LMS J. Comput. Math. 19 (2) (2016) 360–381 © 2017 Authors
doi:10.1112/S1461157016000425

Certain aspects of holomorphic function theory on some genus-zero arithmetic groups

Jay Jorgenson, Lejla Smajlović and Holger Then

Abstract

There are a number of fundamental results in the study of holomorphic function theory associated to the discrete group $\mathrm{PSL}(2,\mathbb{Z})$, including the following statements: the ring of holomorphic modular forms is generated by the holomorphic Eisenstein series of weights four and six, denoted by E_4 and E_6; the smallest-weight cusp form Δ has weight twelve and can be written as a polynomial in E_4 and E_6; and the Hauptmodul j can be written as a multiple of E_4^3 divided by Δ. The goal of the present article is to seek generalizations of these results to some other genus-zero arithmetic groups $\Gamma_0(N)^+$ with square-free level N, which are related to 'Monstrous moonshine conjectures'. Certain aspects of our results are generated from extensive computer analysis; as a result, many of the space-consuming results are made available on a publicly accessible web site. However, we do present in this article specific results for certain low-level groups.

1. *Introduction and statement of results*

Consider the discrete group $\mathrm{PSL}(2,\mathbb{Z})$ which acts on the upper half plane $\mathbb{H}$. The quotient space $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ has one cusp, which can be taken to be at $i\infty$. Let Γ_∞ denote the stabilizer subgroup for the cusp at $i\infty$, which consists of isometries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$$

with $c = 0$. For every integer $k \geqslant 2$, the holomorphic Eisenstein series $E_{2k}(z)$ is defined by the absolutely convergent sum

$$E_{2k}(z) := \sum_{\gamma\in\Gamma_\infty\backslash\,\mathrm{PSL}(2,\mathbb{Z})} (cz+d)^{-2k}, \quad \text{where } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

There is an abundance of important and classical formulae which can be wound back to the holomorphic Eisenstein series E_{2k}. For example, if one defines

$$G_{2k}(z) := \sum_{(n,m)\in\mathbf{Z}^2\backslash\{(0,0)\}} (nz+m)^{-2k},$$

then $E_{2k}(z) = G_{2k}(z)/2\zeta(2k)$, where $\zeta(s)$ is the Riemann zeta function. If we set $g_2 = 60G_4$ and $g_3 = 140G_6$, the modular discriminant

$$\Delta(z) := e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$$

Received 22 May 2015; revised 10 February 2016.

2010 Mathematics Subject Classification 11F12 (primary), 11F20 (secondary).

J. J. acknowledges grant support from NSF and PSC-CUNY grants, and H. T. acknowledges support from EPSRC grant EP/H005188/1.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

can be written as

$$\Delta(z) = g_2^3(z) - 27g_3^2(z) = \tfrac{1}{1728}(E_4^3(z) - E_6^2(z)). \quad (1)$$

The function Δ is a weight-twelve cusp form with respect to $\mathrm{PSL}(2,\mathbb{Z})$, meaning that it vanishes as z approaches $i\infty$. It can be shown that no smaller weight cusp form exists. Furthermore, Δ is related to the algebraic discriminant of the cubic equation $y^2 = 4x^3 - g_2x - g_3$, in the complex projective coordinates $[x, y, 1]$, which defines an elliptic curve associated to the modular parameter z.

All higher weight modular forms associated to $\mathrm{PSL}(2,\mathbb{Z})$, including Eisenstein series, can be written in terms of $E_4(z)$ and $E_6(z)$. For example, the formulae $E_8(z) = E_4^2(z)$, $E_{10}(z) = E_4(z)E_6(z)$, and

$$691E_{12}(z) = 441E_4(z)^3 + 250E_6(z)^2$$

are just the beginning of the never ending list of interesting relations which one can write.

