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THE HAUSDORFF DIMENSION OF AN ERGODIC INVARIANT MEASURE FOR A PIECEWISE MONOTONIC MAP OF THE INTERVAL

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ABSTRACT. We consider a piecewise monotonic and piecewise continuous map T on the interval. If T has a derivative of bounded variation, we show for an ergodic invariant measure μ with positive Ljapunov exponent λ_μ that the Hausdorff dimension of μ equals h_μ / λ_μ.

1. **Introduction.** A map T on the interval $[0, 1]$ is called piecewise monotonic, if there exists a finite or countable set $\mathcal{Z}$ of pairwise disjoint open subintervals of $[0, 1]$ such that $T|Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. If $\mathcal{Z}$ is finite, we say that T is a map with finitely many monotonic pieces. It suffices to define T on $\bigcup_{Z\in\mathcal{Z}} Z$. We are interested in $E_{\mathcal{Z}} := \bigcap_{i=0}^{\infty} T^{-i}(\bigcup_{Z\in\mathcal{Z}} Z)$, the set of all points, for which all iterates of T are defined. We consider an ergodic T-invariant measure μ concentrated on $E_{\mathcal{Z}}$. Its Ljapunov exponent is denoted by λ_μ and its entropy by h_μ. The Hausdorff dimension $\mathrm{HD}(\mu)$ of μ is defined as the infimum of the Hausdorff dimensions $\mathrm{HD}(X)$, the infimum taken over all $X \subset [0, 1]$ with $\mu(X) = 1$. The aim of this paper is to show $\mathrm{HD}(\mu) = h_\mu / \lambda_\mu$.

In order to give the exact statement of the result we define p-variation. For $p > 0$, $g: [0, 1] \longrightarrow \mathbb{R}$ and a subinterval $[a, b]$ of $[0, 1]$ define

$$\mathrm{var}^p_{[a,b]}\, g = \sup\{ \sum_{i=1}^{m} |g(x_{i-1}) - g(x_i)|^p : m \in \mathbb{N}, a \leq x_o < x_1 < \cdots < x_m \leq b\}$$

We say that a piecewise monotonic map has a derivative of bounded p-variation, if there exists a function $g: [0, 1] \longrightarrow \mathbb{R}$ with $\mathrm{var}^p_{[0,1]}\, g < \infty$ and $g(x) = 0$ for $x \in [0, 1] \setminus \bigcup_{Z\in\mathcal{Z}} Z$ such that $T|Z$ is an antiderivative of $g|Z$ for $Z \in \mathcal{Z}$. We denote g by T'. Furthermore, if μ is an ergodic T-invariant measure on $E_{\mathcal{Z}}$, we can define the Ljapunov exponent λ_μ by $\int \log |T'|\, d\mu$. The result of this paper is then

THEOREM 1. *Let T be a map on $[0, 1]$ with finitely many monotonic pieces and a derivative of bounded p-variation for some $p > 0$. If μ is an ergodic T-invariant measure with Ljapunov exponent $\lambda_\mu > 0$, then $\mathrm{HD}(\mu) = h_\mu / \lambda_\mu$.*

REMARK. Actually we shall show a bit more. Theorem 1 also holds, if T is piecewise monotonic with the following additional property: The set of all x, which satisfy $x \in T(Z)$ for infinitely many $Z \in \mathcal{Z}$, is at most countable. If T has finitely many monotonic pieces, there are no such x. Furthermore, this property is also satisfied, if $T: [0, 1] \longrightarrow [0, 1]$ is

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continuous and if the endpoints of the intervals in $\mathcal{Z}$ have only finitely or countably many limit points.

A proof of this formula for C^1-maps T on $[0, 1]$ under additional conditions is given in the appendix of [4], using a result about entropy proved in [1]. If discontinuities are allowed for T, one has the problem that the image of an interval might not be an interval. Hence the proof in this paper follows different lines than that in [4] . In Section 2 we show that, under certain conditions, the Hausdorff dimension does not change, if one only allows certain intervals in its definition. These intervals are chosen such that they behave well with respect to the discontinuities of T. In Sections 3 and 4, this and certain techniques about piecewise monotonic maps, for example Markov extensions, are used to prove the desired formula.

Theorem 1 is trivial, if μ is concentrated on a periodic orbit. If this does not happen, then μ has no atoms, as μ is ergodic. We shall assume this throughout the paper.

Next we prove a lemma which shows how the assumptions of Theorem 1 will be used in the proofs of this paper. A function g: $[0, 1] \longrightarrow \mathbb{R}$ is said to be regular, if $g(x+) := \lim_{y\downarrow x} g(y)$ and $g(x-) := \lim_{y\uparrow x} g(y)$ exist for all $x \in [0, 1]$. It is easy to see that a function is regular, if it is of bounded p-variation. A family $\mathcal{Y}$ of disjoint open subintervals of $[0, 1]$ is called a μ-partition, if $\mu(\bigcup_{Y\in\mathcal{Y}} Y) = 1$.

LEMMA 1. *Let* $\psi\colon [0, 1] \longrightarrow [0, \infty)$ *satisfy* $\operatorname{var}^p_{[0,1]} \psi < \infty$ *and* $\psi(0) = \psi(1) = 0$. *Let* μ *be a probability measure on* $[0, 1]$ *such that* $\varrho := \log \psi \in L^1(\mu)$. *Then, for every* $\varepsilon > 0$ *there is a* μ*-partition* $\mathcal{Y}$ *of* $[0, 1]$ *into intervals, such that*

$$\sup_{x,y\in Y} |\varrho(x) - \varrho(y)| \leq \varepsilon \text{ for all } Y \in \mathcal{Y} \tag{1.1}$$

and such that

$$-\sum_{Y\in\mathcal{Y}} \mu(Y) \log \mu(Y) < \infty \tag{1.2}$$

PROOF. Let I be a subinterval of $[0, 1]$. Then we have $\operatorname{var}^p_I \varrho \leq (\sup \frac{1}{x})^p \operatorname{var}^p_{[0,1]} \psi$, where the supremum is taken over all x in the image of I under ψ. For $\alpha > 0$ and $c := \operatorname{var}^p_{[0,1]} \psi$ we get that

$$\psi \geq \alpha \text{ on } I \Rightarrow \operatorname{var}^p_I \varrho \leq c\alpha^{-p} \tag{1.3}$$

As ψ is regular, we can define $N = \{z \in [0, 1] : \psi(z) = 0, \psi(z+) = 0 \text{ or } \psi(z-) = 0\}$. Then N is closed and contains 0 and 1. Hence $[0, 1] \setminus N$ is the disjoint union of finitely or countably many open intervals. We denote the set of these intervals by $\mathcal{X}$. Since $\int \log \psi \, d\mu$ exists, it follows that $\mu(N) = 0$ and $\mathcal{X}$ is a μ-partition. In order to construct $\mathcal{Y}$, which will be a refinement of $\mathcal{X}$, we define w: $\bigcup_{J\in\mathcal{X}} J \longrightarrow \mathbb{R}$ as follows. For each $J \in \mathcal{X}$ fix an $u_J \in J$. Set $w(x) = \operatorname{var}^p_{[u_J,x]} \varrho$ for $x \in J$, $x \geq u_J$ and set $w(x) = -\operatorname{var}^p_{[x,u_J]} \varrho$ for $x \in J$, $x \leq u_J$. Then $w|J$ is increasing for every $J \in \mathcal{X}$. If I is a compact subinterval of some $J \in \mathcal{X}$, then ψ is bounded away from zero in I by definition of N and w is bounded on I

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by (1.3). Hence w is finite on each $J \in \mathcal{X}$. If $J = (a, b)$ then $w(a+) = -\infty$, if and only if $\psi(a+) = 0$ and $w(b-) = \infty$, if and only if $\psi(b-) = 0$. We can assume that ε is less than one. Fix some $J \in \mathcal{X}$. For $i \in \mathbb{Z}$ let y_i be an inverse image of $i\varepsilon^p$ under the monotonic function $w|J$. Here we say that y is an inverse image of x, if $w(y-) \leq x \leq w(y+)$. As $w|J$ is increasing, there are $\sigma \geq -\infty$ and $\tau \leq \infty$ such that y_i exists for $\sigma < i < \tau$. If $J = (a, b)$, we have $\sigma = -\infty$, if and only if $\psi(a+) = 0$ and $\tau = \infty$, if and only if $\psi(b-) = 0$. If $\sigma > -\infty$, we set $y_\sigma = a$, and if $\tau < \infty$, we set $y_\tau = b$. The intervals (y_j, y_{j+1}) form a partition of J up to a countable set and hence a μ-partition, since μ has no atoms. We do this for all $J \in \mathcal{X}$. As $\mathcal{X}$ is a μ-partition, we get in this way a μ-partition $\mathcal{Y}$ refining $\mathcal{X}$. By definition of w and $\mathcal{Y}$ we have (1.1).

