Canad. Math. Bull. Vol. **32** (4), 1989

PROPERTIES OF THE PRODUCT OF TWO DERIVATIONS OF A C^*-ALGEBRA

BY

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ABSTRACT. Let δ_1, δ_2 be two derivations of a C^*-algebra. We characterize when $\delta_1\delta_2$ is a derivation, a compact, or a weakly compact operator.

1. **Introduction.** A number of years ago, Posner proved in [9] that if the product $\delta_1\delta_2$ of two derivations δ_1, δ_2 of a prime ring of characteristic different from 2 is a derivation, then $\delta_1 = 0$ or $\delta_2 = 0$. This result has been reproved (under stronger assumptions) several times (cf. e.g. [2], [13]). It is also known that if δ is a derivation of a C^*-algebra and δ^2 is also a derivation, then $\delta = 0$ ([3], proof of Lem. 1.1.9). The higher iterates of (inner) derivations were investigated by Martindale and Miers [5] and Miers and Phillips [8]. For instance, they proved that $(ad\, a)^{2n}$ is an inner derivation of a unital C^*-algebra A on a Hilbert space H only if there exists a central element z in the weak closure of A such that $(a - z)^n = 0$ ([5], Thm 5). In Theorem 1 below, we will see how Posner's theorem extends to arbitrary C^*-algebras. In particular, it will follow that $\delta_1\delta_2$ is a derivation only if $\delta_1\delta_2 = 0$.

The compact and the weakly compact derivations of C^*-algebras were characterized by Akemann and Wright [1] (see also [12]). In [6] we studied compact and weakly compact elementary operators on prime C^*-algebras. The techniques developed there and in [7] will yield characterizations of when $\delta_1\delta_2$ is a compact or a weakly compact operator (Theorems 8 and 6, respectively). In particular, the product of two non-zero derivations of a prime C^*-algebra is weakly compact only if either one is weakly compact, and is compact only if both of them are weakly compact. (Note that there are no non-zero *compact* derivations on an infinite dimensional prime C^*-algebra [6].)

We conclude this introduction by recalling some notions and establishing the notation which will be used in the sequel. A C^*-algebra A is called *prime* if the product IJ of any two non-zero ideals I, J of A is a non-zero ideal. Two elements a, b of a W^*-algebra are said to be *centrally orthogonal* if the mapping $x \longmapsto axb$ is identically zero. If δ is a derivation of a C^*-algebra A and (π, H) is a representation of A, then δ^π denotes the induced ultraweakly continuous derivation on $\pi(A)''$. Also δ^{**} denotes the induced derivation on the enveloping von Neumann algebra A^{**}. The ideal $K(A)$ of compact elements of A consists of those $a \in A$ for which $x \longmapsto axa$ is a compact

Received by the editors April 28, 1988 and, in revised form, July 7, 1988.
Research supported by DFG and DAAD.
AMS Subject Classification Primary 46L40, Secondary 47B05, 47B47.

operator on A. Equivalently, $a \in K(A)$ if and only if $x \longmapsto ax(x \longmapsto xa)$ is weakly compact ([14], Thm 3.1). It is well known that if $A = B(H)$, the algebra of all bounded operators on some Hilbert space H, then $K(A)$ coincides with the ideal $K(H)$ of all compact operators on H. Finally, $Z(A)$ stands for the center of A.

2. **The results.** Our first theorem shows how the information from Posner's result applied in irreducible representations of a C^*-algebra can be patched together to obtain a global result. If δ is a derivation of A, the identity $\delta = ad\, a$ with $a \in A^{**}$ will mean that A is considered as a subalgebra of A^{**} modulo *some* faithful representation of A.

THEOREM 1. *Let δ_1, δ_2 be two derivations of a C^*-algebra A. Then $\delta_1\delta_2$ is a derivation if and only if there are centrally orthogonal elements a_1, a_2 in A^{**} such that $\delta_i = ad\, a_i$ for $i = 1, 2$.*

PROOF. "if"-part. Let $\delta_i = ad\, a_i$ for some centrally orthogonal elements $a_i \in A^{**}$. Then $\delta_1\delta_2 = ad\, a_1 \circ ad\, a_2 = 0$ is a derivation.

