16

Gluon emission via the bremsstrahlung process

16.1 Introduction

Bremsstrahlung emission is an inherent property of all gauge field theories. It can be understood even within classical mechanics, at least for the soft part of the spectrum. Suppose that we consider a charge surrounded by its Coulomb field, which necessarily is extended in space outside the charge. Then suppose that there is a sudden change in the state of motion of the charge itself. The result will be that the outlying field will need some time to readjust to the new situation.

Therefore there will be, as in all other situations of sudden change in physics, a brief interlude of compressions and extensions in the field before it comes back to a stable state. *The ensuing radiation field, to be described below, is a bremsstrahlung field.* Its properties depend upon the way in which the charge distribution is changed. For a single charge with a sudden momentum transfer, or for the situation when a charge and anticharge suddenly emerge, the bremsstrahlung is essentially of a *dipole character*. This approximation means that the current contains a direction, the dipole axis, but the size of the interaction region is neglected. We will consider a 'classical' current with these properties.

Some warning is needed against taking the classical picture too far. We have shown in Chapter 2 how the method of virtual quanta describes the Coulomb field of a fast-moving charge. In particular we have shown that the virtual field quanta have a distribution in rapidity and transverse momentum. In this chapter we will meet this again, as the bremsstrahlung distribution.

If we make a measurement on the field that really interacts with one of the quanta then the field will change. This will in turn (i.e. causally) also affect the current-charge itself. Therefore the bremsstrahlung process is difficult to visualise in a classical scenario, i.e. it is not possible to say

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

whether the field quanta exist before a measurement is made on them or whether they come to existence because of the measurement. We will in this chapter consider the bremsstrahlung process in some detail. We derive the cross section from first principles and express it in different ways in order to stress different properties.

Dipole bremsstrahlung contains coherence conditions, i.e. inside some regions the waves stemming from the different parts of the current will interfere constructively and in other destructively. In order to take coherence into account it is necessary to carefully preserve gauge invariance. If the interference diagrams are all taken into account then it is possible to use any gauge to evaluate the result. It turns out that in the emission of coherent dipole bremsstrahlung there is a close connection between the regions with positive interference and the regions allowed by energy-momentum conservation. As in many other cases the laws of nature ensure consistency. In this case one is evidently not allowed to emit more radiation energy than the energy carried by the charge!

These conditions imply that bremsstrahlung emission may only occur inside a chacteristic emission region, which can be expressed in terms of the transverse momenta and rapidities of the emitted quanta. In order that the conditions should be valid in any Lorentz frame these variables are most conveniently expressed in terms of Lorentz invariants. The bremsstrahlung spectrum from unpolarised charges must be independent of the azimuthal angle around the dipole axis in the rest frame. Together with the requirements on rapidity and transverse momentum this requirement translates into certain allowed conar emission regions in a moving frame. As long as one considers the emission from the full dipole these regions are easily traced.

The total bremsstrahlung from the dipole is in many models, e.g. HERWIG [94] and JETSET [105], subdivided into contributions from the individual charges. This is, of course, an allowed operation as long as one avoids double counting, i.e. the total coherence conditions are invoked. We will derive a condition referred to as the *strong angular ordering condition*, [59], in this connection.

We will also indicate that a too-literal application of strong angular ordering means that some, usually soft, emission will be displaced in phase space. Clever model builders, like the authors of the two Monte Carlo models mentioned above, have taken some precautions in this respect.

16.2 The matrix element for dipole emission

We will use a semi-classical picture and assume that the electromagnetic current **j** is suddenly changed, e.g. by an external agent. As a simple model for such a current distribution we assume the shape

$$\mathbf{j}(\mathbf{x}, t) = g\mathbf{v}(t)\delta(\mathbf{x} - \mathbf{x}(t)), \quad \mathbf{v}(t) = \frac{d\mathbf{x}}{dt}(t) \tag{16.1}$$

We also assume that $\mathbf{v}(t)$ suddenly changes at $t = 0$ from $\mathbf{v}(-\delta) = \mathbf{v}_-$ to $\mathbf{v}(+\delta) = \mathbf{v}_+$ so that we are in effect considering the case where a charged particle (charge g) moves in a pointlike way along some straight line $\mathbf{x}(t)$ (with velocity $\mathbf{v}(t) = d\mathbf{x}/dt$) and suddenly during a very short time interval $-\delta < t < +\delta$ changes to another straight-line orbit.