It remains to show (1.2). As ψ is bounded, there is an $r \in \mathbb{Z}$ with $\sup_{[0,1]} \varrho \leq r+1$. We construct a partition $(\mathcal{W}_{j,l})_{1 \leq j \leq q(l), -\infty < l \leq r}$ of $\mathcal{Y}$, such that $J_{j,l} := \bigcup_{Y \in \mathcal{W}_{j,l}} \overline{Y}$ is an interval for $1 \leq j \leq q(l)$ and $-\infty < l \leq r$, such that $l-1 \leq \varrho(x) \leq l+1$ for all $x \in \bigcup_{j=1}^{q(l)} J_{j,l}$, where $-\infty < l \leq r$, and such that there is a constant b satisfying $\sum_{j=1}^{q(l)} \operatorname{card} \mathcal{W}_{j,l} \leq be^{-2p(l-1)}$ for $-\infty < l \leq r$. To this end set $\tilde{\mathcal{W}}_r = \{Y \in \mathcal{Y} : \inf_Y \varrho > r - 1\}$. Let $(\tilde{\mathcal{W}}_{j,r})_{j \geq 1}$ be the coarsest partition of $\tilde{\mathcal{W}}_r$, such that $\tilde{J}_{j,r} := \bigcup_{Y \in \tilde{\mathcal{W}}_{j,r}} \overline{Y}$ is a subinterval of the closure of some element of $\mathcal{X}$. Cancel those $\tilde{\mathcal{W}}_{j,r}$ which satisfy $\sup_{\tilde{J}_{j,r}} \varrho \leq r$. The remaining $\tilde{\mathcal{W}}_{j,r}$ are the sets $\mathcal{W}_{j,r}$. It follows that $r - 1 < \varrho \leq r + 1$ holds on $\bigcup_{j \geq 1} J_{j,r}$. Furthermore, for every $j \geq 1$, there is a point in $J_{j,r}$, whose value under ϱ is larger than r, and for each endpoint of $J_{j,r}$ there is a $Y \in \mathcal{Y}$ with $Y \not\subset J_{j,r}$, which has this point as a common endpoint with $J_{j,r}$ and which contains a point in its closure, whose value is less than or equal to $r-1$ (if we go through this step with m instead of r, for each end point of $J_{j,m}$ either the above holds or there is a $J_{i,s}$ with $1 \leq i \leq q(s)$ and $m < s \leq r$, which has this point as a common endpoint with $J_{j,m}$, such that the $Y \in \mathcal{Y}$, which is contained in $J_{j,m}$ and adjacent to $J_{i,s}$ contains a point in its closure, whose value is less than or equal to $s-1$). Hence, if there are n different $J_{j,r}$, we get $(2n-1)(e^r - e^{r-1})^p \leq \operatorname{var}^p_{[0,1]} \psi =: c$. This implies $2n-1 \leq c(e^x)^{-p}$ for some $x \in (r-1, r)$, and hence $n \leq c(e^{r-1})^{-p}$. This means that there are only finitely many different $J_{j,r}$ and that their number $q(r)$ is bounded by $ce^{-p(r-1)}$. As $J_{j,r}$ is an interval, we get $\operatorname{var}^p_{J_{j,r}} \varrho \leq ce^{-p(r-1)}$ by (1.3), since $\psi \geq e^{r-1}$ on $J_{j,r}$. This and the definition of $\mathcal{Y}$ imply $\operatorname{card} \mathcal{W}_{j,r} \leq c\varepsilon^{-p} e^{-p(r-1)}$. So we get $\sum_{j=1}^{q(r)} \operatorname{card} \mathcal{W}_{j,r} \leq be^{-2p(r-1)}$ with $b = c^2 \varepsilon^{-p}$. We have shown all desired properties of $(\mathcal{W}_{j,l})_{1 \leq i \leq q(l)}$ for $l = r$. As $\varepsilon < 1$ the above construction gives also that $\sup_Y \varrho \leq r$ for all $Y \in \mathcal{Y} \setminus \bigcup_{j=1}^{q(r)} \mathcal{W}_{j,r}$. Suppose we have constructed $(\mathcal{W}_{j,l})_{1 \leq j \leq q(l), m < l \leq r}$ and that $\sup_Y \varrho \leq m+1$ for all $Y \in \mathcal{Y} \setminus \bigcup_{l=m+1}^{r} \bigcup_{j=1}^{q(l)} \mathcal{W}_{j,l}$. Then we can go through the above construction with $\mathcal{Y} \setminus \bigcup_{l=m+1}^{r} \bigcup_{j=1}^{q(l)} \mathcal{W}_{j,l}$ instead of $\mathcal{Y}$ and with m instead of r, which gives $(\mathcal{W}_{j,m})_{1 \leq j \leq q(m)}$ with the desired properties. Since ϱ is bounded on each element of $\mathcal{Y}$, every $Y \in \mathcal{Y}$ will belong to some $\mathcal{W}_{j,l}$. Hence we end with a partition $(\mathcal{W}_{j,l})_{1 \leq j \leq q(l), -\infty < l \leq r}$ of $\mathcal{Y}$.

Set $\beta_Y = 3p(r - l)$ for $Y \in \bigcup_{j=1}^{q(l)} \mathcal{W}_{j,l}$. Since $-l - 1 \leq -\varrho(x)$ if x is in some $Y \in \bigcup_{j=1}^{q(l)} \mathcal{W}_{j,l}$, we get $\sum_{Y \in \mathcal{Y}} \beta_Y \mu(Y) \leq 3p \int r + 1 - \varrho \, d\mu < 3p(r + 1 - \int \varrho \, d\mu) < \infty$. Since

$\sum_{j=1}^{q(l)} \operatorname{card} \mathcal{W}_{j,l} \leq be^{-2p(l-1)}$ we get

$$\sum_{Y\in\mathcal{Y}} e^{-\beta_Y} = \sum_{l\leq r} e^{-3p(r-l)} \sum_{j=1}^{q(l)} \operatorname{card} \mathcal{W}_{j,l} \leq be^{p(2-3r)} \sum_{l\geq -r} e^{-pl} < \infty.$$

Hence the inequality

$$-x\log x \leq \beta x + \frac{1}{e}e^{-\beta} \tag{1.4}$$

implies $-\sum_{Y\in\mathcal{Y}} \mu(Y)\log\mu(Y) \leq \sum_{Y\in\mathcal{Y}} \beta_Y\mu(Y) + \frac{1}{e}\sum_{Y\in\mathcal{Y}} e^{-\beta_y} < \infty$ which is (1.2). ∎

We use Lemma 1 as follows. For a μ-partition $\mathcal{Y}$ set $\mathcal{Y}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{Y} := \{\bigcap_{i=0}^{n-1} T^{-i}Y_i \neq \emptyset : Y_i \in \mathcal{Y}\}$ which is again a μ-partition of $[0,1]$, as μ is T-invariant. Furthermore, set $E_{\mathcal{Y}} = \bigcap_{n=1}^{\infty} \bigcup_{Y\in\mathcal{Y}_n} Y$, which we have already introduced above for $\mathcal{Z}$. Then $\mu(E_{\mathcal{Y}}) = 1$. If $x \in E_{\mathcal{Y}}$, for all n, there is a unique element in $\mathcal{Y}_n$ which contains x. We denote it $Y_n(x)$. The length of an interval J is denoted by $|J|$. Finally set $\varphi = \log|T'|$ and $S_n\varphi = \sum_{i=0}^{n-1} \varphi\circ T^i$.

COROLLARY 1. *Let T be a piecewise monotonic map with respect to $\mathcal{Z}$ which has a derivative of bounded p-variation for some $p > 0$. Let μ be an ergodic T-invariant measure on $E_{\mathcal{Z}}$ with Ljapunov exponent $\lambda_\mu > 0$. Then for every $\varepsilon > 0$ there is a finite or countable μ-partition $\mathcal{Y}$ of $[0,1]$ into intervals, which refines $\mathcal{Z}$, such that for $x \in E_{\mathcal{Y}}$ one has*

$$\left|S_n\varphi(x) - \log\frac{1}{|Y_n(x)|}\int_{Y_n(x)} e^{S_n\varphi(y)}\,dy\right| \leq n\varepsilon \text{ for all } n\geq 1 \tag{1.5}$$

and such that (1.2) holds.

PROOF. We apply Lemma 1 with $\psi = |T'|$. This is possible, since T' is bounded and $\lambda_\mu = \int \log|T'|\,d\mu > 0$, which imply $\varphi = \log|T'| \in L^1(\mu)$. By (1.1) it is impossible, that a zero of T' is in an $Y\in\mathcal{Y}$. This and $T'(x) = 0$ for $x \notin \bigcup_{Z\in\mathcal{Z}} Z$ imply that $\mathcal{Y}$ refines $\mathcal{Z}$. From (1.1) we get $\sup_{y\in Y_n(x)} |S_n\varphi(x) - S_n\varphi(y)| \leq n\varepsilon$ for all $x \in E_{\mathcal{Y}}$ and all $n\in\mathbb{N}$, which implies (1.5). ∎

In the course of the proof of Theorem 1 we only shall use the conclusions of Corollary 1 and the fact that $|T'|$ is bounded. If T has finitely many monotonic pieces and if $|T'|$ is bounded away from zero and regular, then φ is bounded and regular, and hence for every $\varepsilon > 0$ there is a finite μ-partition $\mathcal{Y}$ of $[0,1]$ (it covers $\bigcup_{Z\in\mathcal{Z}} Z$ up to a finite set), such that (1.1) holds. This implies (1.5) for $x\in E_{\mathcal{Y}}$ as in the above proof. Since $\mathcal{Y}$ is finite, (1.2) is trivial. Hence if T has finitely many monotonic pieces, Theorem 1 holds also under the assumption, that $|T'|$ is bounded away from zero and regular.

We give an application of this result to the problem considered in [5]. Suppose that the piecewise monotonic map T is expanding, that is $\inf_{[0,1]}|T'| > 1$, and that $\varphi := \log|T'|$ is regular. One considers a T-invariant perfect subset R of $[0,1]$ and defines $p(t)$ as the pressure of the function $-t\varphi$ on $(R, T|R)$. One can define the pressure in this case approximating φ uniformly by piecewise constant functions (see [5]). Since T is expanding, $t \longmapsto p(t)$ is strictly decreasing with $p(0) = h_{top}(R,T) \geq 0$ and $p(t) < 0$ for some $t > 0$. It is shown in [5] , that HD(R) equals the unique zero t_R of $p\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$. The easier part of the proof is to show that HD(R) $\leq t_R$ (Theorem 1 of [5]). Using the above result, the other inequality is also easy. Let ν be an equilibrium state of $-t_R\varphi$ on $(R, T|R)$, which exists since T is expanding. It means that $0 = p(t_R) = h_\nu - t_R\lambda_\nu$. As T is expanding, we get $\lambda_\nu > 0$ and the above result implies HD$(\nu) = h_\nu / \lambda_\nu$. Hence HD($R$) $\geq$ HD$(\nu) = t_R\lambda_\nu / \lambda_\nu = t_R$, the desired result.

2. **The Hausdorff dimension.** In this section we prepare some results about the Hausdorff dimension. Let $\mathcal{A}$ be a subset of the set of all subintervals of $[0,1]$ and define a diameter of the intervals in $\mathcal{A}$. For $X \subset [0,1]$ let $C(\mathcal{A},\delta,X)$ be the set of all finite or countable covers of X by elements of $\mathcal{A}$ with diameter less than δ. Set

$$\mathrm{H}_{\mathcal{A}}(X) = \inf\{\alpha : \lim_{\delta\to 0} \inf_{\mathcal{R}\in C(\mathcal{A},\delta,X)} \sum_{A\in\mathcal{R}} |A|^\alpha = 0\}$$

Let $\mathcal{U}$ be the set of all open subintervals of $[0,1]$ and define the diameter of an interval in $\mathcal{U}$ as the length of the interval. Then $\mathrm{H}_{\mathcal{U}}(X)$ is the Hausdorff dimension of X.