"only if"-part. Let Γ be a family of disjoint irreducible representations of A with faithful direct sum ρ. Identifying $\rho(A)''$ with $A^{**}c(\rho)$, where $c(\rho)$ is the central cover of ρ, we have $\delta^{**}_{i|A^{**}c(\rho)} = \delta^{\rho}_i$. By [10], 4.1.7, there are $b_i \in A^{**}c(\rho)$ such that $\delta^{\rho}_i = ad\, b_i, i = 1, 2$, thus $\delta_i = ad\, b_i$. Take $\pi \in \Gamma$. Since $\pi(A)$ is prime and $\delta^{\pi}_1\delta^{\pi}_2 = (\delta_1\delta_2)^{\pi}$ is a derivation, it follows from Posner's result ([9], Thm 1) that either $\delta^{\pi}_1 = 0$ or $\delta^{\pi}_2 = 0$. Now $\delta^{\pi}_i = ad(b_ic(\pi)) = 0$ if and only if $b_ic(\pi) \in \mathbf{C}c(\pi)$, thus we obtain complex numbers λ^{π}_i such that $b_ic(\pi) = \lambda^{\pi}_i c(\pi)$ whenever $\delta^{\pi}_i = 0$. We put $\lambda^{\pi}_i = 0$ if $\delta^{\pi}_i \neq 0$. From $|\lambda^{\pi}_i| \leqq \|b_i\|$ we can define

$$z_i = \sum_{\pi\in\Gamma}{}^{\oplus} \lambda^{\pi}_i c(\pi) \in \sum_{\pi\in\Gamma}{}^{\oplus} \mathbf{C}c(\pi) = Z(A^{**}c(\rho)).$$

Putting $a_i = b_i - z_i$ we obtain $a_i \in A^{**}c(\rho)$ satisfying $\delta_i = ad\, a_i$, and $\delta^{\pi}_i = 0$ if and only if $a_ic(\pi) = 0$. Let $x \in A^{**}$. Then

$$a_1xa_2 = a_1c(\rho)xa_2c(\rho) = \sum_{\pi\in\Gamma}{}^{\oplus} a_1c(\pi)xa_2c(\pi) = 0$$

for, if $a_1c(\pi) \neq 0$ then $\delta^{\pi}_1 \neq 0$ and therefore $\delta^{\pi}_2 = 0$ which implies $a_2c(\pi) = 0$. Hence, a_1 and a_2 are centrally orthogonal.

□

COROLLARY 2. *The product of two derivations of a C^*-algebra is a derivation if and only if it is zero.*

Suppose that δ is a derivation of A such that δ^2 is also a derivation. By Theorem 1, $\delta = ad\, a_1 = ad\, a_2$ for some centrally orthogonal elements $a_i \in A^{**}$. Since the range of δ is contained in the intersection of the ultraweakly closed ideals generated by a_1 and a_2 respectively, it follows that $\delta = 0$. This gives another proof for the result cited in the Introduction.

The next result is quoted from [7]; its proof is similar to that of [6], Lem. 3.5. We say that a bounded linear map T on a C^*-algebra A is a *central bimodule homomorphism* of A if its second adjoint T^{**} fixes each closed ideal of A^{**}.

LEMMA 3. *If $T : A \longrightarrow A$ is a weakly compact central bimodule homomorphism of a C^*-algebra A then $TA \subseteq K(A)$.*

In the sequel we will use the equivalence of the following three properties of a derivation δ on $B(H)$ (cf. [1], Thm 3.1 or [6], Cor. 3.3): (i) δ is weakly compact, (ii) $\delta B(H) \subseteq K(H)$, (iii) $\delta = ad\, a$ for some $a \in K(H)$ and $\|a\| \leqq \|\delta\|$.