The number of quanta, i.e. photons, emitted with energy-momentum vector k is as usual given by Fermi's Golden Rule. By means of the methods we have used several times before we obtain, cf. Eq. (3.104)

$$dn_\gamma = \sum_{\mathbf{k}} \frac{w}{\Delta t} = \frac{|\mathscr{M}|^2}{2V\omega} \frac{Vd^3k}{(2\pi)^3} = \frac{dk}{(2\pi)^3}\delta(k^2)|\mathscr{M}|^2 \tag{16.2}$$

The transition matrix element $\mathscr{M}$ is given by

$$\mathscr{M} = \int dtd^3x\mathbf{j}(\mathbf{x}, t) \cdot \mathbf{A}(\mathbf{x}, t) \tag{16.3}$$

The vector potential $\mathbf{A}$ describes a free photon with polarisation vector $\boldsymbol{\epsilon}$, i.e. it corresponds to a transverse wave

$$\mathbf{A} = \boldsymbol{\epsilon}\exp(ikx), \quad k = (\omega, \omega\mathbf{n}), \quad \boldsymbol{\epsilon} \cdot \mathbf{n} = 0 \tag{16.4}$$

Note that the normalisation factor $1/\sqrt{2V\omega}$ already has been used in connection with Eq. (16.2). We will sometimes write the polarisation $\boldsymbol{\epsilon}$ as a four-vector.

Under these assumptions we can immediately obtain a result for $\mathscr{M}$, by means of an integration over time:

$$\begin{aligned}
\mathscr{M} &= \int dtg\mathbf{v} \cdot \boldsymbol{\epsilon}\exp[i\omega(t - \boldsymbol{n} \cdot \mathbf{x}(t))] \\
&= \int \frac{g\mathbf{v} \cdot \boldsymbol{\epsilon}}{i\omega(1 - \boldsymbol{n} \cdot \mathbf{v})} id[\omega(t - \boldsymbol{n} \cdot \mathbf{x}(t))]\exp[i\omega(t - \boldsymbol{n} \cdot \mathbf{x}(t))] \\
&= \int dtig\frac{dX}{dt}(t)\exp[i\omega(t - \boldsymbol{n} \cdot \mathbf{x}(t))]
\end{aligned} \tag{16.5}$$

where we have neglected a surface term in the integral corresponding to times well before or well after the emission and have written

$$X(t) = \frac{\boldsymbol{\epsilon} \cdot \mathbf{v}(t)}{\omega(1 - \boldsymbol{n} \cdot \mathbf{v}(t))} \tag{16.6}$$

In the second line of Eq. (16.5) we have changed the integration variable in an obvious way. The dipole approximation corresponds to the assumption that the quantity X changes much faster than the exponential in the last line of Eq. (16.5) so that we may take the exponential phase factor outside

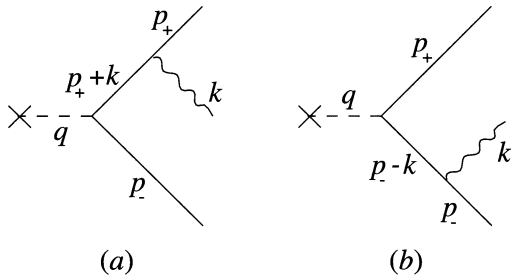


Fig. 16.1. The emission of bremsstrahlung either before or after the encounter with an external-momentum-transfer producing agent at time $t = 0$.

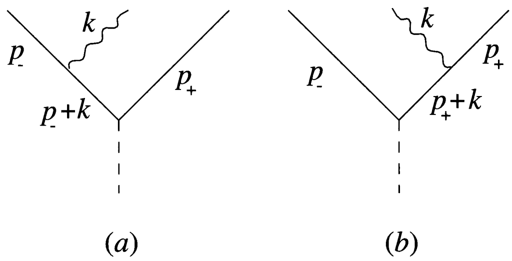


Fig. 16.2. The production of a pair at $t = 0$ and the ensuing bremsstrahlung from each of the charges.

the integral and write

$$\mathscr{M} = \exp(i\Phi)[X(+\delta) - X(-\delta)] \tag{16.7}$$

We will from now on ignore the unobservable phase factor $\exp(i\Phi)$. The two terms in Eq. (16.7) can be written, incorporating the assumptions on $\mathbf{v}$, as

$$X(\pm\delta) \equiv X^{\pm} = \frac{g\epsilon p(\pm)}{kp(\pm)}, \quad p(\pm) = (E_{\pm},\ E_{\pm}\mathbf{v}(\pm)) \tag{16.8}$$

In order to obtain this formula we have multiplied by the energies E_- and E_+ of the current particle before and after the emission in the numerators and denominators of the two terms, respectively.