Whereas the content of the above discussion is classical, there is a very modern component. The function

$$j(z) = \frac{1728E_4^3(z)}{E_4^3(z) - E_6^2(z)} \quad (2)$$

is a weight-zero modular form on $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$, which can be viewed as the biholomorphic function that maps the one-point compactification of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ onto the Riemann sphere $\mathbb{P}^1$. If we set $q = e^{2\pi iz}$, then one can expand $j(z)$ as a function of q, namely one has

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + O(q^3) \quad \text{as } q \to 0. \quad (3)$$

In the 1970s, the coefficients in (3) were observed to be related to the sizes of the irreducible representations of the largest sporadic simple group, which is now known as 'the monster'. The observations were made precise through the 'Monstrous moonshine conjectures', some of which are proven in the celebrated work by Borcherds. We refer the interested reader to [**4**, **5**] for a thorough account of the underlying mathematics and physics surrounding the moonshine conjectures as well as the mathematical history associated to $j(z)$.

Setting to the side the important formulae themselves, one can summarize the above discussion as the three following points. First, the ring of holomorphic modular forms associated to $\mathrm{PSL}(2,\mathbb{Z})$ is generated by E_4 and E_6. Second, the smallest-weight cusp form Δ has weight twelve and hence can be written as a polynomial in E_4 and E_6. Third, the Hauptmodul j is equal to a multiple of E_4^3 divided by Δ and hence is a rational function in E_4 and E_6.

The goal of this article is to seek generalizations of the above three statements to certain other arithmetic groups related to the 'Monstrous moonshine conjectures'. Specifically, for any square-free positive integer N, let

$$\Gamma_0(N)^+ = \left\{ e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) : ad - bc = e,\ a,b,c,d,e \in \mathbb{Z}, \right. \\ \left. e \mid N,\ e \mid a,\ e \mid d,\ N \mid c \right\} \quad (4)$$

and let $\overline{\Gamma_0(N)^+} = \Gamma_0(N)^+/\{\pm\,\mathrm{Id}\}$, where Id denotes the identity matrix. Observe that $\mathrm{PSL}(2,\mathbb{Z}) = \overline{\Gamma_0(1)^+}$. It has been shown that there are 43 square-free integers $N > 1$ such that the quotient space $X_N := \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$ has genus zero (see [**3**]). Each group has one cusp, which we can always choose to be at $i\infty$. As stated in the title, the aim of this paper is to present results in the study of the holomorphic function theory associated to these 43 spaces.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

Let

$$\eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$$

denote the Dedekind eta function. For any square-free N, assume that N has r prime factors, and let $\mathrm{lcm}(\cdot\,,\cdot)$ denote the least common multiple function. Let $\sigma(N)$ equal the sum of divisors of N. It was proven in [7] that the function

$$\Delta_N(z) = \left(\prod_{v|N} \eta(vz) \right)^{\ell_N},$$

where

$$\ell_N = 2^{1-r} \,\mathrm{lcm}\left(4,\ 2^{r-1} \frac{24}{(24, \sigma(N))}\right),$$

is a weight-$k_N = 2^{r-1}\ell_N$ modular form on $\overline{\Gamma_0(N)^+}$, vanishing at the cusp $i\infty$ only. For reasons discussed in [7], we refer to Δ_N as *the Kronecker limit function on* $\overline{\Gamma_0(N)^+}$.

The main results of the present paper are the following statements, which hold true for each square-free N provided that X_N has genus zero.

(1) There is an explicitly computed integer M_N such that $\Delta_N^{M_N}$ is equal to a polynomial Q_N in holomorphic Eisenstein series associated to $\overline{\Gamma_0(N)^+}$.
(2) The Hauptmodul j_N associated to $\overline{\Gamma_0(N)^+}$ is equal to a rational function whose numerator is a polynomial P_N in holomorphic Eisenstein series and whose denominator is $\Delta_N^{M_N}$.
(3) The polynomials P_N and Q_N are explicitly computed; hence, we determine, for each N, a finite set $\mathcal{T}^{(N)}$ of holomorphic Eisenstein series such that any meromorphic form with at most polynomial growth at $i\infty$ can be expressed as a rational function involving elements of $\mathcal{T}^{(N)}$.