LEMMA 4. *Let δ_1, δ_2 be two derivations of a prime C^*-algebra A. If $\delta_1\delta_2$ is weakly compact then δ_1 is weakly compact or δ_2 is weakly compact.*

PROOF. Since $\delta_1\delta_2$ is clearly a central bimodule homomorphism of A, we have $\delta_1\delta_2 A \subseteq K(A)$ by Lemma 3. If $K(A) = 0$ then $\delta_1\delta_2 = 0$, whence by Posner's result $\delta_1 = 0$ or $\delta_2 = 0$. If A contains non-zero compact elements, it is primitive (cf. e.g. [6], Prop. 2.3). Let (π, H) be a faithful irreducible representation of A with $\pi(K(A)) = K(H)$. A standard argument shows that $\delta_1^\pi\delta_2^\pi$ is weakly compact on $\pi(A)'' = B(H)$ and that $\delta_1^\pi\delta_2^\pi B(H) \subseteq K(H)$ (compare [6], Lem. 3.4). If $\tilde{\delta}_i$ denotes the induced derivation on the Calkin algebra $C(H) = B(H)/K(H)$ for $i = 1, 2$, we conclude that $\tilde{\delta}_1\tilde{\delta}_2 = 0$. Posner's result applied to the prime algebra $C(H)$ yields $\tilde{\delta}_1 = 0$ or $\tilde{\delta}_2 = 0$, i.e. $\delta_1^\pi B(H) \subseteq K(H)$ or $\delta_2^\pi B(H) \subseteq K(H)$. Therefore either δ_1 or δ_2 has to be weakly compact.

□

The next technical lemma may be viewed as an asymptotic version of Posner's theorem.

LEMMA 5. *Let $(H_n)_{n \in \mathbf{N}}$ be a sequence of Hilbert spaces, $A = \sum^{\oplus} B(H_n)$ and $\delta^{(1)}, \delta^{(2)}$ be two derivations of A such that $\lim_{n \to \infty} \|\delta_n^{(1)}\delta_n^{(2)}\| = 0$ where $\delta_n^{(1)}$ denotes the restriction of $\delta^{(i)}$ to $B(H_n)$, $i = 1, 2$. Then $\lim_{n \to \infty} \|\delta_n^{(1)}\| \, \|\delta_n^{(2)}\| = 0$.*

PROOF. If $x \in A$ then x_n will mean its component in $B(H_n) \subseteq A$, thus $\delta_n^{(i)}(x_n) = (\delta^{(i)}(x))_n$. Given $\epsilon > 0$ take $n_0 \in \mathbf{N}$ such that $\|\delta_n^{(1)}\delta_n^{(2)}\| < \epsilon$ for all $n \geqq n_0$. Since, for all $x, y \in A$,

$$\delta^{(1)}\delta^{(2)}(xy) = (\delta^{(1)}\delta^{(2)}x)y + (\delta^{(2)}x)(\delta^{(1)}y) + (\delta^{(1)}x)(\delta^{(2)}y) + x(\delta^{(1)}\delta^{(2)}y)$$

it follows that

$$\begin{aligned} &\|(\delta_n^{(2)}x_n)(\delta_n^{(1)}y_n) + (\delta_n^{(1)}x_n)(\delta_n^{(2)}y_n)\| \\ &= \|\delta_n^{(1)}\delta_n^{(2)}(x_n y_n) - (\delta_n^{(1)}\delta_n^{(2)}x_n)y_n - x_n(\delta_n^{(1)}\delta_n^{(2)}y_n)\| \\ &\leqq \|\delta_n^{(1)}\delta_n^{(2)}\| \, \|x_n y_n\| + \|\delta_n^{(1)}\delta_n^{(2)}\| \, \|x_n\| \, \|y_n\| + \|x_n\| \, \|\delta_n^{(1)}\delta_n^{(2)}\| \, \|y_n\|, \end{aligned}$$

thus

$$\|(\delta_n^{(2)}x_n)(\delta_n^{(1)}y_n) + (\delta_n^{(1)}x_n)(\delta_n^{(2)}y_n)\| < 3\epsilon\|x_n\| \, \|y_n\| \tag{1}$$

for all $n \geqq n_0$.