The result is rather easy to interpret in terms of Feynman graphs (see Figs. 16.1 and 16.2). In Fig. 16.1(a) we see a particle coming in on-shell with energy-momentum p_-, suddenly changing during the encounter with the external agent to a virtual particle with momentum $p' = p_+ + k$, and

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

with propagator proportional to

$$\frac{1}{(p_+ + k)^2 - m^2} = \frac{1}{2p_+k} \tag{16.9}$$

and finally emitting the photon. In Fig. 16.1(b) the particle emits the photon before it meets the external agent, thereby becoming virtual with propagator proportional to

$$\frac{1}{(p_- - k)^2 - m^2} = \frac{-1}{2p_-k} \tag{16.10}$$

Similarly we may interpret Figs. 16.2 as an emission from a produced pair with charges $\pm g$ and with energy-momenta $p_\pm$,

$$\pm g \frac{\epsilon p_\pm}{(p_\pm + k)^2 - m^2} = \frac{\pm g \epsilon p_\pm}{2p_\pm k} \tag{16.11}$$

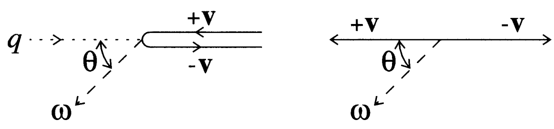
The appearance of the numerator $g\epsilon p$, i.e. a coupling between the particle momentum times the charge and the polarisation vector of the radiated photon, corresponds to the QED current-vector-potential interaction in Eq (16.3). The four-vector potential A is determined up to a gradient (cf. Chapter 2) owing to the freedom to perform local gauge transformations. Thus we may make the change (for an arbitrary $\tilde{\Lambda}$)

$$\epsilon \to \epsilon + \tilde{\Lambda}(k)k \tag{16.12}$$

It is essential to have a difference between X^+ and X^- because then each term will obtain the same contribution $\tilde{\Lambda}(k)$, which vanishes in the difference. Thus *in order that the matrix element $\mathscr{M}$ should be gauge-invariant the contributions must occur with a relative minus sign.*

We have evidently obtained the same gauge-invariant result whether we imagine a sudden change in the equations of motion of a single particle with charge g, or the equally sudden production of a pair of particles with charges $\pm g$. In both cases the Coulomb field changes, in the first by rebuilding and in the second by starting up. We will come back to this picture again in Chapter 20. For now we note that the result that the matrix elements of the two processes are the same is a general one in relativistic quantum field theory, corresponding to the property of *crossing symmetry* for the S-matrix.

A final remark at this point is that it is often dangerous to make too-literal interpretations of Feynman graph techniques. According to the discussion in Chapter 3 the Feynman propagator is completely satisfactory with respect to the requirements of Lorentz covariance, causality and the quantum conditions put up by the Heisenberg indeterminacy relations. In that case, however, the interpretations are well defined.



(*a*) (*b*)

Fig. 16.3. The two different situations when bremsstrahlung γ's are emitted either (*a*) from a single charge bouncing back (the Breit frame) or (*b*) in connection with the production of a pair in the cms.

16.3 The dipole cross section

1 *The dependence on energy and rapidity*

We will provide several forms for the cross section in Eq. (16.2) in order to stress different properties of the process. We will start with a description in terms of the photon's energy ω and rapidity y:

$$y = \frac{1}{2} \log\left(\frac{\omega + k_3}{\omega - k_3}\right) = \frac{1}{2} \log\left(\frac{1 + \cos\theta}{1 - \cos\theta}\right) = \log\cot(\theta/2) \quad (16.13)$$

where θ is the angle between the dipole (3-)axis and the photon direction $\mathbf{n}$. We will for now work in a Lorentz frame in which the two particle momenta, $\mathbf{p}(_\pm)$, are along the 3-axis, equal in size and oppositely directed.

This means that when there is a sudden momentum transfer to bring the incoming state particle with $\mathbf{p}(_+)$ to the final state $\mathbf{p}(_-)$ this momentum transfer must be directed oppositely to the particle's momentum so that it comes in and bounces back again (in the Breit frame). For the case when a pair, $(\mathbf{p}(_+), \mathbf{p}(_-))$, is produced the two particles move in opposite directions along the 3-axis (see Fig. 16.3). The rapidities of the two particles will be called $\pm y_0$.

We next choose two independent directions to describe the polarisation vector $\boldsymbol{\epsilon}$. For simplicity we choose one of them to be in the plane of $\mathbf{n}$ and the 3-axis and the other out of this plane. Therefore we will only obtain a contribution to the matrix element from the one in the plane. That contribution is given by

$$\boldsymbol{\epsilon} \cdot \mathbf{v}(_\pm) = \pm \tanh y_0 \frac{1}{\cosh y} \quad (16.14)$$

where we have used the formula

$$\sin\theta = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sin^2(\theta/2) + \cos^2(\theta/2)} = \frac{1}{\cosh y} \quad (16.15)$$

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

(An exercise for the reader: prove that one may chose the two polarisation directions in any orthogonal way and still obtain the same result!).