Points 1 and 2 are direct generalizations of the formulae (1) and (2). Point 3 is a weak generalization of the result that the ring of holomorphic modular forms associated to $\mathrm{PSL}(2,\mathbb{Z})$ is generated by the holomorphic Eisenstein series of weights four and six. For certain small levels, we are able to compute generators of the ring of holomorphic forms; however, for general N, and for future investigations we plan to undertake, we are content with point 3 as stated.

The present article is organized as follows. In § 2, we establish notation and cite appropriate background material. In particular, we recall the Kronecker limit formula associated to the non-parabolic Eisenstein series on $X_N = \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$ and a computer algorithm of [7]. In § 3, we prove some basic results regarding low-weight modular forms for any level $N > 1$. In § 4, we present a variant of the algorithm of [7] from which we prove that for every square-free N, provided that X_N has genus zero, there is an integer M_N such that $\Delta_N^{M_N}$ can be written as a polynomial in holomorphic Eisenstein series. Let j_N denote the biholomorphic map from the one-point compactification of X_N to the Riemann sphere $\mathbb{P}^1$ which maps $i\infty$ to zero. The algorithm described in § 4 allows us to prove that $j_N\Delta_N^{M_N}$ can be written as a polynomial in holomorphic Eisenstein series. The data provided by the algorithm is presented in Table 1, as is a comparison of the results of the original algorithm of [7] and the modified variant thereof. From the algorithm developed in this paper, we are able to determine for each level N a set of holomorphic Eisenstein series which generate $\mathcal{T}^{(N)}$, the ring of holomorphic modular forms associated to X_N; this information is given in Table 2. It is important to note that the entries in Table 2 may not be a minimal set of generators, meaning that for each N there may exist further relations amongst the elements of sets listed in Table 2. In § 5, we present results regarding the ring of holomorphic forms for certain small levels.

As N grows, so does the complexity of the formulae for Δ_N and j_N. For example, when $N = 17$, our algorithm shows that the five holomorphic Eisenstein series of weights four

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

through twelve generate $\mathcal{T}^{(17)}$ and $M_{17} = 9$, meaning that Δ_{17}^9 and $j_{17}\Delta_{17}^9$ can be written as a polynomial in these five Eisenstein series. As an indication of the complexity of the formulae, we present these two examples in § 5. The formulae for Δ_{17} and j_{17} each occupy approximately one page.

We note that the Tables 3 and 4a of [2] describe, in their notation, how one can express each Hauptmodul j_N in terms of holomorphic forms. In Table 3, we translate the aforementioned data from [2], related to 43 groups defined by (4) with square-free N and genus zero, such that we explicitly write these formulae in terms of the Dedekind eta function and theta function attached to quadratic forms. By combining our formulae for j_N and the formulae from [2], one has the prospect of obtaining further identities involving holomorphic Eisenstein series and theta functions.

As in [7], the theoretical work developed in this article is supplemented by extensive computer analysis and, quite frankly, some of the results are not printable. For example, for $N = 119$, the formula for j_{119} from [7] occupies nearly 60 pages. Nonetheless, in order to disseminate the results obtained by our algorithms, we have posted all formulae to a web site [8].

2. *Background material*

2.1. *Holomorphic modular forms*

Let Γ be a Fuchsian group of the first kind. Following [10], we define a weakly modular form f of weight $2k$ for $k \geqslant 1$ associated to Γ to be a function f which is meromorphic on $\mathbb{H}$ and satisfies the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Assume that Γ has at least one class of parabolic elements. By transforming coordinates, if necessary, we may always assume that the parabolic subgroup of Γ has a fixed point at $i\infty$, with identity scaling matrix. In this situation, any weakly modular form f will satisfy the relation $f(z+1) = f(z)$, so we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad \text{where } q = e^{2\pi i z}.$$

If $a_n = 0$ for all $n < 0$, then f is said to be holomorphic in the cusp $i\infty$. The form f is said to vanish in the cusp $i\infty$ if $a_n = 0$ for all $n \leqslant 0$. A holomorphic modular form with respect to Γ is a weakly modular form which is holomorphic on $\mathbb{H}$ and is holomorphic in all of the cusps of Γ. A holomorphic cusp form is a holomorphic form which vanishes in all of the cusps of Γ.