Replacing x by xz in (1) and using (1) we obtain

$$\begin{aligned}\|(\delta_n^{(2)}x_n)z_n(\delta_n^{(1)}y_n) + (\delta_n^{(1)}x_n)z_n(\delta_n^{(2)}y_n)\| \\ &= \|\delta_n^{(2)}(x_nz_n)(\delta_n^{(1)}y_n) + \delta_n^{(1)}(x_nz_n)(\delta_n^{(2)}y_n) \\ &\quad - x_n(\delta_n^{(2)}z_n)(\delta_n^{(1)}y_n) - x_n(\delta_n^{(1)}z_n)(\delta_n^{(2)}y_n)\| \\ &< 3\epsilon\|x_nz_n\|\,\|y_n\| + 3\epsilon\|x_n\|\,\|z_n\|\,\|y_n\|\end{aligned}$$

whence

$$\|(\delta_n^{(2)}x_n)z_n(\delta_n^{(1)}y_n) + (\delta_n^{(1)}x_n)z_n(\delta_n^{(2)}y_n)\| < 6\epsilon\|x_n\|\,\|y_n\|\,\|z_n\|. \tag{2}$$

The identity

$$\begin{aligned}2(\delta^{(1)}x)(\delta^{(2)}w)(\delta^{(1)}y) &= \delta^{(1)}x((\delta^{(2)}w)(\delta^{(1)}y) + (\delta^{(1)}w)(\delta^{(2)}y)) \\ &\quad + ((\delta^{(1)}x)(\delta^{(2)}w) + (\delta^{(2)}x)(\delta^{(1)}w))\delta^{(1)}y \\ &\quad - (\delta^{(2)}x)(\delta^{(1)}w)(\delta^{(1)}y) - (\delta^{(1)}x)(\delta^{(1)}w)(\delta^{(2)}y)\end{aligned}$$

together with (1) and (2) applied to $z = \delta^{(1)}w$ yields

$$\begin{aligned}2\|(\delta_n^{(1)}x_n)(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| &< 3\epsilon\|\delta_n^{(1)}x_n\|\,\|w_n\|\,\|y_n\| \\ &\quad + 3\epsilon\|x_n\|\,\|w_n\|\,\|\delta_n^{(1)}y_n\| \\ &\quad + 6\epsilon\|x_n\|\,\|y_n\|\,\|\delta_n^{(1)}w_n\| \\ &\leqq 12\epsilon\|x_n\|\,\|y_n\|\,\|w_n\|\,\|\delta_n^{(1)}\|\end{aligned}$$

for all $n \geqq n_0$.

From this and the identity

$$(\delta^{(1)}x)z(\delta^{(2)}w)(\delta^{(1)}y) = \delta^{(1)}(xz)(\delta^{(2)}w)(\delta^{(1)}y) - x(\delta^{(1)}z)(\delta^{(2)}w)(\delta^{(1)}y)$$

it follows that

$$\begin{aligned}&\|(\delta_n^{(1)}x_n)z_n(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| \\ &\leqq \|\delta_n^{(1)}(x_nz_n)(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| + \|x_n\|\,\|(\delta_n^{(1)}z_n)(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| \\ &< 12\epsilon\|x_n\|\,\|y_n\|\,\|z_n\|\,\|w_n\|\,\|\delta_n^{(1)}\|.\end{aligned}$$

Using the fact that $\|L_{a_n}R_{b_n}\| = \|a_n\|\,\|b_n\|$ for all $a_n, b_n \in B(H_n)$ we conclude that

$$\|\delta_n^{(1)}x_n\|\,\|(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| \leqq 12\epsilon\|x_n\|\,\|y_n\|\,\|w_n\|\,\|\delta_n^{(1)}\|$$

and hence, by taking the supremum over $\|x_n\| \leqq 1$,

$$\|(\delta_n^{(2)}w_n)(\delta_n^{(1)}y_n)\| \leqq 12\epsilon\|y_n\|\,\|w_n\|.$$

Replacing w by wz we obtain

$$\begin{aligned}\|(\delta_n^{(2)}w_n)z_n(\delta_n^{(1)}y_n)\| &\leqq \|\delta_n^{(2)}(w_nz_n)(\delta_n^{(1)}y_n)\| + \|w_n\|\,\|(\delta_n^{(2)}z_n)(\delta_n^{(1)}y_n)\| \\ &\leqq 24\epsilon\|y_n\|\,\|w_n\|\,\|z_n\|\end{aligned}$$

which finally gives

$$\|\delta_n^{(2)}w_n\|\,\|\delta_n^{(1)}y_n\| \leqq 24\epsilon\|w_n\|\,\|y_n\|$$

for all $n \geqq n_0$.