For the denominators in the matrix element we obtain directly

$$\omega(1 - \mathbf{n} \cdot \mathbf{v}_{(\pm)}) = \omega(1 \mp \tanh y_0 \tanh y) \tag{16.16}$$

using a similar trick to express $\cos\theta$ in terms of the rapidity variable y. Putting it together we obtain the result of summing over the polarisation directions:

$$|\mathscr{M}|^2 = \frac{2\cosh 2y_0 \cosh^2 y}{\omega^2 \cosh(y + y_0)\cosh(y - y_0)} \tag{16.17}$$

The size of a phase-space element is

$$dk\delta(k^2) = \tfrac{1}{2}\omega d\omega d\phi \sin\theta d\theta \tag{16.18}$$

where ϕ is the azimuthal angle (over which we can evidently integrate to give 2π) and where the θ-dependence easily transforms to a rapidity dependence:

$$\sin\theta d\theta = \frac{dy}{\cosh^2 y} \tag{16.19}$$

Therefore the number of γ's per unit energy and unit rapidity is

$$dn_\gamma = \left(\frac{g^2}{4\pi^2}\right)\frac{d\omega}{\omega} dy \frac{\cosh(y_0 + y_0)}{\cosh(y + y_0)\cosh(y - y_0)} \tag{16.20}$$

The somewhat fancy way we have used to write the arguments in the hyperbolic sine and cosine functions is made in order to exhibit the Lorentz invariance of the formula: it only depends upon the rapidity differences $y_0 - (-y_0)$ and $y - (\pm y_0)$. Therefore it is the same in any Lorentz frame obtained by boosting along the dipole axis.

A closer examination of the rapidity-dependent factor also reveals that it is basically a constant for rapidities

$$|y| < |y_0| \tag{16.21}$$

and that it falls off exponentially fast outside this region. Therefore the spectrum is, to a good approximation

$$dn_\gamma = \left(\frac{2\alpha}{\pi}\right)\frac{d\omega}{\omega} dy \tag{16.22}$$

where the requirement in Eq. (16.21) must be incorporated. We have here as usual introduced the fine structure constant $\alpha = g^2/4\pi$ under the assumption that we are dealing with electrons and positrons. We will shortly come back to the difference when we consider color-charged q- and $\bar{q}$-particles.

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

If we rewrite the energy dependence in terms of a transverse momentum dependence for fixed rapidity,

$$k_\perp \equiv \omega \sin\theta = \frac{\omega}{\cosh y} \tag{16.23}$$

we obtain the formula found in connection with the method of virtual quanta in Chapter 2:

$$dn_\gamma = \frac{\alpha}{\pi}\frac{dk_\perp^2}{k_\perp^2}dy \tag{16.24}$$

(we may of course also make the change $dy \to dx/x$ in the same way).

Consequently, *the bremsstrahlung spectrum arising from a change in the current distribution is equivalent to the flux of virtual quanta which can be used to describe the electromagnetic field around a fast-moving charge.* Quantum mechanics does not tell you before you measure what you may find in your detector!

2 *The invariant cross section for dipole emission*

If we neglect the particle masses we may write for the two denominators in Eq. (16.8), using the conventional notation and so calling the positive charged particle's energy-momentum $p(_+) \equiv p_1$, that of the negative charge $p(_-) \equiv p_3$ and that of the emitted photon $k \equiv p_2$:

$$p(_+)k \equiv \frac{s_{12}}{2}, \quad p(_-)k \equiv \frac{s_{23}}{2} \tag{16.25}$$

where we have introduced the squared masses of the particle pairs.

Squaring the matrix element and summing over the polarisation directions we obtain for the polarisation sum (cf. Eq. (4.40))

$$\sum_{polarisation} \epsilon_j \epsilon_l = \delta_{jl} - \frac{k_j k_l}{\mathbf{k}^2} \tag{16.26}$$

There will then be three terms in the squared matrix element. The first one can be written as (using $\sqrt{\mathbf{k}^2} \equiv \omega$)

$$\frac{4\left[(\mathbf{p}_1)^2 - (\mathbf{p}_1 \cdot \mathbf{k})^2/\mathbf{k}^2\right]}{s_{12}^2} = -\frac{1}{\omega^2} + \frac{4E_1}{\omega s_{12}} \tag{16.27}$$

The second term is the same but with the obvious exchange of index $1 \to 3$. The third becomes

$$\frac{2}{\omega^2} + \frac{8p_1p_3}{s_{12}s_{23}} - \frac{4E_1}{\omega s_{12}} - \frac{4E_3}{\omega s_{23}} \tag{16.28}$$

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

so that the total result is

$$\sum_{polarisation} |\mathscr{M}|^2 = \frac{4s_{13}}{s_{12}s_{23}} \tag{16.29}$$