LEMMA 2. *If $X_k \subset [0,1]$ for $k \geq 1$ and $X = \bigcup_{k=1}^\infty X_k$, then $\mathrm{H}_{\mathcal{A}}(X) = \sup_{k\geq 1} \mathrm{H}_{\mathcal{A}}(X_k)$.*

PROOF. Since $X_k \subset X$, we get $\mathrm{H}_{\mathcal{A}}(X_k) \leq \mathrm{H}_{\mathcal{A}}(X)$ for all k. On the other hand, choose $\alpha > \sup_{k\geq 1} \mathrm{H}_{\mathcal{A}}(X_k)$ arbitrary. Then for every $\delta > 0$ and $\eta > 0$ there is an $\mathcal{R}_k \in C(\mathcal{A},\delta,X_k)$ with $\sum_{A\in\mathcal{R}_k} |A|^\alpha < \frac{\eta}{2^k}$. Set $\mathcal{R} = \bigcup_{k=1}^\infty \mathcal{R}_k$. Then $\mathcal{R} \in C(\mathcal{A},\delta,X)$ and $\sum_{A\in\mathcal{R}} |A|^\alpha \leq \sum_{k=1}^\infty \sum_{A\in\mathcal{R}_k} |A|^\alpha < \eta$. Hence $\mathrm{H}_{\mathcal{A}}(X) < \alpha$. By the choice of α we get $\mathrm{H}_{\mathcal{A}}(X) \leq \sup_{k\geq 1} \mathrm{H}_{\mathcal{A}}(X_k)$. ∎

Let T be a piecewise monotonic map with respect to $\mathcal{Z}$ and let $\mathcal{Y}$ be a μ-partition of $[0,1]$, which refines $\mathcal{Z}$. Remember that $\mathcal{Y}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{Y}$ and $E_{\mathcal{Y}} = \bigcap_{n=1}^\infty \bigcup_{Y\in\mathcal{Y}_n} Y$. We have $\mu(E_{\mathcal{Y}}) = 1$. Furthermore set $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{Y}_n$. For $Y \in \mathcal{V}$ we define the diameter as follows. The diameter of Y is less than or equal to δ if and only if $Y \in \mathcal{Y}_n$ for some $n \geq \frac{1}{\delta}$. If $X \subset E_{\mathcal{Y}}$, then $C(\mathcal{V},\delta,X) \neq \emptyset$ for all $\delta > 0$ and $\mathrm{H}_{\mathcal{V}}(X)$ is defined. In order to prove an equality of this dimension and the Hausdorff dimension, we need the following result. Recall that, if $x \in E_{\mathcal{Y}}$, then $Y_n(x)$ denotes the unique element of $\mathcal{Y}_n$, which contains x. Throughout the paper we fix $\gamma \in (0,\lambda_\mu)$. For a μ-partition $\mathcal{Y}$ set $M_{\mathcal{Y}} = \{x \in E_{\mathcal{Y}} :$ there is a $c(x) > 0$ with $|Y_n(x)| \leq c(x)e^{-\gamma n}$ for all $n \geq 1\}$.

LEMMA 3. *Fix $\varepsilon \in (0,\lambda_\mu - \gamma)$ and let $\mathcal{Y}$ be a μ-partition such that (1.5) holds for all $x \in E_{\mathcal{Y}}$. Then $\mu(M_{\mathcal{Y}}) = 1$.*

PROOF. By the ergodic theorem we have $\lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \lambda_\mu$ for almost every x. Hence the set $G := \{x : \exists\, m(x) \in \mathbb{N}$ with $\frac{1}{n}S_n\varphi(x) > \gamma + \varepsilon\ \forall\, n \geq m(x)\}$ has μ-measure one, since $\gamma + \varepsilon < \lambda_\mu$. Then for $x \in G \cap E_{\mathcal{Y}}$ and $n \geq m(x)$ one has

$$|Y_n(x)| = \left(\frac{1}{|Y_n(x)|}\int_{Y_n(x)} e^{S_n\varphi(y)}\,dy\right)^{-1} |T^nY_n(x)| \leq e^{-S_n\varphi(x)+n\varepsilon} < e^{-\gamma n}$$

since $T^nY_n(x)$ is a subinterval of $[0,1]$. This shows $G \cap E_{\mathcal{Y}} \subset M_{\mathcal{Y}}$. As $\mu(G) = \mu(E_{\mathcal{Y}}) = 1$ we get $\mu(M_{\mathcal{Y}}) = 1$. ∎

LEMMA 4. *Fix* $\varepsilon \in (0, \lambda_\mu - \gamma)$. *Let* $\mathcal{Y}$ *be a* μ*-partition, such that (1.5) for all* $x \in E_{\mathcal{Y}}$ *and (1.2) hold. Then* $\mathrm{HD}(\mu) = \inf \mathrm{H}_{\mathcal{V}}(X)$ *the infimum taken over all measurable subsets* X *of* $M_{\mathcal{Y}}$ *with* $\mu(X) = 1$.

PROOF. We show first, that for every measurable $X \subset [0,1]$ and for every $\vartheta > 0$ there is an $\tilde{X} \subset M_{\mathcal{Y}}$ with

$$\tilde{X} \subset X\,,\ \mu(X \setminus \tilde{X}) = 0 \text{ and } \mathrm{H}_{\mathcal{V}}(\tilde{X}) - \vartheta \leq \mathrm{H}_{\mathcal{U}}(\tilde{X}) \leq \mathrm{H}_{\mathcal{V}}(\tilde{X}) \tag{2.1}$$

To this end set $\beta_Y = -\log \mu(Y) \in \mathbb{R}^+ \cup \{\infty\}$ for $Y \in \mathcal{Y}$ and $\mathcal{W}_i = \{Y \in \mathcal{Y} : \beta_Y \leq i\vartheta\gamma\}$ for $i \geq 1$. Then $\mathcal{W}_i \subset \mathcal{W}_{i+1}$ and $\bigcup_{i=1}^\infty \mathcal{W}_i = \tilde{\mathcal{Y}} := \{Y \in \mathcal{Y} : \mu(Y) > 0\}$. Set $\mathcal{W}_0 = \emptyset$ and for $i \geq 1$ set $d_Y = i$, if $Y \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}$. Then we have $d_Y - 1 < \frac{1}{\vartheta\gamma}\beta_Y \leq d_Y$. This implies that $\sum_{i=1}^\infty \sum_{Y \notin \mathcal{W}_i} \mu(Y) = \sum_{Y \in \tilde{\mathcal{Y}}} (d_Y - 1)\mu(Y) \leq \frac{1}{\vartheta\gamma} \sum_{Y \in \tilde{\mathcal{Y}}} \beta_Y \mu(Y) = -\frac{1}{\vartheta\gamma} \sum_{Y \in \mathcal{Y}} \mu(Y) \log \mu(Y) < \infty$ by (1.2). Furthermore we get $\sum_{i=1}^\infty e^{-\vartheta\gamma i} \operatorname{card} \mathcal{W}_i = \sum_{Y \in \tilde{\mathcal{Y}}} \sum_{i=d_Y}^\infty e^{-\vartheta\gamma i} = \frac{1}{1-e^{-\vartheta\gamma}} \sum_{Y \in \tilde{\mathcal{Y}}} e^{-\vartheta\gamma d_Y} \leq \frac{1}{1-e^{-\vartheta\gamma}} \sum_{Y \in \tilde{\mathcal{Y}}} e^{-\beta_Y} < \infty$, since $\sum_{Y \in \tilde{\mathcal{Y}}} e^{-\beta_Y} = \sum_{Y \in \mathcal{Y}} \mu(Y) = 1$.

Set $M_k = \{x \in E_{\mathcal{Y}} : |Y_n(x)| \leq ke^{-\gamma n} \text{ for } n \geq 1\}$. Then $M_k \subset M_{k+1}$ and $\bigcup_{k=1}^\infty M_k = M_{\mathcal{Y}}$. Lemma 3 implies $\mu(M_k) \longrightarrow 1$ for $k \longrightarrow \infty$. Furthermore set $\mathcal{Y}_n^k = \{\bigcap_{i=0}^{n-1} T^{-i} Y_i \neq \emptyset : Y_i \in \mathcal{W}_{i+k}\}$, which is a subset of $\mathcal{Y}_n$, and set $E_k = \bigcap_{n=1}^\infty \bigcup_{Y \in \mathcal{Y}_n^k} Y$, which is a subset of $E_{\mathcal{Y}}$. Since $\mu(\bigcup_{Y \in \mathcal{Y}} Y) = 1$, which implies $\mu(\bigcup_{Y \in \mathcal{Y}_n} Y) = 1$ for all n, we get $\mu([0,1] \setminus E_k) \leq \sum_{i=0}^\infty \sum_{Y \notin \mathcal{W}_{i+k}} \mu(T^{-i}Y) = \sum_{i=k}^\infty \sum_{Y \notin \mathcal{W}_i} \mu(Y)$. Since $\sum_{i=1}^\infty \sum_{Y \notin \mathcal{W}_i} \mu(Y) < \infty$, as shown above, we get $\mu([0,1] \setminus E_k) \longrightarrow 0$ for $k \longrightarrow \infty$. Set $X_k = X \cap E_k \cap M_k$ and $\tilde{X} = \bigcup_{k=1}^\infty X_k$. Then $\tilde{X} \subset M_{\mathcal{Y}}$, $\tilde{X} \subset X$ and $\mu(X \setminus \tilde{X}) = 0$.

Since $Y \cap M_k \neq \emptyset$ for $Y \in \mathcal{Y}_r$ implies $|Y| \leq ke^{-\gamma r}$, we get from $X_k \subset M_k$ that $C(\mathcal{V}, \frac{1}{r}, X_k) \subset C(\mathcal{U}, ke^{-\gamma r}, X_k)$. Hence $\mathrm{H}_{\mathcal{U}}(X_k) \leq \mathrm{H}_{\mathcal{V}}(X_k)$. As this holds for all k, and as $\bigcup_{k=1}^\infty X_k = \tilde{X}$, by Lemma 2 we get $\mathrm{H}_{\mathcal{U}}(\tilde{X}) \leq \mathrm{H}_{\mathcal{V}}(\tilde{X})$, which is one half of (2.1).