This proves that $\lim_{n\to\infty}\|\delta_n^{(1)}\|\,\|\delta_n^{(2)}\| = 0$.

□

THEOREM 6. *Let δ_1, δ_2 be two derivations of a C^*-algebra A. Then $\delta_1\delta_2$ is weakly compact if and only if there are $a_i \in A^{**}$ such that $\delta_i = ad\, a_i$ for $i = 1,2$ and there exist orthogonal central projections e_j in $A^{**}, j = 1,2,3$, with $e_1 + e_2 + e_3 = 1$, $c \in Z(A^{**})$ and $\tilde{a}_i \in A^{**}$ such that $ca_ie_i = \tilde{a}_ie_i$ is compact for $i = 1,2$, $c\tilde{a}_ie_j = a_ie_j$ for $i,j = 1,2, i \neq j$, and a_1e_3 and a_2e_3 are centrally orthogonal.*

PROOF. "if"-part. We have

$$\begin{aligned} ad\,a_1 \circ ad\,a_2 &= ad\,a_1 \circ ad\,a_2e_1 + ad\,a_1e_2 \circ ad\,a_2 \\ &= ad\,a_1 \circ ad\,c\tilde{a}_2e_1 + ad\,c\tilde{a}_1e_2 \circ ad\,a_2 \\ &= ad\,ca_1e_1 \circ ad\,\tilde{a}_2 + ad\,\tilde{a}_1 \circ ad\,ca_2e_2. \end{aligned}$$

Since $ca_ie_i \in K(A^{**})$ it follows that the left multiplication $L_{ca_ie_i}$ and the right multiplication $R_{ca_ie_i}$ are weakly compact ([14], Thm 3.1). Therefore, $\delta_1\delta_2$ is weakly compact.

"only if"-part. We adopt the notation used in the proof of Theorem 1. Thus we may assume that $\delta_i = ad\,b_i$ with $b_i \in A^{**}c(\rho)$. The weak compactness of $\delta_1\delta_2$ implies that the set $\Gamma_\epsilon = \{\pi \in \Gamma \mid \|\delta_1^\pi\delta_2^\pi\| > \epsilon\}$ is finite for each $\epsilon > 0$, whence $\Gamma_0 = \{\pi \in \Gamma \mid \delta_1^\pi\delta_2^\pi \neq 0\}$ is countable (see [7]; cf. also [1], Lem. 3.2). Let e_3' be the central cover of $\oplus_{\pi\in\Gamma\setminus\Gamma_0}\pi$ and $e_3 = e_3' + 1 - c(\rho)$; then $\delta_1\delta_{2|Ae_3} = 0$. By Theorem 1, we can perturb b_i by central elements in order to obtain $a_i' \in A^{**}c(\rho)$ such that $\delta_i^{**} = ad\,a_i'$ and $a_1'e_3$ and $a_2'e_3$ are centrally orthogonal.

Suppose that $\pi \in \Gamma_0$. By Lemma 4, δ_1^π is weakly compact or δ_2^π is weakly compact. Put $\Gamma_1 = \{\pi \in \Gamma_0 \mid \delta_1^\pi$ is weakly compact$\}$ and $\Gamma_2 = \Gamma_0 \setminus \Gamma_1 = \{\pi \in \Gamma_0 \mid \delta_2^\pi$ is weakly compact and δ_1^π is not weakly compact$\}$, and let e_i be the central cover of $\oplus_{\pi\in\Gamma_i}\pi$ for $i = 1,2$. Without restriction we assume that Γ_1 is denumerable, say $\Gamma_1 = \{\pi_n \mid n \in \mathbf{N}\}$. Since $\lim_{n\to\infty}\|\delta_1^{\pi_n}\delta_2^{\pi_n}\| = 0$, it follows from Lemma 5 that $\lim_{n\to\infty}\|\delta_1^{\pi_n}\|\,\|\delta_2^{\pi_n}\| = 0$ (observe that $A^{**}e_1 = \sum^{\oplus} B(H_{\pi_n})$). By the aforementioned result, we may perturb a_1' by a central element in $A^{**}e_1$ to obtain $a_1'' \in A^{**}$ such that $\delta_1 = ad\,a_1''$, $a_1''p_n \in K(H_{\pi_n})$, and $\|a_1''p_n\| \leqq \|\delta_1^{\pi_n}\|$, where p_n is the central cover of π_n, and by [11], Thm 4 we may perturb a_2' centrally to obtain $a_2'' \in A^{**}$ such that $\delta_2 = ad\,a_2''$ and $\|\delta_2^{\pi_n}\| = 2\|a_2''p_n\|$ for