For the phase-space factors we obtain, by fixing the two squared masses s_{12} and s_{23},

$$\int dk\delta(k^2)\delta(2kp_1 - s_{12})\delta(2kp_3 - s_{23}) = \frac{\pi}{2s_{13}} \tag{16.30}$$

Then the total γ-multiplicity is given by

$$dn_\gamma = \left(\frac{\alpha}{\pi}\right)\frac{ds_{12}ds_{23}}{s_{12}s_{23}} \tag{16.31}$$

Although the result in Eq. (16.31) is derived by semi-classical methods it agrees in detail with the results of a complete quantum mechanical calculation *for soft γ-radiation.* But when it comes to hard bremsstrahlung, i.e. when $s_{13} \simeq s_{12}$ and/or s_{23}, there are corrections. The formula used for the current, cf. Eq. (16.1), in the calculation of the matrix element does not account for the fact that the electrons and positrons are spin 1/2 particles. There are then, just as in connection with the Rutherford scattering matrix elements in section 5.5, also contributions from the spin structure. Further, the treatment of the phase space in Eq. (16.2) leading to Eq. (16.31) is also too simple. As subsequently we will need a formula also for the hard radiation we will briefly exhibit the steps necessary to obtain a more precise formula.

Firstly, the current in Eq. (16.1) should be changed as follows:

$$\frac{g}{E}\mathbf{p}\delta(\mathbf{x} - \mathbf{x}(t)) \to \frac{g}{E}(\mathbf{p} + \boldsymbol{\sigma} \times \nabla)\delta(\mathbf{x} - \mathbf{x}(t)) \tag{16.32}$$

(we have for simplicity written $\mathbf{v} \equiv d\mathbf{x}(t)/dt = \mathbf{p}/E$) with $\boldsymbol{\sigma}$ describing the spin (cf. section 4.4) of the fermions. It is an axial vector, which means that the term $\boldsymbol{\sigma} \times \nabla$ corresponds to a proper vector, as is $\mathbf{p}$, and therefore it is an 'allowed' contribution to the current in a parity conserving theory. Further for a massless fermion the helicity can only take on two values ($\pm 1/2$) corresponding to spin 'along' and 'opposite to' the direction of motion and we must sum over the two values in the final squared matrix element if we have unpolarised fermions.

It is evident that this extra contribution will change the result in Eq. (16.8) into

$$X = \frac{\boldsymbol{\epsilon} \cdot (\mathbf{p} + i\boldsymbol{\sigma} \times \mathbf{k})}{pk} \tag{16.33}$$

(with appropriate indices). When we square the matrix element using this expression for the X-factors we obtain extra terms as compared to Eq.

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

(16.29), which, after summing over the photon polarisations according to Eq. (16.26), will be

$$\begin{aligned}\mathscr{D}_1(\pm) &= \frac{(\boldsymbol{\sigma}(_\pm) \times \mathbf{k})^2}{(p(_\pm)k)^2} \\ \mathscr{D}_2 &= -2\frac{[\boldsymbol{\sigma}(_+) \times \mathbf{k}] \cdot [\boldsymbol{\sigma}(_-) \times \mathbf{k}]}{(p(_+)k)(p(_-)k)}\end{aligned} \tag{16.34}$$

(the remaining interference terms, as e.g. those proportional to

$$\mathbf{p}(_\pm) \cdot [\boldsymbol{\sigma}(_\pm) \times \mathbf{k}]$$

vanish because $\mathbf{p}(_\pm) \times \boldsymbol{\sigma}(_\pm) = 0$ as we mentioned above). The result in Eq. (16.34) should then be summed over the possible values of $\boldsymbol{\sigma}(_\pm)$; only the two quantities $\sum_{spins} \mathscr{D}_1(_\pm) \to k_\perp^2/[p(_\pm)k]^2$ are nonvanishing (with $k_\perp^2$ defined in Eq. (16.39) below). Therefore the result in Eq. (16.29) is changed as follows:

$$\frac{4s_{13}}{s_{12}s_{23}} \to \frac{4s_{13}}{s_{12}s_{23}} + \frac{2s_{12}}{ss_{23}} + \frac{2s_{23}}{ss_{12}} \tag{16.35}$$

For soft radiation the last two terms are negligible compared to the first term.