Now set $d(r,k,\vartheta) = k^\vartheta e^{-\gamma r\vartheta} \operatorname{card} \mathcal{Y}_r^k + 2k^\vartheta \sum_{l=r+1}^\infty e^{-\vartheta\gamma l} \operatorname{card} \mathcal{W}_{l+k}$, which is finite for $\vartheta > 0$, as $\sum_{i=1}^\infty e^{-\vartheta\gamma i} \operatorname{card} \mathcal{W}_i < \infty$ is shown above. Choose $\varrho > \mathrm{H}_{\mathcal{U}}(\tilde{X})$ arbitrary. Since $X_k \subset \tilde{X}$, we have $\mathrm{H}_{\mathcal{U}}(X_k) < \varrho$ for all k. Fix $r \in \mathbb{N}$ arbitrary. As $\varrho > \mathrm{H}_{\mathcal{U}}(X_k)$, for all $\vartheta > 0$, for all $\eta > 0$ and for all $\delta > 0$, there is an $\mathcal{R} \in C(\mathcal{U}, \delta, X_k)$ with

$$\sum_{A \in \mathcal{R}} |A|^\varrho < \frac{\eta}{d(r,k,\vartheta)} \tag{2.2}$$

Fix an $A \in \mathcal{R}$. For $x \in A \cap X_k$ let $m(x) \geq r$ be minimal, such that $Y_{m(x)}(x) \subset A$. We set $B(x) = Y_{m(x)}(x)$. Such an $m(x)$ exists, since $X_k \subset M_k$. As $x \in B(x) \cap X_k$, we have $B(x) \in \mathcal{Y}_{m(x)}^k$, because $Y \cap E_k = \emptyset$ for $Y \in \mathcal{Y}_{m(x)} \setminus \mathcal{Y}_{m(x)}^k$. Set $\mathcal{X}_A = \{B(x) : x \in A \cap X_k\}$. Since every $x \in A \cap X_k$ is in $B(x)$, we get $\mathcal{X}_A \in C(\mathcal{V}, \frac{1}{r}, A \cap X_k)$. We show that $\operatorname{card}\{B(x) : m(x) = l\} \leq 2 \operatorname{card} \mathcal{W}_{l+k}$ for $l > r$. If $B(x) \in \mathcal{Y}_l^k$, then there is a unique $G \in \mathcal{Y}_{l-1}$ with $B(x) \subset G$. Suppose that, for $1 \leq i \leq 3$, there are different $G_i \in \mathcal{Y}_{l-1}$ and $x_i \in A \cap X_k$ with $m(x_i) = l$ and $B(x_i) \subset G_i$. Suppose that G_2 is between G_1 and G_3 in $[0,1]$. As $x_1 \in G_1$ and $x_3 \in G_3$ are both in A and as A is an interval, we have $G_2 \subset A$, which contradicts $m(x_2) = l$. Hence there are at most two $G_1, G_2 \in \mathcal{Y}_{l-1}$, which contain all $B(x)$ with $m(x) = l$ and all $B(x)$ satisfy $B(x) = G_i \cap T^{-l}Y$ with $i = 1$ or 2 and with

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$Y \in \mathcal{W}_{l+k}$. This implies $\operatorname{card}\{B(x) : m(x) = l\} \leq 2 \operatorname{card} \mathcal{W}_{l+k}$ for $l > r$. As $B(x) \in \mathcal{Y}_r^k$, if $m(x) = r$, we get $\operatorname{card}\{B(x) : m(x) = r\} \leq \operatorname{card} \mathcal{Y}_r^k$. Because of $B(x) \subset A$ we have $|B(x)| \leq |A|$. Since $B(x) = Y_{m(x)}(x)$ and $x \in M_k$ we have $|B(x)| \leq ke^{-\gamma m(x)}$. These assertions give

$$\begin{aligned}\sum_{B\in\mathcal{X}_A} |B|^{\varrho+\vartheta} &\leq \sum_{B\in\mathcal{X}_A} |B|^{\varrho}|B|^{\vartheta} \leq |A|^{\varrho} \sum_{B\in\mathcal{X}_A} |B|^{\vartheta} \\ &\leq |A|^{\varrho}\Big(k^{\vartheta} e^{-\gamma r\vartheta} \operatorname{card} \mathcal{Y}_r^k + \sum_{l=r+1}^{\infty} k^{\vartheta} e^{-\gamma l\vartheta} 2 \operatorname{card} \mathcal{W}_{l+k}\Big) \\ &= |A|^{\varrho} d(r,k,\vartheta)\end{aligned}$$

Now set $\mathcal{X} = \bigcup_{A\in\mathcal{R}} \mathcal{X}_A$. Then $\mathcal{X} \in C(\mathcal{V}, \frac{1}{r}, X_k)$. The above estimate and (2.2) imply $\sum_{B\in\mathcal{X}} |B|^{\varrho+\vartheta} \leq d(r,k,\vartheta) \sum_{A\in\mathcal{R}} |A|^{\varrho} < \eta$. As r and η were arbitrary, we get $\mathrm{H}_{\mathcal{V}}(X_k) \leq \varrho + \vartheta$. As $\varrho > \mathrm{H}_{\mathcal{U}}(\tilde{X})$ was arbitrary, we get $\mathrm{H}_{\mathcal{V}}(X_k) \leq \mathrm{H}_{\mathcal{U}}(\tilde{X}) + \vartheta$. Lemma 2 implies $\mathrm{H}_{\mathcal{V}}(\tilde{X}) - \vartheta \leq \mathrm{H}_{\mathcal{U}}(\tilde{X})$.

Hence we have shown that for all measurable $X \subset [0,1]$ and for all $\vartheta > 0$ there is an $\tilde{X} \subset M_{\mathcal{Y}}$ such that (2.1) holds. This implies that $\inf \mathrm{H}_{\mathcal{V}}(X) - \vartheta \leq \inf \mathrm{H}_{\mathcal{U}}(X) := \mathrm{HD}(\mu) \leq \inf \mathrm{H}_{\mathcal{V}}(X)$ where the first and third infimum is taken over all $X \subset M_{\mathcal{Y}}$ with $\mu(X) = 1$ and the second infimum is taken over all $X \subset [0,1]$ with $\mu(X) = 1$. As ϑ was arbitrary, we get the desired result. ∎

The following result is similar to Proposition 2.2 of [6].

LEMMA 5. *Let $\mathcal{Y}$ be a μ-partition and X a measurable subset of $M_{\mathcal{Y}}$ with $\mu(X) > 0$. Suppose that, for all $x \in X$, we have*

$$\underline{\delta} \leq \liminf_{n\to\infty} \frac{\log \mu\big(Y_n(x)\big)}{\log |Y_n(x)|} \leq \limsup_{n\to\infty} \frac{\log \mu\big(Y_n(x)\big)}{\log |Y_n(x)|} \leq \overline{\delta} \tag{2.3}$$

Then $\underline{\delta} \leq \mathrm{H}_{\mathcal{V}}(X) \leq \overline{\delta}$.

PROOF. Let $\vartheta > 0$ be arbitrary and set $X_k = \Big\{x \in X : \underline{\delta} - \vartheta \leq \frac{\log \mu(Y_n(x))}{\log |Y_n(x)|} \leq \overline{\delta} + \vartheta \ \forall n \geq k\Big\}$. Then $X_k \subset X_{k+1}$ and $X = \bigcup_{k=1}^{\infty} X_k$. In particular, $\mu(X_k) \longrightarrow \mu(X) > 0$. By definition of X_k we get

$$B \in \bigcup_{n\geq k} \mathcal{Y}_n \text{ and } B \cap X_k \neq \emptyset \Rightarrow |B|^{\underline{\delta}-\vartheta} \geq \mu(B) \geq |B|^{\overline{\delta}+\vartheta} \tag{2.4}$$

Let $\mathcal{X} \in C(\mathcal{V}, \frac{1}{r}, X_k)$ be arbitrary. By (2.4), we get $\sum_{B\in\mathcal{X}} |B|^{\underline{\delta}-\vartheta} \geq \sum_{B\in\mathcal{X}} \mu(B) \geq \mu(X_k)$ for $r \geq k$. If k is large enough, such that $\mu(X_k) > 0$, this implies that $\underline{\delta} - \vartheta \leq \mathrm{H}_{\mathcal{V}}(X_k)$ and hence $\underline{\delta} - \vartheta \leq \mathrm{H}_{\mathcal{V}}(X)$. Since ϑ was arbitrary, this gives $\underline{\delta} \leq \mathrm{H}_{\mathcal{V}}(X)$.

Now set $M_k = \{x \in E_{\mathcal{Y}} : |Y_n(x)| \leq ke^{-\gamma n} \text{ for } n \geq 1\}$ and choose $\eta > 0$ and $r \in \mathbb{N}$ arbitrary. Set $\mathcal{X}_r = \{B \in \mathcal{Y}_r : B \cap X_k \cap M_k \neq \emptyset\}$. Then $\mathcal{X}_r \in C(\mathcal{V}, \frac{1}{r}, X_k \cap M_k)$. By (2.4) we get $\sum_{B\in\mathcal{X}_r} |B|^{\overline{\delta}+\vartheta+\eta} \leq \sup_{B\in\mathcal{X}_r} |B|^{\eta} \sum_{B\in\mathcal{X}_r} \mu(B) \leq \sup_{B\in\mathcal{X}_r} |B|^{\eta}$, since the elements of $\mathcal{X}_r$ are pairwise disjoint. As $B \cap M_k \neq \emptyset$, if $B \in \mathcal{X}_r$, this implies $\sum_{B\in\mathcal{X}_r} |B|^{\overline{\delta}+\vartheta+\eta} \leq$

$k^{\eta} e^{-\eta\gamma r}$, which tends to zero for $r \longrightarrow \infty$. Hence $\mathrm{H}_{\mathcal{V}}(X_k \cap M_k) \leq \overline{\delta} + \vartheta + \eta$. As $M_{\mathcal{Y}} = \bigcup_{k=1}^{\infty} M_k$ and as $X \subset M_{\mathcal{Y}}$, this and Lemma 2 imply $\mathrm{H}_{\mathcal{V}}(X) \leq \overline{\delta} + \vartheta + \eta$. Since $\vartheta > 0$ and $\eta > 0$ were arbitrary, we get $\mathrm{H}_{\mathcal{V}}(X) \leq \overline{\delta}$. ∎

3. **Proof of the formula for the Hausdorff dimension.** In this section we put the lemmas together in order to get the proof of Theorem 1. One step is still missing. It will need results about a Markov extension of piecewise monotonic maps. We introduce it in Section 4 and show first, how the proof of Theorem 1 follows.