https://doi.org/10.4153/CMB-1989-072-4 Published online by Cambridge University Press

each $n \in \mathbf{N}$. Therefore, $\lim_{n\to\infty} \|a_1'' p_n\| \, \|a_2'' p_n\| = 0$. We now put

$$\begin{aligned} c_{11} &= \sum_{n\in\mathbf{N}}{}^{\oplus} \|a_2'' p_n\|^{1/2} p_n \in Z(A^{**}e_1), \\ a_{11} &= c_{11}a_1'' + a_1''(1-e_1), \\ a_{21} &= \sum_{n\in\mathbf{N}}{}^{\oplus} \|a_2'' p_n\|^{-1/2} a_2'' p_n \in A^{**}e_1 \end{aligned}$$

(observe that $\|a_2'' p_n\| > 0$ for all n since $\delta_2^{\pi_n}$ is non-zero).

As $a_{11}p_n = c_{11}a_1''p_n = \|a_2''p_n\|^{1/2}a_1''p_n$ is compact and $\|a_{11}p_n\| = \|a_2''p_n\|^{1/2} \cdot \|a_1''p_n\| \longrightarrow 0$, we conclude from Propositon 2.1 in [6] that $a_{11}e_1$ is a compact element of $A^{**}e_1$. We obviously have $c_{11}a_{21} = a_2''e_1$.

Applying the same arguments to Γ_2 we will change a_i'' into a_i''', enjoying the corresponding properties, by perturbing with central elements in $A^{**}e_2$ in order to obtain $c_{22} \in Z(A^{**}e_2), a_{22} = a_2'''(1-e_2) + c_{22}a_2''', a_{22}e_2 \in K(A^{**}e_2)$, and $a_{12} \in A^{**}e_2$ satisfying $c_{22}a_{12} = a_1'''e_2$.

If we now define $a_i = a_i''e_1 + a_i'''e_2 + a_i'e_3, \tilde{a}_i = a_{i1}e_1 + a_{i2}e_2, i+1,2$, and $c_i = c_{11}+c_{22} \in Z(A^{**})$, then a_1e_3 and a_2e_3 are still centrally orthogonal, $\delta_i = ad\, a_i, ca_ie_i = a_{ii}e_i = \tilde{a}_ie_i$ is compact for $i = 1,2$, and clearly $c\tilde{a}_ie_j = a_ie_j$ for $i \neq j$. The proof is complete.

□

The next result appeared for the case $A = B(H)$ in [4], however with an incorrect proof. Fong and Sourour used a version of Posner's theorem and a result on elementary operators to give a new proof in [2], p. 854. The following lemma uses similar techniques to obtain an extension to prime C^*-algebras.

LEMMA 7. *Let δ_1, δ_2 be two non-zero derivations of an infinite dimensional prime C^*-algebra A. Then $\delta_1\delta_2$ is compact if and only if there exist $a_i \in K(A)$ such that $\delta_i = ad\, a_i$ for $i = 1,2$ and $a_1a_2 = a_2a_1 = 0$.*