For $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, the full modular surface, there is no weight-two holomorphic modular form. Nonetheless, one defines the function $E_2(z)$ by the q-expansion

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{5}$$

It can be shown that $E_2(z)$ transforms according to the formula

$$E_2(\gamma z) = (cz+d)^2 E_2(z) + \frac{6}{\pi i} c(cz+d) \quad \text{for } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{6}$$

From this, it is elementary to show that for a prime p, the function

$$E_{2,p}(z) := \frac{pE_2(pz) - E_2(z)}{p-1}$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

is a weight-two holomorphic form associated to the congruence subgroup $\Gamma_0(p)$ of $\mathrm{SL}(2,\mathbb{Z})$. The q-expansion of $E_{2,p}$ is

$$E_{2,p}(z) = 1 + \frac{24}{p-1}\sum_{n=1}^{\infty}\sigma(n)(q^n - pq^{pn}).$$

2.2. *Certain arithmetic groups related to 'moonshine'*

For any square-free integer N, the subset of $\mathrm{SL}(2,\mathbb{R})$ defined by (4) is an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. As shown in [**3**], there are precisely 44 such groups which have genus zero and which appear in 'Monstrous moonshine conjectures'. In this article we will focus on the 43 genus-zero groups for which $N > 1$.

We denote by $\overline{\Gamma_0(N)^+} = \Gamma_0(N)^+/\{\pm\,\mathrm{Id}\}$ the corresponding subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Basic properties of $\Gamma_0(N)^+$ for square-free N are derived in [**6**] and references therein. In particular, we use that the surface $X_N = \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$ has exactly one cusp, which can be taken to be at $i\infty$.

Let $\mathcal{T}^{(N)}$ denote the ring of holomorphic modular forms associated to X_N, and let $\mathcal{T}^{(N)}_{2k}$ denote the holomorphic modular forms of weight $2k$. We will denote the subspace of cusp forms on X_N of weight $2k$ by $S^{(N)}_{2k}$.

2.3. *Holomorphic Eisenstein series on* $\overline{\Gamma_0(N)^+}$

In the case when $N > 1$ is square-free, the holomorphic Eisenstein series associated to $\overline{\Gamma_0(N)^+}$ are defined for $k \geqslant 2$ by

$$E^{(N)}_{2k}(z) := \sum_{\gamma\in\Gamma_\infty(N)\backslash\Gamma_0(N)^+}(cz+d)^{-2k} \quad \text{with } \gamma = \begin{pmatrix} * & * \\ c & d\end{pmatrix},$$

where $\Gamma_\infty(N)$ denotes the stabilizer group of the cusp at $i\infty$. Note that for all N, $\Gamma_\infty(N)$ is independent of N, namely one has that $\Gamma_\infty(N) = \Gamma_\infty$. In [**7**], it is proven that $E^{(N)}_{2k}(z)$ may be expressed as a linear combination of forms $E_{2k}(z)$, the holomorphic Eisenstein series associated to $\mathrm{PSL}(2,\mathbb{Z})$. Namely, it is known that

$$E^{(N)}_{2k}(z) = \frac{1}{\sigma_k(N)}\sum_{v\mid N} v^k E_{2k}(vz), \tag{7}$$

where σ_α denotes the generalized divisor function

$$\sigma_\alpha(m) = \sum_{\delta\mid m}\delta^\alpha.$$

Formula (7), together with a well-known q-expansion of classical forms E_{2k}, yields that the q-expansion of $E^{(N)}_{2k}$ is given by

$$E^{(N)}_{2k}(z) = \frac{1}{\sigma_k(N)}\sum_{v\mid N} v^k\left(1 - \frac{4k}{B_{2k}}\sum_{j=1}^{\infty}\sigma_{2k-1}(j)q^{vj}\right), \tag{8}$$

where B_k denotes the kth Bernoulli number.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

2.4. *Kronecker limit function on $\overline{\Gamma_0(N)^+}$*

Associated to the cusp of $\overline{\Gamma_0(N)^+}$ one has a non-holomorphic Eisenstein series denoted by $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$, which is defined for $z \in \mathbb{H}$ and $\mathrm{Re}(s) > 1$ by