Secondly, the phase-space factor in Eq. (16.2) should be exchanged for the three-particle phase space we obtained in Eq. (4.14) (with the modification that we have defined this phase space with a factor $(2\pi)^3$ too large according to Eq. (4.4)). Putting it all together (with the right numerical factors) and introducing the squared pair-masses in terms of the x_j-variables:

$$x_j = \frac{2E_j}{\sqrt{s}}, \quad \sum_{j=1}^{3} x_j = 2 \tag{16.36}$$

For example, we have for s_{12}

$$s_{12} = (p_1 + k)^2 = (P_{tot} - p_3)^2 = s - 2P_{tot}p_3 = s(1 - x_3) \tag{16.37}$$

We obtain after straightforward algebra (note that $2(1 - x_2) + (1 - x_1)^2 + (1 - x_3)^2 = x_1^2 + x_3^2$)

$$dn_\gamma = \frac{\alpha}{2\pi} \frac{x_1^2 + x_3^2}{(1 - x_1)(1 - x_3)} dx_1 dx_3 \tag{16.38}$$

The introduction of the fermion spin means that we exchange 1 for $(x_1^2 + x_3^2)/2$ but the new factor is in general close to unity because of the two pole factors in Eq. (16.38). In section 17.7, when we consider collinear bremsstrahlung, we will discuss the results of this modification.

We end this subsection with a few comments. We firstly note that, while the *spin* (for massless particles) is along (or opposite to) the direction of

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

motion, the *polarisation* of the current, i.e. the added cross-product in Eq. (16.33), is transverse to this direction. This is the same behaviour as for the electromagnetic fields.

A vector product $\mathbf{a} \times \mathbf{b}$ is not a true vector but instead describes the components of an antisymmetric tensor (which has the same transformation properties with respect to rotations as a vector but is different with respect to space reflections). There is a single 3-tensor $\epsilon_{jlm} = \pm 1$, depending upon whether the permutation jlm among the numbers 123 is even or odd (e.g. 231 is even and 213 is odd); the axial vector $(\mathbf{a} \times \mathbf{b})_j$ can be written as $\epsilon_{jlm} a_l b_m$ with a sum over repeated indices. Actually this latter quantity can also be described as the $\mu = 0$ (the 'time') component of the antisymmetric 4-tensor $\epsilon_{\mu\nu\sigma\lambda}$, which is defined in the same way in terms of the four indices 0123. It is interesting to note that the relationship to the electromagnetic fields can in this way be taken even further because the polarisation term of the current is then $\epsilon_{0jlm}\sigma_l \nabla_m$. The polarisation of the electromagnetic field is conventionally taken along the electric field $\mathscr{E}_j$; this is likewise the $0j$-component of the field tensor.

The use of an axial vector to describe the polarisation also means some loss of gauge and Lorentz invariance (although these symmetries may be restored by a more elaborate formalism). But while the current term based upon the true vector $(\mathbf{p}, E) \propto (d\mathbf{x}(t)/dt, 1)\delta(\mathbf{x} - \mathbf{x}(t)$ may easily be seen to fulfil a Lorentz-invariant current conservation requirement, $\nabla \mathbf{j} + \partial j_0/\partial t = 0$, the added axial vector term obeys only space-current conservation $\nabla \mathbf{j} = 0$ as well as the corresponding invariance under 'transverse' gauge transforms $\boldsymbol{\epsilon} \cdot \mathbf{j} \equiv (\boldsymbol{\epsilon} + i\mathbf{k}\Lambda) \cdot \mathbf{j}$. Nevertheless we may use the shape of the current we have introduced above to derive the tensors $(T_1 + T_2)_{\mu\nu} \propto \sum_{spins} \langle 0 | j_\mu | k_1, k_2 \rangle \langle k_1, k_2 | j_\nu | 0 \rangle$, which we discussed in section 4.4. (What are the necessary normalisation factors?)

3 *The invariant transverse momentum, the rapidity and phase space*

It is useful to introduce the invariant transverse momentum and rapidity for the photon,

$$\begin{aligned} k_\perp^2 &= \frac{s_{12}s_{23}}{s} \equiv s(1 - x_1)(1 - x_3) \\ y &= \frac{1}{2} \log\left(\frac{1 - x_1}{1 - x_3}\right) \end{aligned} \tag{16.39}$$

in terms of which we may obtain the inclusive photon multiplicity distribution from Eq. (16.31),

$$dn_\gamma = \frac{\alpha}{\pi} \frac{dk_\perp^2}{k_\perp^2} dy \tag{16.40}$$

https://doi.org/10.1017/9781009401296.016 Published online by Cambridge University Press

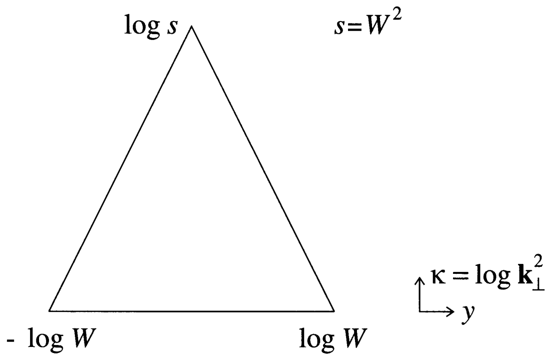


Fig. 16.4. The phase space for photon emission in terms of the logarithmic variables κ and y described in the text.

i.e. the same result as in Eq. (16.24). This time it is, however, expressed in terms of invariants.