LEMMA 6. *Suppose that T is piecewise monotonic with respect to $\mathcal{Z}$ and that T has a bounded derivative (defined in the same way as derivative of bounded variation). Set $W_{\mathcal{Z}} := \{x : x \in T(Z)$ for infinitely many $Z \in \mathcal{Z}\}$ and suppose that $W_{\mathcal{Z}}$ is at most countable. Suppose further that μ is an ergodic T-invariant measure on $E_{\mathcal{Z}}$ with $h_\mu > 0$. Let $\mathcal{Y}$ be a μ-partition refining $\mathcal{Z}$ which satisfies (1.2). For $x \in E_{\mathcal{Y}}$ let $r_n(x)$ be the distance of $T^n(x) \in T^n\big(Y_{n+1}(x)\big)$ to the nearer endpoint of the interval $T^n\big(Y_{n+1}(x)\big)$ (to one of the endpoints, if both have the same distance). Then $\lim_{n\to\infty} \frac{1}{n}\log r_n(x) = 0$ for μ-almost all $x \in [0,1]$.*

PROOF. As $\mathcal{Y}$ refines $\mathcal{Z}$, T is also piecewise monotonic with respect to $\mathcal{Y}$ and as $\mathcal{Y}$ is a μ-partition, μ is concentrated on $E_{\mathcal{Y}}$. Since T is strictly monotone on each interval in $\mathcal{Z}$, we get $W_{\mathcal{Y}} \subset W_{\mathcal{Z}}$. Hence the desired result follows immediately from Proposition 2 in Section 4. ∎

LEMMA 7. *Suppose that the assumptions about T in Lemma 6 hold and that μ is an ergodic T-invariant measure on $E_{\mathcal{Z}}$. Fix $\varepsilon \in (0, \lambda_\mu - \gamma)$. Let $\mathcal{Y}$ be a μ-partition, which refines $\mathcal{Z}$, such that (1.5) for all $x \in E_{\mathcal{Y}}$ and (1.2) hold. Then there is a set $L \subset M_{\mathcal{Y}}$ with $\mu(L) = 1$, such that (2.3) holds for all $x \in L$ with $\underline{\delta} = \frac{h_\mu}{\lambda_\mu + \varepsilon}$ and with $\overline{\delta} = \frac{h_\mu}{\lambda_\mu - \varepsilon}$.*

PROOF. By (1.2) and the Shannon-McMillan-Breiman-Theorem there is a set $L_1 \subset E_{\mathcal{Y}}$ with $\mu(L_1) = 1$, such that for all $x \in L_1$ the sequence $-\frac{1}{n}\log \mu\big(Y_n(x)\big)$ converges to $h_\mu(T, \mathcal{Y})$. This equals h_μ since, by Lemma 3, $\mathcal{Y}$ is a generator for μ.

Now consider $-\frac{1}{n}\log|Y_n(x)| = \frac{1}{n}\log\frac{1}{|Y_n(x)|}\int_{Y_n(x)} e^{S_n\varphi(y)}\,dy - \frac{1}{n}\log|T^n Y_n(x)|$. Let L_2 be the set of all $x \in E_{\mathcal{Y}}$, for which $\lim_{n\to\infty}\frac{1}{n}S_n\varphi(x) = \lambda_\mu$ holds. Since $\mu(E_{\mathcal{Y}}) = 1$, the ergodic theorem implies $\mu(L_2) = 1$. By (1.5) we have

$$\Big|\frac{1}{n}S_n\varphi(x) - \frac{1}{n}\log\frac{1}{|Y_n(x)|}\int_{Y_n(x)} e^{S_n\varphi(y)}\,dy\Big| < \varepsilon$$

for $x \in L_2 \subset E_{\mathcal{Y}}$ and $n \geq 1$. This gives for $x \in L_2$ that the sequence

$$\Big(\frac{1}{n}\log\frac{1}{|Y_n(x)|}\int_{Y_n(x)} e^{S_n\varphi(y)}\,dy\Big)_{n\geq 1}$$

has its limit points in $[\lambda_\mu - \varepsilon, \lambda_\mu + \varepsilon]$.

Because of $T^n\big(Y_n(x)\big) \subset [0,1]$, we get that $-\frac{1}{n}\log\big|T^n\big(Y_n(x)\big)\big| \geq 0$, and for $x \in L_1 \cap L_2$ that $\limsup_{n\to\infty}\frac{\log\mu\left(Y_n(x)\right)}{\log|Y_n(x)|} \leq \frac{h_\mu}{\lambda_\mu - \varepsilon}$. Since $\mu\big(Y_n(x)\big) \leq 1$ and $|Y_n(x)| < 1$ for

$n \geq 1$ we have $\liminf_{n\to\infty} \frac{\log \mu\left(Y_n(x)\right)}{\log |Y_n(x)|} \geq 0 = \frac{h_\mu}{\lambda_\mu+\varepsilon}$, if $h_\mu = 0$. If $h_\mu > 0$, it follows from Lemma 6, that there is a set $L_3 \subset E_{\mathcal{Y}}$ with $\mu(L_3) = 1$, such that $\lim_{n\to\infty} \frac{1}{n} \log r_n(x) = 0$ for $x \in L_3$. As $T^n(x) \in T^n\left(Y_{n+1}(x)\right) \subset T^n\left(Y_n(x)\right)$ and hence $r_n(x) \leq |T^n\left(Y_n(x)\right)| \leq 1$, we get $-\frac{1}{n} \log |T^n\left(Y_n(x)\right)| \longrightarrow 0$ for all $x \in L_3$, which implies $\liminf_{n\to\infty} \frac{\log \mu\left(Y_n(x)\right)}{\log |Y_n(x)|} \geq \frac{h_\mu}{\lambda_\mu+\varepsilon}$ for $x \in L_1 \cap L_2 \cap L_3$. Hence the desired result follows for $L = L_1 \cap L_2 \cap L_3 \cap M_{\mathcal{Y}}$. Using also Lemma 3, we get $\mu(L) = 1$. ∎

Now we can give the proof of Theorem 1 and the Remark following it.

PROOF. Let $\varepsilon \in (0, \lambda_\mu - \gamma)$ be arbitrary. By Corollary 1 there is a μ-partition $\mathcal{Y}$ refining $\mathcal{Z}$, such that (1.5) for all $x \in E_{\mathcal{Y}}$ and (1.2) hold. Lemma 4 says that $\mathrm{HD}(\mu) = \inf \mathrm{H}_{\mathcal{V}}(X)$, where the infimum is taken over all $X \subset M_{\mathcal{Y}}$ with $\mu(X) = 1$. By Lemmas 5 and 7 there is a set $L \subset M_{\mathcal{Y}}$ with $\mu(L) = 1$ and $\mathrm{H}_{\mathcal{V}}(L) \leq \frac{h_\mu}{\lambda_\mu-\varepsilon}$. Hence $\mathrm{HD}(\mu) \leq \frac{h_\mu}{\lambda_\mu-\varepsilon}$. On the other hand, if $X \subset M_{\mathcal{Y}}$ is arbitrary such that $\mu(X) = 1$, and if L is as in Lemma 7, then $\mu(X \cap L) = 1$ and (2.3) holds for all $x \in X \cap L$ with $\underline{\delta} = \frac{h_\mu}{\lambda_\mu+\varepsilon}$. Hence $\mathrm{H}_{\mathcal{V}}(X) \geq \mathrm{H}_{\mathcal{V}}(X \cap L) \geq \frac{h_\mu}{\lambda_\mu+\varepsilon}$ by Lemma 5. This implies $\mathrm{HD}(\mu) \geq \frac{h_\mu}{\lambda_\mu+\varepsilon}$. As ε was arbitrary, we get $\mathrm{HD}(\mu) = h_\mu / \lambda_\mu$. ∎

4. **Markov extensions.** This section is independent of the previous part of the paper. We introduce a Markov extension of piecewise monotonic maps, an important tool for their investigation. We use it to prove a result, namely Proposition 2 and Corollary 2 below, which is needed to show Lemma 6, but which might also be of independent interest. First we give the definition of the Markov extension, as it is done in [3] and describe its basic properties. Let $\mathcal{Z}$ be the collection of intervals, with respect to which T is piecewise monotonic. Recall that $\mathcal{Z}_k = \bigvee_{i=0}^{k-1} T^{-i}\mathcal{Z}$. The map T^{k-1} is strictly monotone on each $Z \in \mathcal{Z}_k$ and $T^{k-1}(Z)$ is an open subinterval of some element of $\mathcal{Z}$ for $Z \in \mathcal{Z}_k$. Set $\mathcal{D}_1 = \mathcal{Z}$, for $k \geq 2$ set $\mathcal{D}_k = \mathcal{D}_{k-1} \cup \{T^{k-1}(Z) : Z \in \mathcal{Z}_k\}$ and finally set $\mathcal{D} = \bigcup_{k=1}^\infty \mathcal{D}_k$. For $D \in \mathcal{D}$ define $D' = \{(x, D) : x \in D\}$ so that the sets D' are disjoint copies of the sets $D \in \mathcal{D}$. Now define $\hat{X} = \bigcup_{D\in\mathcal{D}} D'$ and two projections $\pi : \hat{X} \longrightarrow [0,1]$ by $\pi(x, D) = x$ and $\psi : \hat{X} \longrightarrow \mathcal{D}$ by $\psi(x, D) = D$. Since each D' is canonically isomorphic to the open interval D, and since $\hat{X}$ is the disjoint union of the sets D', $\hat{X}$ is in a natural way a locally compact, σ-compact, metric space, on which π is continuous. Furthermore set $E_k = \bigcup_{Z\in\mathcal{Z}_k} Z$. We have $E_k = \bigcap_{l=0}^{k-1} T^{-l}(E_1)$. For $x \in E_k$, remember that $Z_k(x)$ is the unique element of $\mathcal{Z}_k$, which contains x. Define $i : E_1 \longrightarrow \hat{X}$ by $i(x) = \left(x, Z_1(x)\right)$, which is in $\hat{X}$, since $Z_1(x) \in \mathcal{Z} \subset \mathcal{D}$ for all $x \in E_1$. Finally set $\hat{T}(x, D) = \left(Tx, T(D) \cap Z_1(Tx)\right)$. Since T is defined on E_1 and since $Z_1(Tx)$ exists only for $Tx \in E_1$, $\hat{T}$ is defined only on $\pi^{-1}\left(T^{-1}(E_1) \cap E_1\right) = \pi^{-1}(E_2)$. We show that $\hat{T}$ is well defined. Since $D = T^{k-1}(Z)$ for some $k \geq 1$ and some $Z \in \mathcal{Z}_k$, we get $T(D) \cap Z_1(Tx) = T^k\left(Z \cap T^{-k}\left(Z_1(Tx)\right)\right)$ and $Z \cap T^{-k}\left(Z_1(Tx)\right) \in \mathcal{Z}_{k+1}$. Hence $T(D) \cap Z_1(Tx) \in \mathcal{D}$ and $Tx \in T(D) \cap Z_1(Tx)$ follows trivially from $x \in D$, such that $\left(Tx, T(D) \cap Z_1(Tx)\right) \in \hat{X}$. The definition of $\hat{T}$ implies $T \circ \pi = \pi \circ \hat{T}$. As $\hat{T}$ is defined on $\pi^{-1}(E_2)$, this implies that $\hat{T}^j$ is defined on $\pi^{-1}(E_{j+1})$.

Hence all iterates of $\hat{T}$ are defined on $\pi^{-1}(E_{\mathcal{Z}})$. We compute $\hat{T}^j\big(i(x)\big)$. As i is defined on E_1, the above implies that $\hat{T}^j \circ i$ is defined on E_{j+1}. We have $i(x) = \big(x, Z_1(x)\big)$. It follows by induction that

$$\hat{T}^j\big(i(x)\big) = \Big(T^j x, T^j\big(Z_{j+1}(x)\big)\Big) \text{ for } x \in E_{j+1} \tag{4.1}$$

since $T\Big(T^{l-1}\big(Z_l(x)\big)\Big) \cap Z_1(T^l x) = T^l\Big(Z_l(x) \cap T^{-l}\big(Z_1(T^l x)\big)\Big) = T^l\big(Z_{l+1}(x)\big)$. The definitions in [2] and in [3] are slightly different. But the results we shall use from these two papers are not affected by these differences.