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = \sum_{\eta \in \Gamma_\infty \backslash \Gamma_0(N)^+} \mathrm{Im}(\eta z)^s.$$

In [7] it is proven that, for any square-free N which has r prime factors, the parabolic Eisenstein series $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$ admits a Taylor series expansion of the form

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = 1 + s \cdot \log\left(\sqrt[2^r]{\prod_{v \mid N} |\eta(vz)|^4} \cdot \mathrm{Im}(z) \right) + O(s^2) \quad \text{as } s \to 0,$$

where $\eta(z)$ is Dedekind's eta function associated to $\mathrm{PSL}(2,\mathbb{Z})$. As stated above, it is proven that the function

$$\Delta_N(z) = \left(\prod_{v \mid N} \eta(vz) \right)^{\ell_N}, \tag{9}$$

where

$$\ell_N = 2^{1-r} \operatorname{lcm}\left(4,\ 2^{r-1} \frac{24}{(24, \sigma(N))}\right)$$

and $\operatorname{lcm}(\cdot\,,\cdot)$ denotes the least common multiple function, is a weight-$k_N = 2^{r-1}\ell_N$ modular form on $\overline{\Gamma_0(N)^+}$, vanishing at the cusp $i\infty$ only. We call the function $\Delta_N(z)$ defined by (9) the *Kronecker limit function on* $\overline{\Gamma_0(N)^+}$.

2.5. *The algorithm*

Let $X_N = \overline{\Gamma_0(N)^+} \backslash \mathbb{H}$ have genus g. For any positive integer M, the function

$$F_b(z) = \prod_\nu (E^{(N)}_{m_\nu}(z))^{b_\nu} / (\Delta_N(z))^M, \quad \text{where } \sum_\nu b_\nu m_\nu = M k_N \quad \text{and} \quad b = (b_1, \ldots) \tag{10}$$

is a holomorphic modular function on X_N, meaning a weight-zero modular form with polynomial growth near $i\infty$. The q-expansion of F_b follows from substituting the q-expansions of $E^{(N)}_k$ and Δ_N.

Let $\mathcal{S}_M$ denote the set of all possible rational functions defined in (10) for all vectors $b = (b_\nu)$ and $m = (m_\nu)$ with fixed M. In [7], we implemented the following algorithm, which we refer to as the **JST2** algorithm.

Choose a non-negative integer κ. Let $M = 1$ and set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_0$.

(1) Form the matrix $A_{\mathcal{S}}$ of coefficients from the q-expansions of all elements of $\mathcal{S}$, where each element in $\mathcal{S}$ is expanded along a row with each column containing the coefficient of a power, negative, zero, or positive, of q. The expansion is recorded out to order q^κ.
(2) Apply Gauss elimination to $A_{\mathcal{S}}$, thus producing a matrix $B_{\mathcal{S}}$ which is in row-reduced echelon form.
(3) Implement the following decision to determine if the algorithm is complete: if the $g+1$ lowest non-trivial rows at the bottom of $B_{\mathcal{S}}$ correspond to q-expansions whose leading terms have precisely g gaps, meaning zero coefficients, in the set $\{q^{-1}, \ldots, q^{-2g}\}$, then the algorithm is completed. If the indicator to stop fails, then replace M by $M+1$, $\mathcal{S}$ by $\mathcal{S}_M \cup \mathcal{S}$, and reiterate the algorithm.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

If $g = 0$, then the algorithm stops if the lowest non-trivial row at the bottom of $B_{\mathcal{S}}$ has a q-expansion which begins with q^{-1}. We also denote by M_N the value of M for the group of level N at which Step (3) shows that the algorithm is completed.