The total phase space is, in terms of the invariants $k_\perp$ and y (we will from now on only use the variables in that sense so we drop the word invariant),

$$s \geq s_{12} + s_{23} = 2k_\perp \sqrt{s} \cosh y \tag{16.41}$$

which can be conveniently approximated by

$$|y| \leq (L - \kappa)/2 \tag{16.42}$$

with $\kappa = \log(k_\perp^2/s_0)$ and $L \equiv \kappa(k_\perp^2 = s)$. Here s_0 is some scale which is not determined by our present considerations. We note that the phase space has in this way changed to the interior of a triangle (Fig. 16.4). The meaning of the cross section is evidently now that there is a density of photons given by α/π inside the triangular phase space because the cross section is $dn_\gamma = \alpha d\kappa dy/\pi$.

We will use this picture extensively in the following. The energy-momentum conservation requirement in Eqs. (16.41) and (16.42) is evidently very similar to the results from the coherence calculations in Eq. (16.21). In that case we found that radiation is only allowed inside a certain (pseudo)rapidity region determined by the rapidity of the emitters.

The result in Eq. (16.41) is valid for massless emitters. For massive ones, there should be a region, close to the rapidity endpoints for the massless case, where there is suppression for photon emission. Although we will not consider this situation we note that dipole emission is only allowed within an angular cone which is characteristic for the dipole.

It is worthwhile to note that the $k_\perp$-variable in Eq. (16.39), although defined in a rather abstract way, is nevertheless a reasonable measure of the 'true' transverse momentum of the photon with respect to some dynamical axis. This is in particular so if the photon is soft. We will now show that there is always a direction **e** such that the transverse component of the cms momentum of the γ with respect to **e** is equal to $k_\perp$. If the angle between the γ's momentum direction and **e** is θ we obtain the requirement

$$E_2^2 \sin^2\theta = k_\perp^2 = s(1-x_1)(1-x_3) \tag{16.43}$$

Using the relation in Eq. (16.36) to express x_2 we obtain

$$\tan^2\theta = \frac{4(1-x_1)(1-x_3)}{(x_1-x_3)^2} \tag{16.44}$$

This means that e.g. when the electron and positron afterwards have the same energies then the direction of **e** is at an angle $\pi/2$ to the γ's direction and the whole γ-momentum is transverse to **e**. Note, however, that in order to conserve momentum the charged emitters will afterwards also move at an angle to the **e**-axis (for the recoil problems in the emissions, cf. section 17.8).

We have in Chapter 4 described the changes necessary when we go from QED to QCD. The number of color configurations which contribute for a color-$(3,\bar{3})$ dipole is $N_c - 1/N_c$ with $N_c = 3$ the number of colors. There is also the unfortunate definition of the QCD charge to take into account so that we should change α_{QED} to $(N_c - 1/N_c)\alpha_s/2$; all in all this leads to

$$\begin{aligned} \alpha_{QED} &\to \alpha_{QCD,effective}(q\bar{q} \to qg\bar{q}) = \frac{4\alpha_s}{3} \\ dn_g &= \left(\frac{2\alpha_s}{3\pi}\right)\frac{x_1^2 + x_3^2}{(1-x_1)(1-x_3)}dx_1 dx_3 \end{aligned} \tag{16.45}$$

16.4 The antenna pattern of dipole emission

In this section we will describe the physics corresponding to the *strong angular ordering condition* [59]. We assume that the dipole is boosted transversely to its axis as described by Figs. 16.5(*b*) and (*c*). This means that the angle between the directions of motion of the charges is no longer π as in the rest frame but 2ψ, with $v = \cos\psi$ as the relative velocity of the frames.

From Eqs. (16.2) and (16.29) we may obtain an angular emission pattern, which is called the *antenna pattern* in [27]. When this is expressed in angular variables (or rather in the scalar products betwen unit vectors) we obtain, using e.g. $s_{12} = 2E_1k(1 - \mathbf{n}_1 \cdot \mathbf{n}_2)$, the following angular dependence

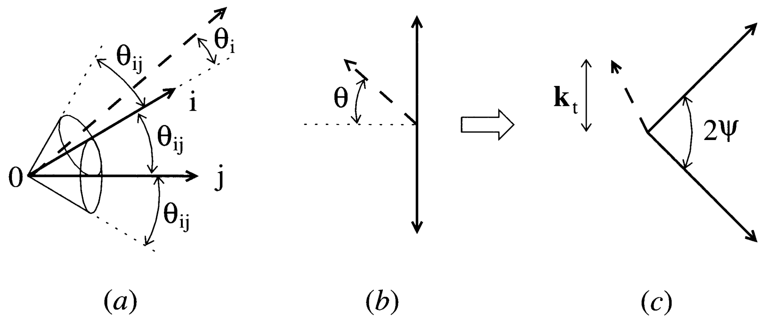


Fig. 16.5. (a) The angular emittance cones around the partons i and j and the region O with no emission according to the strong angular condition. (b) A gluon with a certain k_t is emitted at an angle $\pi/2-\theta$ from a dipole in its cms. (c) The system is boosted to a frame with velocity $v=\cos\psi$, for the notation see the text.