What we have to do, is to lift the ergodic T-invariant measure μ with $h_\mu > 0$ to $(\hat{X}, \hat{T})$ as it is done in [3] for maps with finitely many monotonic pieces. To this end set $\hat{\mu}_1 = \mu \circ i^{-1}$ and $\hat{\mu}_n = \frac{1}{n}\sum_{k=0}^{n-1} \hat{\mu}_1 \circ \hat{T}^{-k}$. Then $\hat{\mu}_n$ is a well defined probability measure on $\hat{X}$, since $\hat{T}^k \circ i$ is defined on E_{k+1}, which is a set of μ-measure one.

LEMMA 8. *Suppose that T is piecewise monotonic with respect to $\mathcal{Z}$ and that μ is an ergodic T-invariant measure on $E_{\mathcal{Z}}$ with $h_\mu > 0$ and $-\sum_{Z \in \mathcal{Z}} \mu(Z)\log\mu(Z) < \infty$. If for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there is a finite subset $\mathcal{F}$ of $\mathcal{D}_k$ with $\mu(\bigcup_{D \in \mathcal{D}_k \setminus \mathcal{F}} D) < \varepsilon$, then the zero-measure is not a weak limit point of $(\hat{\mu}_n)_{n \geq 1}$.*

PROOF. We assume that the zero-measure is a weak limit point of $(\hat{\mu}_n)_{n\geq 1}$ and shall arrive at a contradiction. To this end define $f: E_1 \longrightarrow [1, \infty)$ by $f(x) = -\log\mu\big(Z_1(x)\big) + 1$. As $\mu(E_1) = 1$, f is defined μ-almost everythere. By assumption we have $\int f\, d\mu < \infty$. Set $\hat{X}_k = \bigcup_{D \in \mathcal{D}_k} D' \subset \hat{X}$ and define $\hat{f}_k: \hat{X} \longrightarrow [1, \infty)$ by $\hat{f}_k = 1_{\hat{X}_k}(f \circ \pi)$. Remark that $\pi(\hat{X}) = E_1$. First we show that for all $k \geq 1$ and for all $\delta > 0$ there is an infinite subset $I(\delta)$ of $\mathbb{N}$ such that

$$\int \hat{f}_k \, d\hat{\mu}_n < \delta, \ \forall n \in I(\delta). \tag{4.2}$$

To this end, fix a finite $\mathcal{F} \subset \mathcal{D}_k$, such that $\mu(G)$ is so small that $\int_G f\, d\mu < \frac{\delta}{4}$, where $G = \bigcup_{D \in \mathcal{D}_k \setminus \mathcal{F}} D$. Set $N = \operatorname{card} \mathcal{F}$. For $D \in \mathcal{F}$ choose $f_D: E_1 \longrightarrow \mathbb{R}^+$, such that $f_D \leq f$, such that $\operatorname{supp} f_D$ is a compact subset of D and such that $\int_D f - f_D\, d\mu < \frac{\delta}{4N}$. Set $g = 1_G f + \sum_{D \in \mathcal{F}}(f - f_D)$. Then $g \geq 0$ and $\int g \circ \pi \, d\hat{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} \int g \circ T^i d\mu = \int g\, d\mu < \frac{\delta}{2}$ for all n. Furthermore, $\hat{f}_k \leq \sum_{D \in \mathcal{F}} \hat{f}_D + g \circ \pi$ where $\hat{f}_D = 1_{D'}(f_D \circ \pi)$. As $\operatorname{supp} f_D$ is a compact subset of D and as $\pi | D': D' \longrightarrow D$ is a homeomorphism, $\hat{f}_D$ has compact support in $\hat{X}$. As we assume that the zero-measure is a weak limit point of $(\hat{\mu}_n)_{n \geq 1}$, a subsequence of $(\int \sum_{D \in \mathcal{F}} \hat{f}_D \, d\hat{\mu}_n)_{n \geq 1}$ converges to zero. As $\int g \circ \pi \, d\hat{\mu}_n < \frac{\delta}{2}$ for all n, the sequence $(\int \hat{f}_k \, d\hat{\mu}_n)_{n\geq 1}$ has a limit point in $[0, \frac{\delta}{2}]$. This implies (4.2).

There is a map $\vartheta : [0, 1] \longrightarrow [0, 1]$ with $\vartheta(r) \downarrow 0$ for $r \downarrow 0$ such that

$$\sup_{j \geq 0} \int_A f \circ T^j d\mu \leq \vartheta\big(\mu(A)\big) \text{ for all measurable } A \subset [0, 1] \tag{4.3}$$

This holds with $\vartheta(r) = \sqrt{r} + \mu\big(\{x : f(x) \geq \frac{1}{\sqrt{r}}\}\big)$ since

$$\int_A f \circ T^j d\mu \leq \frac{1}{\sqrt{r}}\mu(A) + \mu\Big(\{x : f(T^j(x)) \geq \frac{1}{\sqrt{r}}\}\Big) = \frac{1}{\sqrt{r}}\mu(A) + \mu\Big(\{x : f(x) \geq \frac{1}{\sqrt{r}}\}\Big)$$

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for $r = \mu(A)$.

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Choose $\eta > 0$ such that $\vartheta(\eta) < \varepsilon$. Set $I = I(\varepsilon\eta)$ and fix $n \in I$. Set $\mathcal{B}_n = \{Z \in \mathcal{Z}_n : \frac{1}{n}S_n\hat{f}_k \circ i \leq \varepsilon \text{ on } Z\}$ and $\mathcal{A}_n = \mathcal{Z}_n \setminus \mathcal{B}_n$. As $\frac{1}{n}S_n\hat{f}_k \circ i$ is constant on each element of $\mathcal{Z}_n$, we get $\frac{1}{n}S_n\hat{f}_k \circ i > \varepsilon$ on $A_n := \bigcup_{Z\in\mathcal{A}_n} Z$. Since $n \in I = I(\varepsilon\eta)$, we get $\int \frac{1}{n}S_n\hat{f}_k \circ i\, d\mu < \varepsilon\eta$ by (4.2) and hence $\mu(A_n) < \eta$. This, the definition of η and (4.3) imply $\int_{A_n} \frac{1}{n}S_n f d\mu < \varepsilon$. As $S_n f$ is constant on elements of $\mathcal{Z}_n$, for $Z \in \mathcal{Z}_n$ we can define $\beta_Z = S_n f(x)$, where $x \in Z$ is arbitrary. By (1.4) we get $-\sum_{Z\in\mathcal{A}_n} \mu(Z)\log\mu(Z) \leq \sum_{Z\in\mathcal{A}_n} \beta_Z\mu(Z) + \frac{1}{e}\sum_{Z\in\mathcal{A}_n} e^{-\beta_Z}$. Furthermore $\sum_{Z\in\mathcal{A}_n} e^{-\beta_Z} \leq \left(\sum_{Z\in\mathcal{Z}_1} e^{\log\mu(Z)-1}\right)^n = e^{-n} \leq 1$ and $\frac{1}{n}\sum_{Z\in\mathcal{A}_n} \beta_Z\mu(Z) \leq \int_{A_n} \frac{1}{n}S_n f d\mu < \varepsilon$. Hence $-\frac{1}{n}\sum_{Z\in\mathcal{Z}_n} \mu(Z)\log\mu(Z) \leq -\frac{1}{n}\sum_{Z\in\mathcal{B}_n} \mu(Z)\log\mu(Z) + \frac{1}{ne} + \varepsilon$. Set $\chi := \sum_{Z\in\mathcal{B}_n} \mu(Z)$. If $\chi > 0$, we get $-\sum_{Z\in\mathcal{B}_n} \mu(Z)\log\mu(Z) = -\chi\sum_{Z\in\mathcal{B}_n} \frac{\mu(Z)}{\chi}\log\frac{\mu(Z)}{\chi} - \chi\log\chi \leq \log\operatorname{card}\mathcal{B}_n + \frac{1}{e}$. This implies

$$(4.4)\qquad -\frac{1}{n}\sum_{Z\in\mathcal{Z}_n} \mu(Z)\log\mu(Z) \leq \frac{1}{n}\log\operatorname{card}\mathcal{B}_n + \frac{2}{en} + \varepsilon \quad \text{for } n \in I$$

If $\chi = 0$, (4.4) holds without the first term in its right hand side. Then one needs no estimate of card $\mathcal{B}_n$ and the last paragraph of this proof with $\alpha(\varepsilon, k) = 0$ leads to the desired contradiction. If $\chi > 0$, it remains to estimate card $\mathcal{B}_n$ for $n \in I$.

To this end remember that the map f is constant on each element of $\mathcal{Z}$ and let $\beta_Z \in [1, \infty)$ be its value on $Z \in \mathcal{Z}$. For $l \geq 1$ set $\mathcal{X}_l = \{Z \in \mathcal{Z} : l \leq \beta_Z < l+1\}$. Then $\sum_{l\geq 1} e^{-l}\operatorname{card}\mathcal{X}_l = \sum_{l\geq 1}\sum_{Z\in\mathcal{X}_l} e^{-l} \leq \sum_{Z\in\mathcal{Z}} e^{-\beta_Z+1} = \sum_{Z\in\mathcal{Z}} \mu(Z) = 1$ and hence $\operatorname{card}\mathcal{X}_l \leq e^l$ for $l \geq 1$. Let $P(\varepsilon, n)$ be the set of all partitions $(M_l)_{l\geq 0}$ of $\{0, 1, \ldots, n-1\}$ with $0 \notin M_0$ and $\sum_{l\geq 0} l\operatorname{card} M_l \leq \varepsilon n$. We define a map $Q : \mathcal{B}_n \longrightarrow P(\varepsilon, n)$ as follows. Fix $Z \in \mathcal{B}_n$ and let $Z_m \in \mathcal{Z}$ be such that $Z = \bigcap_{m=0}^{n-1} T^{-m}Z_m$. For $0 \leq m \leq n-1$ set $D_m = T^m(\bigcap_{j=0}^{m} T^{-j}Z_j) \in \mathcal{D}$. Now define $Q(Z) = (M_l)_{l\geq 0}$ where $M_0 = \{m : D_m \notin \mathcal{D}_k\}$ and $M_l = \{m : D_m \in \mathcal{D}_k, Z_m \in \mathcal{X}_l\}$ for $l \geq 1$. As $D_0 = Z_0 \in \mathcal{D}_1$, we get $0 \notin M_0$. By definition of M_l and $\mathcal{X}_l$ we get $\sum_{l\geq 0} l\operatorname{card} M_l \leq \sum_{l\geq 1}\sum_{m\in M_l} l \leq \sum_{m\notin M_0} \beta_{Z_m}$. By (4.1) we have $\hat{T}^m\big(i(x)\big) \in D'_m$ for all $x \in Z$. As $D_m \subset Z_m$, we get $\hat{f}_k \circ \hat{T}^m\big(i(x)\big) = \beta_{Z_m}$, if $m \notin M_0$ which means $D_m \in \mathcal{D}_k$. This implies $\sum_{m\notin M_0} \beta_{Z_m} \leq S_n\hat{f}_k\big(i(x)\big)$ for $x \in Z$. Hence the definition of $\mathcal{B}_n$ implies that $\sum_{l\geq 0} l\operatorname{card} M_l \leq n\varepsilon$ showing $Q(Z) = (M_l)_{l\geq 0} \in P(\varepsilon, n)$.

