcal{A}$ be a set of holomorphic functions on the disk $D(R)$, uniformly bounded in norm by X. Suppose that for points z in a finite set $\mathcal{S} \subset D(R)$ at most r_z values are assumed at z by functions in $\mathcal{A}$. Suppose also that for any two function $f, g \in \mathcal{A}$, $f \neq g$, we have $|f(0) - g(0)| > \epsilon$. Then*

$$|\mathcal{A}| \le \frac{\sum_{z\in\mathcal{S}} \log\left|\frac{R}{z}\right| - \log\frac{2X}{\epsilon}}{\sum_{z\in\mathcal{S}} \frac{1}{r_z}\log\left|\frac{R}{z}\right| - \log\frac{2X}{\epsilon}},$$

provided that the denominator is positive.

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PROOF. Let f and g be two distinct elements of $\mathcal{A}$. Consider Jensen's formula applied to $f-g$:

$$0 = \log|f(0)-g(0)| - \int_0^{2\pi} \log|f(Re^{i\theta})-g(Re^{i\theta})|\frac{d\theta}{2\pi} + \sum_{z\in D(R)} \mathrm{ord}_z(f-g)\cdot\log\left|\frac{R}{z}\right|$$
$$\geq \log(\epsilon) - \log(2X) + \sum_{z\in\mathcal{S}} \mathrm{ord}_z(f-g)\cdot\log\left|\frac{R}{z}\right|.$$

Averaging over all pairs of distinct $f,g\in\mathcal{A}$ and throwing away multiplicities gives us

$$0 \geq -\log\frac{2X}{\epsilon} + \frac{1}{|\mathcal{A}|(|\mathcal{A}|-1)}\sum_{f\neq g}\sum_{z:f(z)=g(z)}\log\left|\frac{R}{z}\right|$$
$$\geq -\log\frac{2X}{\epsilon} + \frac{1}{|\mathcal{A}|(|\mathcal{A}|-1)}\sum_{z\in\mathcal{S}}\sum_{\substack{f(z)=g(z)\\ f\neq g}}\log\left|\frac{R}{z}\right|.$$

The inner sum may be bounded below by using the fact that the functions of $\mathcal{A}$ have only r_z distinct values at z. By Cauchy-Schwarz, or some related inequality, the number of (not necessarily distinct) pairs (f,g) of elements of $\mathcal{A}$ that are equal at z is at least $|\mathcal{A}|^2/r_z$. Hence

$$0 \geq -\log\frac{2X}{\epsilon} + \frac{1}{|\mathcal{A}|(|\mathcal{A}|-1)}\sum_{z\in\mathcal{S}}\log\left|\frac{R}{z}\right|\left(\frac{|\mathcal{A}|^2}{r_z} - |\mathcal{A}|\right)$$
$$= -\log\frac{2X}{\epsilon} + \frac{1}{|\mathcal{A}|-1}\sum_{z\in\mathcal{S}}\log\left|\frac{R}{z}\right|\left(\frac{|\mathcal{A}|}{r_z}-1\right).$$

It follows that

$$\sum_{z\in\mathcal{S}}\log\left|\frac{R}{z}\right|\left(\frac{|\mathcal{A}|}{r_z}-1\right) \leq (|\mathcal{A}|-1)\log\frac{2X}{\epsilon},$$

or

$$\left[\sum_{z\in\mathcal{S}}\frac{1}{r_z}\log\left|\frac{R}{z}\right| - \log\frac{2X}{\epsilon}\right]|\mathcal{A}| \leq \sum_{z\in\mathcal{S}}\log\left|\frac{R}{z}\right| - \log\frac{2X}{\epsilon},$$

from which the lemma follows immediately.

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