PROOF. If $a_1a_2 = a_2a_1 = 0$ then $ad\, a_1 \circ ad\, a_2 = -L_{a_1}R_{a_2} - L_{a_2}R_{a_1}$, which is compact if $a_1, a_2 \in K(A)$. To prove the "only if"-part observe that $\delta_1\delta_2 A \subseteq K(A)$ by Lemma 3. By Posner's result, $\delta_1\delta_2 \neq 0$ hence $K(A) \neq 0$. We may therefore assume that A acts irreducibly on a Hilbert space H and that $K(A) = K(H)$. Let $\tilde{\delta}_i$ denote the ultraweak extension of δ_i to $B(H)$. Since $\tilde{\delta}_1\tilde{\delta}_2$ is compact, $\delta_1 = ad\, a$ and $\delta_2 = ad\, b$ where a or b is compact by Lemma 4 and the remarks preceding it. We now have to show that a and b can be replaced by elements $a_1, a_2 \in K(H)$ such that $a_1a_2 = a_2a_1 = 0$. This needs the same arguments as in [2], p. 854, which for the convenience of the reader are given here. Suppose that $a \in K(H)$. By [2], Thm 2, the compactness of

$$\tilde{\delta}_1\tilde{\delta}_2 = L_{ab} - L_aR_b - L_bR_a + R_{ba} \tag{3}$$

and $\dim H = \infty$ imply that the set $\{1, b, a, ba\}$ is linearly dependent. Thus $\lambda 1 + \mu b \in K(H)$ for some complex numbers λ, μ. If $\mu \neq 0$, we put $a_1 = a$ and $a_2 = (\lambda/\mu)1 + b$.

By equation (3), $L_{a_1a_2}+R_{a_2a_1}$ is then compact. By [2], Thm 2 again, the set $\{1, a_2a_1\}$ is linearly dependent, but since $a_2a_1 \in K(H)$, this implies $a_2a_1 = 0$. Similarly, $a_1a_2 = 0$. If $\mu = 0$ then $\lambda = 0$, too. Therefore $\{a, ba\}$ is linearly dependent, say $\lambda' a + \mu' ba = 0$ with $\mu' \neq 0$. Replacing b by $b - (\lambda'/\mu')1$ we may assume that $ba = 0$. Then $\tilde{\delta}_1\tilde{\delta}_2 = L_{ab} - L_aR_b - L_bR_a$, and [2], Thm 2 entails that $\{1, b, a\}$ is linearly dependent. Therefore, $\lambda''1 + \mu''b \in K(H)$ where $\mu'' \neq 0$, and we are back to the first case.

□

The above lemma together with Theorem 6 yields our final result.

THEOREM 8. *Let δ_1, δ_2 be two derivations of a C^*-algebra A. Then $\delta_1\delta_2$ is compact if and only if there are $a_i \in A^{**}$ such that $\delta_i = ad\, a_i, i = 1, 2,$ as well as orthogonal central projections e_j in $A^{**}, j = 1, 2, 3,$ with $e_1 + e_2 + e_3 = 1$ and elements $\tilde{a}_i \in A^{**}, c_i \in Z(A^{**})_+, i = 1, 2,$ such that a_1e_3 and a_2e_3 are centrally orthogonal, $c_ia_i(1 - e_3)$ is compact for $i = 1, 2, c_2\tilde{a}_1 = a_1(1 - e_3), c_1\tilde{a}_2 = a_2(1 - e_3), a_1a_2e_1 = a_2a_1e_1 = 0,$ and $c_i\delta_{i|Ae_2}$ is compact.*

PROOF. Under the hypotheses on $a_i, \tilde{a}_i, c_i$ and e_j we put $b_i = (c_1c_2)^{1/2}\tilde{a}_i$ and obtain

$$
\begin{aligned}
ad\, a_1 \circ ad\, a_2 &= ad\, a_1(1 - e_3) \circ ad\, a_2(1 - e_3) \\
&= ad\, c_2\tilde{a}_1 \circ ad\, c_1\tilde{a}_2 \\
&= ad(c_1c_2)^{1/2}\tilde{a}_1 \circ ad(c_1c_2)^{1/2}\tilde{a}_2 \\
&= ad\, b_1e_1 \circ ad\, b_2e_1 + ad\, b_1e_2 \circ ad\, b_2e_2 \\
&= -L_{b_1e_1}R_{b_2e_1} - L_{b_2e_1}R_{b_1e_1} + ad\, b_1e_2 \circ ad\, b_2e_2,
\end{aligned}
$$

since $b_1b_2e_1 = c_1c_2\tilde{a}_1\tilde{a}_2e_1 = a_1a_2e_1 = 0$ and similarly $b_2b_1e_1 = 0$.