As stated in [7], the rationale for the stopping decision in Step (3) above is based on two ideas, one factual and one hopeful. First, the Weierstrass gap theorem states that for any point P on a compact Riemann surface there are precisely g gaps in the set of possible orders from 1 to $2g$ of functions whose only pole is at P. Second, for any genus, the assumption which is hopeful is that the function field is generated by the set of holomorphic modular functions defined in (10), which is related to the question of whether the field of meromorphic modular forms on $\overline{\Gamma_0(N)^+}$ is generated by holomorphic Eisenstein series and the Kronecker limit function. The latter assumption is not obvious and, indeed, the assumption itself is equivalent to the statement that the rational function field on X_N is generated by the holomorphic Eisenstein series. As it turned out, for all groups $\overline{\Gamma_0(N)^+}$ that we have studied so far, which includes all groups of genus zero, genus one, genus two, and genus three, the algorithm stopped. Therefore, we conclude that, in particular, the rational function field associated to all genus-zero groups $\overline{\Gamma_0(N)^+}$ is generated by a finite set of holomorphic Eisenstein series.

We described the algorithm with choice of arbitrary κ and g. For reasons of efficiency, we initially selected κ to be zero, so that all coefficients for q^{ν} for $\nu \leqslant \kappa$ are included in $A_{\mathcal{S}}$. In [7], it is shown that for each N, there is an explicitly computable $\kappa = \kappa_N$ such that if a modular form associated to $\overline{\Gamma_0(N)^+}$ has integral coefficients in its q-expansion out to q^{κ_N}, then all remaining coefficients are also integral. The list of κ_N for square-free levels N provided that $\overline{\Gamma_0(N)^+}$ has genus zero is given in Table 1 of [7]. In the implementation of the above algorithm, both in the present article and in [7], the value of κ was finally increased to κ_N.

In the present article, we implemented a slight variant of the above algorithm, which we refer to as the **JST3** algorithm. The difference between the **JST2** and the **JST3** algorithm is the following action should the decision in Step (3) fail.

Replace M by $M + 1$, $\mathcal{S}$ by $\mathcal{S}_M$, and reiterate the algorithm.

In other words, the **JST3** algorithm studies the q-expansions of the space of rational functions of the form (10) with a fixed denominator $(\Delta_N(z))^M$. Should the **JST3** algorithm successfully complete, then the row in $B_{\mathcal{S}}$ with q-expansion beginning with q^{-1} would correspond to a formula for j_N with denominator $(\Delta_N(z))^M$ and numerator given as a polynomial in Eisenstein series. Furthermore, any lower row in $B_{\mathcal{S}}$ would correspond to a q-expansion beginning with q^0, which would yield, upon clearing the denominator, a formula for $(\Delta_N(z))^M$ in terms of Eisenstein series.

As we will report below, the **JST3** algorithm has successfully completed for all genus-zero groups $\overline{\Gamma_0(N)^+}$ with square-free level N.

3. *Modular forms on surfaces X_N*

From [1, Proposition 7, p. II-7], we immediately obtain the following formula, which relates the number of zeros of a modular form, counted with multiplicity, with its weight and the volume of X_N.

LEMMA 1. *Let f be a modular form on X_N of weight $2k$, not identically zero. Let $\mathcal{F}_N$ denote the fundamental domain of $\overline{\Gamma_0(N)^+}$ and let $v_z(f)$ denote the order of zero z of f. Then*

$$k\frac{\mathrm{Vol}(X_N)}{2\pi} = v_{i\infty}(f) + \sum_{e\in\mathcal{E}_N}\frac{1}{n_e}v_e(f) + \sum_{z\in\mathcal{F}_N\setminus\mathcal{E}_N} v_z(f), \tag{11}$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

where $\mathcal{E}_N$ denotes the set of elliptic points in $\mathcal{F}_N$ and n_e is the order of the elliptic point $e \in \mathcal{E}_N$.

Lemma 1 enables us to deduce the following proposition.

PROPOSITION 2. *Let N be a square-free number such that the surface X_N has genus zero. Then there are no weight-two holomorphic modular forms on X_N.*

Proof. From the tables in [3], one determines that all genus-zero groups $\overline{\Gamma_0(N)^+}$ for a square-free N have at most one elliptic point of order three, four, or six and a various number of order-two elliptic points. Let $e_N(2)$ denote the number of order-two elliptic points on X_N, and let $n_N \in \{3, 4, 6\}$ denote the order of the additional elliptic point on X_N. Since all surfaces X_N have one cusp and genus zero, the Gauss–Bonnet formula for the volume of the surface X_N becomes

$$\frac{\mathrm{Vol}(X_N)}{2\pi} = \frac{1}{2}e_N(2) + \left(1 - \frac{1}{n_N}\right)\delta(N) - 1, \tag{12}$$

where $\delta(N)$ is equal to 1 if X_N has an elliptic point of order different from two, and zero otherwise.