Having defined Q: $\mathcal{B}_n \longrightarrow P(\varepsilon, n)$ we get

$$(4.5)\qquad \operatorname{card}\mathcal{B}_n \leq \operatorname{card} P(\varepsilon, n) \sup_{P(\varepsilon,n)} \operatorname{card} Q^{-1}(M_l)_{l\geq 0}$$

We begin with the estimation of card $Q^{-1}((M_l)_{l\geq 0})$ for a fixed $(M_l)_{l\geq 0} \in P(\varepsilon, n)$. We have to fill $\{0, 1, \ldots, n-1\}$ with $Z_m \in \mathcal{Z}$ according to the definition of Q. Let $I_1, I_2, \ldots, I_s$ be the maximal disjoint intervals in $\{0, 1, \ldots, n-1\}$, such that $\bigcup_{j=1}^{s} I_j = M_0$. As $0 \notin M_0$ we get $s \leq \operatorname{card}(\bigcup_{l\geq 1} M_l) \leq \sum_{l\geq 1} l\operatorname{card} M_l \leq n\varepsilon$. Remember that $Z_0, Z_1, \ldots, Z_j$ determine $D_j = T^j(\bigcap_{q=0}^{j} T^{-q}Z_q)$ and that Z_j can be recovered from D_j as it is the unique element of $\mathcal{Z}$ which contains D_j. Consider some I_r with $1 \leq r \leq s$. Suppose that $I_r = \{u, u+1, \ldots, u+v-1\}$ and that Z_m and hence also D_m are already chosen for $0 \leq m < u$. As $0 \notin M_0$, we have $u > 0$. We have to find Z_m for $u \leq m < u+v$, such that $D_m \notin \mathcal{D}_k$. Theorem 9

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of [2] implies the following for $u \leq m < u+v$. If D_{m-1} is already determined then there are at most two possibilities for D_m, and if there are two possibilities for D_m, then there is only one possibility for D_{m+j} with $1 \leq j < \min\{k, u+v-m\}$. As Z_m is uniquely determined by D_m, there are at most $2^{\frac{v}{k}+1}$ possibilities to fill $I_r = \{u, u+1, \ldots, u+v-1\}$ with sets Z_m. Furthermore, if $m \in M_l$ and $l \geq 1$ then Z_m has to be in $\mathcal{X}_l$. Hence we get $\operatorname{card} Q^{-1}\big((M_l)_{l\geq 0}\big) \leq \big(\prod_{l\geq 1}\prod_{m\in M_l}\operatorname{card}\mathcal{X}_l\big)\big(\prod_{j=1}^{s} 2^{(\operatorname{card} I_j/k)+1}\big)$. Because of $\operatorname{card}\mathcal{X}_l \leq e^l$ we get

$$\log \prod_{l\geq 1}\prod_{m\in M_l}\operatorname{card}\mathcal{X}_l = \sum_{l\geq 1}\operatorname{card} M_l \log\operatorname{card}\mathcal{X}_l \leq \sum_{l\geq 1} l \operatorname{card} M_l \leq n\varepsilon$$

as $(M_l)_{l\geq 0} \in P(\varepsilon, n)$. Because of $s \leq n\varepsilon$ we get

$$\log \prod_{j=1}^{s} 2^{(\operatorname{card} I_j/k)+1} \leq s\log 2 + \frac{1}{k}\operatorname{card} M_0 \log 2 \leq n\varepsilon + \frac{n}{k}.$$

Hence we have shown that

$$\text{(4. 6)} \qquad \log\operatorname{card} Q^{-1}\big((M_l)_{l\geq 0}\big) \leq 2n\varepsilon + \frac{n}{k} \text{ for all } (M_l)_{l\geq 0} \in P(\varepsilon, n)$$

In addition to this, (4.5) requires the estimation of $\operatorname{card} P(\varepsilon, n)$.

To this end set $R = \{(n_j)_{j\geq 0} : n_j \geq 0, \sum_{j\geq 0} n_j = n, \sum_{j\geq 0} j n_j \leq \varepsilon n\}$. Then $\operatorname{card} P(\varepsilon, n) \leq \sum_{(n_j)_{j\geq 0}\in R} n!/\prod_{j\geq 0} n_j!$. We have for every $(n_j)_{j\geq 0} \in R$ that $n_j \leq \frac{\varepsilon n}{j}$ and $\max\{j : n_j \neq 0\} \leq \varepsilon n$. Hence $\operatorname{card} R \leq \frac{(\varepsilon n)^{[\varepsilon n]}}{[\varepsilon n]!}$, where $[\varepsilon n] = \max\{m \in \mathbb{N} : m \leq \varepsilon n\}$. By Stirling's formula we get the existence of a $c_1 > 0$ and a $c_2 < \infty$ with $c_1 n^{n+\frac{1}{2}} e^{-n} \leq n! \leq c_2 n^{n+\frac{1}{2}} e^{-n}$ for all $n \in \mathbb{N}$, which implies $\log\operatorname{card} R \leq d_1 + \varepsilon n$ for some constant $d_1 < \infty$. It implies also that $\log(n!/\prod_{j\geq 0} n_j!) \leq \log c_2 - \log c_1 + (n+\frac{1}{2})\log\frac{n}{n_0} + (n-n_0)\log n_0 - \sum(\log c_1 + n_j \log n_j)$, where the sum is taken over all $j \geq 1$ with $n_j \neq 0$. For $(n_j)_{j\geq 0} \in R$ we estimate the right hand side of this inequality in three steps. The definition of R gives $n - n_0 \leq \sum_{j\geq 1} j n_j \leq \varepsilon n$ and $\frac{n}{n_0} \leq \frac{1}{1-\varepsilon}$, which implies $(n+\frac{1}{2})\log\frac{n}{n_0} \leq -(n+\frac{1}{2})\log(1-\varepsilon)$. As $\operatorname{card}\{j : n_j \neq 0\} \leq \varepsilon n$ we get $-\sum_{n_j\neq 0}\log c_1 \leq -\varepsilon n \log c_1$. Finally using (1.4), the definition of R and $\sum_{j\geq 1} n_j = n - n_0 \leq \varepsilon n$ imply $(n-n_0)\log n_0 - \sum_{j\geq 1} n_j \log n_j = -n_0 \sum_{j\geq 1}\frac{n_j}{n_0}\log\frac{n_j}{n_0} \leq n_0\sum_{j\geq 1}(j - \log\varepsilon)\frac{n_j}{n_0} + \frac{n_0}{e}\sum_{j\geq 1} e^{-j+\log\varepsilon} \leq n\varepsilon - n\varepsilon\log\varepsilon + \frac{n}{e}\varepsilon\frac{1}{e-1}$. Setting $d_2 = \log c_2 - \log c_1$ and $d_3 = 1 + \frac{1}{e(e-1)} - \log c_1$, these three inequalities imply $\log(n!/\prod_{j\geq 0} n_j!) \leq d_2 - (n+\frac{1}{2})\log(1-\varepsilon) + d_3 n\varepsilon - n\varepsilon\log\varepsilon$ for all $(n_j)_{j\geq 0} \in R$. As $\log\operatorname{card} R \leq d_1 + n\varepsilon$ we get $\log\operatorname{card} P(\varepsilon, n) \leq d_1 + d_2 - (n+\frac{1}{2})\log(1-\varepsilon) + (1+d_3)n\varepsilon - n\varepsilon\log\varepsilon$.

This estimate of $P(\varepsilon, n)$, (4.5) and (4.6) imply that all limit points of the sequence $(\frac{1}{n}\log\operatorname{card}\mathcal{B}_n)_{n\in I}$ are in an interval $[0, \alpha(\varepsilon, k)]$, where $\alpha(\varepsilon, k)$ tends to zero, if $\varepsilon \longrightarrow 0$ and $k \longrightarrow \infty$ and where I is the infinite subset of $\mathbb{N}$ introduced above. Together with (4.4) we get that $\liminf_{n\to\infty}\big(-\frac{1}{n}\sum_{Z\in\mathcal{Z}_n}\mu(Z)\log\mu(Z)\big) \in [0, \alpha(\varepsilon, k) + \varepsilon]$. Since $\varepsilon > 0$ and $k \in \mathbb{N}$ were arbitrary, we get $h_\mu(T, \mathcal{Z}) = 0$. We show that $\mathcal{Z}$ is a generator for μ, which then implies $h_\mu = 0$, the desired contradiction. To this end we show

$$\text{(4. 7)} \qquad \mu(I) = 0 \text{ for all } I \in \mathcal{Z}_\infty := \bigvee_{i=0}^{\infty} T^{-i}\mathcal{Z}$$

Suppose that $\mu(I) > 0$ for some $I \in \mathcal{Z}_\infty$. As μ is T-invariant, this implies $T^k(I) \cap I \neq \emptyset$ for some $k \geq 1$. As $T^k(I) \subset J$ for some $J \in \mathcal{Z}_\infty$ and as the elements of $\mathcal{Z}_\infty$ are pairwise disjoint, this gives $T^k(I) \subset I$. As μ is ergodic, we get $\mu(\bigcup_{i=0}^{k-1} T^i(I)) = 1$. But $T|T^i(I)$ is monotone for $0 \leq i \leq k-1$, which implies that $h_\mu = 0$, a contradiction. Hence (4.7) holds and $\mathcal{Z}$ is a generator. ∎

PROPOSITION 1. *Suppose that T is piecewise monotonic with respect to $\mathcal{Z}$ and that μ is an ergodic T-invariant measure on $E_\mathcal{Z}$ with $h_\mu > 0$ and $-\sum_{Z \in \mathcal{Z}} \mu(Z) \log \mu(Z) < \infty$. If the set $W_\mathcal{Z} := \{x : x \in T(Z)$ for infinitely many $Z \in \mathcal{Z}\}$ is at most countable, then there is an ergodic $\hat{T}$-invariant probability measure $\hat{\mu}$ on $\hat{X}$ satisfying $\hat{\mu} \circ \pi^{-1} = \mu$.*