Observe that $c_ia_ie_1 = c_1c_2\tilde{a}_ie_1$ is compact, thus b_ie_1 is compact. Therefore, $L_{b_1e_1}R_{b_2e_1}$ and $L_{b_2e_1}R_{b_1e_1}$ are both compact operators. The identity

$$ad\, b_1e_2 \circ ad\, b_2e_2 = ad\, c_2\tilde{a}_1e_2 \circ ad\, c_1\tilde{a}_2e_2 = ad\, c_1a_1e_2 \circ ad\, \tilde{a}_2e_2$$

shows that $ad\, b_1e_2 \circ ad\, b_2e_2$ is compact, too. This proves the "if"-part.

In the proof of the "only if"-part we begin as in Theorem 6 to obtain a central projection e_3 and $a_i' \in A^{**}$ such that $\delta_i = ad\, a_i'$, and $a_1'e_3$ and $a_2'e_3$ are centrally orthogonal. Since, by Lemma 7, both δ_1^π and δ_2^π are weakly compact for each $\pi \in \Gamma_0$, which we may write as $\Gamma_0 = \{\pi_n\ n \in \mathbf{N}\}$, we can proceed further in one step (instead of two steps) and obtain $a_i'' \in A^{**}$ satisfying $\delta_i = ad\, a_i'', a_i''p_n \in K(H_{\pi_n}), \|a_1''p_n\|\, \|a_2''p_n\| > 0$, and $\lim_{n\to\infty} \|a_1''p_n\|\, \|a_2''p_n\| = 0$ (recall that $p_n = c(\pi_n)$).

Put $a_i = a_i''(1 - e_3) + a_i'e_3, c_1 = \sum^\oplus \|a_2''p_n\|^{1/2}p_n$ and $c_2 = \sum^\oplus \|a_1''p_n\|^{1/2}p_n$. Then, c_i are positive central elements in A^{**} such that $c_ia_i \in K(A^{**}(1 - e_3))$. As in the proof of Theorem 6 define $\tilde{a}_i$ by the relations $c_2\tilde{a}_1 = a_1(1 - e_3)$ and $c_1\tilde{a}_2 = a_2(1 - e_3)$. Since $c_1c_2\tilde{a}_i = c_ia_i$ is compact, it follows that $b_i = (c_1c_2)^{1/2}\tilde{a}_i$ is compact. Let $\Gamma_f = \{\pi \in \Gamma_0 |\dim H_\pi < \infty\}$ and put $e_2 = c(\oplus_{\pi\in\Gamma_f}\pi), e_1 = 1 - e_2 - e_3$. If $\pi \in \Gamma_f$, then δ_i^π is compact (in fact, finite-rank) and thus $c_i\delta_{i|Ae_2} = ad\, c_ia_ie_2$ as the norm limit

https://doi.org/10.4153/CMB-1989-072-4 Published online by Cambridge University Press

of the compact mappings $ad(c_i a_i(p_1 + \ldots + p_n)e_2)$ is compact. If $\pi \in \Gamma_0 \setminus \Gamma_f$, then $a_1 a_2 c(\pi) = a_2 a_1 c(\pi) = 0$ by Lemma 7. Therefore, $a_1 a_2 e_1 = a_2 a_1 e_1 = 0$. □

In view of Lemma 5 we like to conclude with the following question.

Problem. What is the norm of the product $ad\,a \circ ad\,b$ if a, b are elements in a prime C^*-algebra A?

Even in the case when $ab = ba = 0$ so that $\|ad\,a \circ ad\,b\| = \|L_a R_b + L_b R_a\|$, the answer is not evident since simple examples show that $\|L_a R_b + L_b R_a\|$ can be strictly less than $2\|a\|\,\|b\|$. However, it seems reasonable to conjecture that it is always at least $\|a\|\,\|b\|$.

ACKNOWLEDGEMENTS. The research on this paper was initiated during a stay at the Mathematical Department of the University of Victoria, B.C. The author is grateful to his colleagues for the warm hospitality which they extended to him.

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