For an arbitrary, square-free N and $e \mid N$, the elliptic element of $\overline{\Gamma_0(N)^+}$ is of the form

$$\begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ (cN)/\sqrt{e} & d\sqrt{e} \end{pmatrix}, \tag{13}$$

where $a, b, c, d \in \mathbb{Z}$ are such that $|(a + d)\sqrt{e}| < 2$ and $ade - (bcN)/e = 1$. The first condition implies that either $a + d = 0$ or $|a + d| = 1$ and $e \in \{1, 2, 3\}$.

If $|a + d| = 1$, then $d = \pm 1 - a$; hence,

$$\begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ (cN)/\sqrt{e} & d\sqrt{e} \end{pmatrix}^2 = \begin{pmatrix} \pm ae - 1 & \pm b \\ \pm cN & \pm ae - 1 \end{pmatrix} \neq \pm \mathrm{Id}$$

for any choice of $a, b, c \in \mathbb{Z}$ such that $a(\pm 1 - a)e - (bcN)/e = 1$. Therefore, there are no order-two elements in $\overline{\Gamma_0(N)^+}$ such that $|a + d| = 1$.

On the other hand, if $a + d = 0$, then $-a^2 e - (bcN)/e = 1$; hence,

$$\begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ (cN)/\sqrt{e} & d\sqrt{e} \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In other words, any elliptic element (13) of $\overline{\Gamma_0(N)^+}$ has order two if and only if $a + d = 0$. Let

$$\eta = \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ (cN)/\sqrt{e} & -a\sqrt{e} \end{pmatrix}$$

denote an arbitrary elliptic element of $\overline{\Gamma_0(N)^+}$ of order two, and let z_η be its fixed point. Solving the equation $\eta(z_\eta) = z_\eta$ leads to the conclusion that $(z_\eta cN/\sqrt{e} - a\sqrt{e})^2 = -1$.

Assume that $f_{2,N}$ is a holomorphic modular form on X_N of weight two. By the transformation rule, we have that

$$f_{2,N}(z_\eta) = f_{2,N}(\eta z_\eta) = (-1)f_{2,N}(z_\eta)$$

and hence z_η is a vanishing point of $f_{2,N}$. Since this holds true for any order-two elliptic element of $\overline{\Gamma_0(N)^+}$, we conclude that all order-two elliptic points of X_N are vanishing points of $f_{2,N}$. Applying Lemma 1 to $f_{2,N}$, we arrive at the inequality

$$\frac{\mathrm{Vol}(X_N)}{2\pi} \geqslant \frac{1}{2}e_N(2),$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

which contradicts (12). Therefore, there are no weight-two holomorphic modular forms on X_N. □

Though there are no weight-two holomorphic forms on $\Gamma_0(N)^+$, we may construct forms that transform almost like a weight-two form, up to an order-two character.

PROPOSITION 3. *Let $N = p_1 \ldots p_r$ be a square-free positive integer with $N > 1$. Let $\mu(\nu)$ denote the Möbius function and E_2 the series defined in (5). Then the holomorphic function*

$$E_{2,N}(z) := \frac{(-1)^r}{\varphi(N)} \sum_{v|N} \mu(v) v E_2(vz)$$

satisfies the transformation rule

$$E_{2,N}(\gamma_e z) = \mu(e)\left(c\frac{N}{\sqrt{e}}z + d\sqrt{e}\right)^2 E_{2,N}(z)$$

for any

$$\gamma_e = \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ (cN)/\sqrt{e} & d\sqrt{e} \end{pmatrix} \in \Gamma_0(N)^+.$$