PROOF. We apply Theorem 2 of [3]. The first one of the two assumptions of this theorem is that for $A \subset [0,1]$ the measurability of $\pi^{-1}(A)$ modulo a set $\pi^{-1}(N)$ with $\mu(N) = 0$ implies the measurability of A up to a μ-nullset. This follows since $i^{-1} \circ \pi^{-1}(A) = A \cap E_1$ and $\mu(E_1) = 1$. The second one is the existence of a set N of μ-measure zero such that

$$\text{(4.8)} \qquad \hat{x}, \hat{y} \in \hat{X} \setminus \pi^{-1}(N) \text{ and } \pi(\hat{x}) = \pi(\hat{y}) \Rightarrow \hat{T}^n(\hat{x}) = \hat{T}^n(\hat{y}) \text{ for some } n$$

It follows from (ii) of Theorem 1 in [2], that N can be chosen as the union of those sets of $\mathcal{Z}_\infty := \bigvee_{i=0}^\infty T^{-i}\mathcal{Z}$, which contain an endpoint of some $D \in \mathcal{D}$. This is a countable union of sets, each of which has μ-measure zero by (4.7). Hence $\mu(N) = 0$ and (4.8) follows. Furthermore, since $\hat{T}$ is defined only on $\hat{X} \setminus \pi^{-1}(E_2)$, in order to make the proof of Theorem 2 in [3] work, we have to show the following continuity result. Consider some $D \in \mathcal{D}$. As D is a subset of some $Z \in \mathcal{Z}$, $\hat{T}$ maps $D' \cap \pi^{-1}(E_2)$ monotonically and bijectively to $\bigcup_{C \in \mathcal{H}} C'$, where $\mathcal{H}$ is a finite or countable subset of $\mathcal{D}$. If K is a compact subset of some C' with $C \in \mathcal{H}$, this implies that $\hat{T}^{-1}(K) \cap D'$ is compact and contained in $D' \cap \pi^{-1}(E_2)$. This fact shows the following. If $g : \hat{X} \longrightarrow \mathbb{R}$ has compact support, extend $g \circ \hat{T}$ to all of $\hat{X}$, setting it equal to zero outside $\pi^{-1}(E_2)$, which is the set, on which $\hat{T}$ is not defined. Using the above result for $K = \operatorname{supp} g \cap C'$ for all $C \in \mathcal{H}$, we get that the extended $g \circ \hat{T}$ is continuous on D' and hence on all of $\hat{X}$. Now Theorem 2 of [3] implies that $(\hat{\mu}_n)_{n\geq 1}$ has a weak limit point $\hat{\mu}$. Furthermore, if $\hat{\mu}$ is not the zero-measure, it has the desired properties. In order to show this, we check that the zero-measure is not a limit point of $(\hat{\mu}_n)_{n\geq 1}$ using Lemma 8.

To this end set $V_k = \{x : x \in D$ for infinitely many $D \in \mathcal{D}_k\}$. We show by induction that V_k is at most countable. As $\mathcal{D}_1 = \mathcal{Z}$, we have $V_1 = \emptyset$. Hence suppose that the assertion is shown for $k = l-1$. Choose $x \in V_l \setminus V_{l-1}$. This means that there is a sequence $(D_i)_{i\geq 1}$ in $\mathcal{D}_l \setminus \mathcal{D}_{l-1}$ with $x \in D_i$. By definition of $\mathcal{D}_l$ there are $A_i \in \mathcal{Z}_l$ with $D_i = T^{l-1}(A_i)$. Since $A_i = B_i \cap T^{-(l-1)}(Z_i)$ for some $B_i \in \mathcal{Z}_{l-1}$ and $Z_i \in \mathcal{Z}$, we get $D_i = T(E_i) \cap Z_i$, where $E_i = T^{l-2}(B_i) \in \mathcal{D}_{l-1}$. For each E_i there is a unique $Y_i \in \mathcal{Z}$ with $E_i \subset Y_i$. We consider two cases. The first case is that there is a $Y \in \mathcal{Z}$ and an infinite subset I of $\mathbb{N}$ with $Y_i = Y$ for all $i \in I$. As $x \in D_i \subset T(E_i) \subset T(Y)$ for $i \in I$, there is a $y \in Y$ with $T(y) = x$. As $T|Y$ is strictly monotone and continuous, $x \in T(E_i)$ implies $y \in E_i$ for $i \in I$. Hence $y \in V_{l-1}$ and $x \in T(V_{l-1})$. The second case is that each

$Y \in \mathcal{Z}$ occurs only finitely many times in $(Y_i)_{i\geq 1}$. Hence $(Y_i)_{i\geq 1}$ contains infinitely many different elements of $\mathcal{Z}$. As $x \in T(Y_i)$ for all i, we get $x \in W_{\mathcal{Z}}$. We have shown that $x \in T(V_{l-1}) \cup W_{\mathcal{Z}}$, which gives $V_l \subset V_{l-1} \cup T(V_{l-1}) \cup W_{\mathcal{Z}}$. This implies that V_l is at most countable, finishing the induction. For all k we have $\bigcap \bigcup_{D\in\mathcal{D}_k\setminus\mathcal{F}} D = V_k$, where the intersection is taken over all finite subsets $\mathcal{F}$ of $\mathcal{D}_k$. As μ is ergodic and $h_\mu > 0$ and hence μ has no atoms, we get $\mu(V_k) = 0$. This implies the condition in Lemma 8. The zero-measure is not a weak limit point of $(\hat{\mu}_n)_{n\geq 1}$. ∎

Now we can show

PROPOSITION 2. *Suppose that T is piecewise monotonic with respect to $\mathcal{Z}$ and has a bounded derivative. Suppose further that the set $W_{\mathcal{Z}} := \{x : x \in T(Z)$ for infinitely many $Z \in \mathcal{Z}\}$ is at most countable. For $x \in E_{\mathcal{Z}}$ let $r_n(x)$ be the distance of $T^n(x) \in T^n\big(Z_{n+1}(x)\big)$ to the nearer endpoint of the interval $T^n\big(Z_{n+1}(x)\big)$. If μ is an ergodic T-invariant measure on $E_{\mathcal{Z}}$ with $h_\mu > 0$ and $-\sum_{Z\in\mathcal{Z}}\mu(Z)\log\mu(Z) < \infty$, we have that* $\lim_{n\to\infty}\frac{1}{n}\log r_n(x) = 0$ *for μ-almost all $x \in E_{\mathcal{Z}}$.*

PROOF. Let $\hat{\mu}$ be as in Proposition 1. Since $\hat{X} = \bigcup_{D\in\mathcal{D}} D'$, there is an $E \in \mathcal{D}$ with $\hat{\mu}(E') > 0$ and an interval $C \subset E$ with $\hat{\mu}(C') > 0$ where $C' = E' \cap \pi^{-1}(C)$, such that the distance from C to the endpoints of E is greater than or equal to some $d > 0$. All iterates of $\hat{T}$ are defined on the set $\pi^{-1}(E_{\mathcal{Z}})$, whose $\hat{\mu}$-measure equals $\mu(E_{\mathcal{Z}}) = 1$. Set $\hat{L} = \{\hat{x} \in \pi^{-1}(E_{\mathcal{Z}}\setminus N) : \lim_{j\to\infty}\frac{1}{j}\sum_{l=0}^{j-1} 1_{C'}\big(\hat{T}^l(\hat{x})\big) = \hat{\mu}(C')\}$, where N is as in (4.8). As $\hat{\mu}$ is ergodic and as $\hat{\mu}(\pi^{-1}(N)) = \mu(N) = 0$, the ergodic theorem implies that $\hat{\mu}(\hat{L}) = 1$. By (4.8), $\hat{x} \in \hat{L}$ and $\pi(\hat{x}) = \pi(\hat{y})$ imply that $\hat{y} \in \hat{L}$. Hence the set $L := \pi(\hat{L})$ satisfies $\hat{L} = \pi^{-1}(L)$. We have $\mu(L) = \hat{\mu}(\hat{L}) = 1$.

Fix $x \in L$. Let $n_1, n_2, \ldots$ be the succesive integers, such that $\hat{T}^{n_k} \circ i(x) \in C'$ for $k \geq 1$. As $i(x) \in \hat{L}$, this sequence is infinite and $\lim_{k\to\infty}\frac{k}{n_k} = \hat{\mu}(C') > 0$, which implies $\lim_{k\to\infty}\frac{n_k}{n_{k-1}} = 1$. For $j \geq 0$ set $D_j = \psi\big(\hat{T}^j \circ i(x)\big)$, which is $T^j(Z_{j+1}(x))$ by (4.1). For $k \geq 0$ we have $r_{n_k}(x) \geq d$, as $\hat{T}^{n_k} \circ i(x) \in C'$ and $D_{n_k} = E$. Choose $c \in (1, \infty)$, such that $c \geq \sup_{x\in[0,1]}|T'|$. Suppose that a is an endpoint of D_j with $r_j(x) = |T^j(x) - a|$. As D_{j+1} is a subinterval of the interval $T(D_j)$, which can be seen from the definition of $\hat{T}$, we get that $r_{j+1}(x) \leq |T^{j+1}(x) - T(a)| \leq c|T^j(x) - a| \leq cr_j(x)$. For fixed j let k be minimal, such that $n_k \geq j$. Then we get $r_j(x) \geq c^{-(n_k-j)}r_{n_k}(x) \geq c^{-(n_k-n_{k-1})}d$. This implies $\frac{1}{j}\log r_j(x) \geq -\frac{n_k-n_{k-1}}{n_{k-1}}\log c + \frac{1}{j}\log d$. As $x \in L$, we have $\lim_{k\to\infty}\frac{n_k}{n_{k-1}} = 1$ and hence the right hand side of the above inequality tends to zero for $j \to \infty$. As $\log r_j(x) \leq 0$, this implies that $\frac{1}{j}\log r_j(x) \to 0$ for $j \to \infty$. This holds for all $x \in L$ and $\mu(L) = 1$. ∎

COROLLARY 2. *Set $C = [0,1] \setminus \bigcup_{Z\in\mathcal{Z}} Z$. Under the conditions of Proposition 2 we have* $\lim_{n\to\infty}\frac{1}{n}\log\operatorname{dist}(T^n(x), C) = 0$ *for μ-almost all x.*

PROOF. This follows, since $T^n\big(Z_{n+1}(x)\big)$ is a subinterval of some interval in $\mathcal{Z}$ and hence $r_n(x) \leq \operatorname{dist}(T^n(x), C) \leq 1$. ∎

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