on the gluon direction (note $dk\delta^+(k^2)=kdk\sin\theta\, d\theta d\phi/2$):

$$W_{1,3}(\mathbf{n})=\frac{a_{13}}{a_1a_2}$$
$$a_{ij}=1-\mathbf{n}_i\cdot\mathbf{n}_j,\quad a_{i2}\equiv a_i,\quad i=1,3 \tag{16.46}$$

The angular distribution $W_{1,3}$ contains a dependence both on the relative angle between the two emitters 1 and 3, to be called θ_{13}, and on the polar and azimuthal angles θ and ϕ of the emitted g, see Fig. 16.5(a). It can be written as a sum of two terms:

$$W_{1,3}=U_{1,3}+U_{3,1},\quad U_{i,j}=\frac{1}{2}\left(\frac{1}{a_i}+\frac{a_{ij}-a_i}{a_ia_j}\right) \tag{16.47}$$

We can calculate the polar angle θ with respect to either the i-direction or the j-direction; an index on θ will indicate which one we are using. For the expression $U_{i,j}$ we note that if we fix θ_i and θ_{ij} the only azimuthal angular dependence is that of the second term in the bracket.

The numerator of the second term is $2[\sin^2(\theta_{ij}/2)-\sin^2(\theta_i/2)]$. This provides a positive or negative contribution depending upon whether θ_i is smaller or larger than θ_{ij}. It is useful for the reader to check for himself/herself that the partitioning is done in such a way that this numerator will have no pole in $U_{i,j}$ if $\theta_j=0$.

The expression for $U_{i,j}$ is therefore only large when the emitted g is close in angle to the parton i. The same is evidently also valid for the corresponding term $U_{j,i}$ with respect to the parton j.

One may integrate $U_{i,j}$ over all values of ϕ and obtain

$$\int \frac{d\phi}{2\pi} U_{i,j} = \frac{\Theta(\theta_{ij} - \theta_i)}{a_i} \tag{16.48}$$

This means that *the average emission from the term $U_{i,j}$ is the same as if there had only been emission from the parton i inside a cone such that the following angular relation in the Θ-distribution is fulfilled*:

$$\theta_{ij} \geq \theta_i \tag{16.49}$$

Thus the two terms in $U_{i,j}$ turn out to give equal and opposite contributions outside this 'mother' cone. This makes it possible to interpret the dipole emission formula as terms of independent emission from either the i- or the j-parton.

We will now investigate the way in which such an angular condition works. We consider Fig. 16.5 and first concentrate on the condition for emission from the parton i. The condition in Eq. (16.49) then means that i can emit inside the upper angular cone around the direction i. Similarly j can emit inside the lower angular cone around j.

Therefore both can emit in the region between them and neither can emit in the region indexed O. Due to the partitioning above we also know that the amount which is lost inside the region O is gained in between the partons. Thus a literal use of the angular condition means that some, in general soft, gluon radiation is 'misplaced' in phase space.

In order to inform ourselves about the size of this problem we will make the following calculation. We assume that in its rest frame a dipole emits a gluon at an angle $\pi/2 - \theta$ to the dipole axis; see Fig. 16.5(b). Then in a coordinate system in which the dipole moves with velocity $v = \cos\psi$ transverse to its axis and at an azimuthal angle π with respect to the gluon, (see Fig. 16.5(c)) all those gluons with $\theta \leq \theta_{max}$, where

$$\cos\theta_{max} = \frac{\cos\psi - \cos 3\psi}{1 - \cos\psi\cos 3\psi} \tag{16.50}$$

will be forbidden by the strong angular condition.

This means that when $v \to 1$ then $\theta_{max} \to 0.64$ while $\theta_{max} = 0$ for $v = 0.5$. *The strong angular condition is an inclusive statement* in the sense that if all possible gluon emissions are allowed then the errors compensate. In a Monte Carlo simulation of single events the errors can be appreciable event for event, however.

A clever model builder can to some extent compensate for the error. In particular the most popular Monte Carlo models on the market, JETSET [105], HERWIG [94] and ARIADNE [92] implement the full dipole matrix element in the first emission (in different ways). While in ARIADNE emission continues according to the dipole formula, in the other programs the conar conditions are applied later on in the cascades. If the conar conditions are neglected, however, then there will be considerable double-counting and far too many gluons emitted.