Proof. Choose and fix any $e \mid N$. For any $v \mid N$, let (e, v) denote the greatest common divisor of e and v. Then, using the transformation formula (6) for E_2, it is easy to deduce that

$$vE_2(v(\gamma_e z)) = \frac{ev}{(e,v)^2}\left(c\frac{N}{\sqrt{e}}z + d\sqrt{e}\right)^2 E_2\left(\frac{ev}{(e,v)^2}z\right) + \frac{6}{\pi i}cN\left(c\frac{N}{e}z + d\right).$$

Since N is square-free with r prime factors, it is easy to see that

$$\sum_{v|N} \mu(v)\frac{6}{\pi i}cN\left(c\frac{N}{e}z + d\right) = \frac{6}{\pi i}cN\left(c\frac{N}{e}z + d\right)\sum_{j=1}^{r}\binom{r}{j}(-1)^j = 0$$

and hence

$$\sum_{v|N} \mu(v) v E_2(v(\gamma_e z)) = \sum_{v|N} \mu(v)\frac{ev}{(e,v)^2}\left(c\frac{N}{\sqrt{e}}z + d\sqrt{e}\right)^2 E_2\left(\frac{ev}{(e,v)^2}z\right).$$

We claim that $\{v : v \mid N\} = \{ev/(e,v)^2 : v \mid N\}$, which is easily deduced by induction in r. Furthermore, when e has an even number of prime factors, the parity of the number of factors of $ev/(e,v)^2$ remains the same as the parity of the number of factors of v, while when e has an odd number of factors, the parity changes, meaning that $\mu(v) = \mu(e)\mu(ev/(e,v)^2)$. Therefore,

$$\sum_{v|N} \mu(v) v E_2(v(\gamma_e z)) = \mu(e)\left(c\frac{N}{\sqrt{e}}z + d\sqrt{e}\right)^2 \sum_{v|N} \mu(v) v E_2(vz)$$

and the proof is complete. □

PROPOSITION 4. *The smallest even integer k_N such that there exists a weight-k_N cusp form f_N vanishing only at the cusp $i\infty$ is given by the formula*

$$k_N = \operatorname{lcm}\left(4, 2^{r-1}\frac{24}{(24, \sigma(N))}\right), \tag{14}$$

where lcm *denotes the least common multiple and $(\cdot\,, \cdot)$ stands for the greatest common divisor.*

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

Proof. From [7], one has that the volume of the surface X_N is given by

$$\mathrm{Vol}(X_N) = \frac{\pi\sigma(N)}{6 \cdot 2^{r-1}}, \tag{15}$$

where r is the number of (distinct) prime factors of N. By combining (15) with (11), we have that

$$k_N \cdot \frac{\sigma(N)}{24 \cdot 2^{r-1}} = v_{i\infty}(f_N)$$

and hence $2^{r-1}(24/(24, \sigma(N))) \mid k_N$.

On the other hand, the cusp form f_N does not vanish at order-two elliptic points. As proven above, every surface X_N for a square-free N has at least one order-two elliptic point that is a fixed point of the Atkin–Lehner involution $\tau_N : z \mapsto -1/(Nz)$. Since

$$f_N(\tau_N(i/\sqrt{N})) = f_N(i/\sqrt{N}) = (i)^{k_N} f_N(i/\sqrt{N}),$$

it follows that $4 \mid k_N$. The smallest k_N divisible by both 4 and $2^{r-1}(24/(24, \sigma(N)))$ is given by (14). Therefore, the proof is complete. □

The above proposition, together with Theorem 16 from [7], yields the following corollary.

COROLLARY 5. *Let $\ell_N = 2^{1-r} k_N$, where k_N is given by (14). Then the function*

$$\Delta_N(z) := \left(\prod_{v \mid N} \eta(vz) \right)^{\ell_N}$$

is the smallest-weight cusp form on X_N vanishing at the cusp only. Furthermore, the order of vanishing of Δ_N at the cusp is given by

$$v_{i\infty}(\Delta_N) = \frac{\sigma(N)\ell_N}{24} = k_N \cdot \frac{\sigma(N)}{24 \cdot 2^{r-1}}.$$

The next proposition determines the smallest-weight $\widetilde{k}_N$ for square-free N such that the space $S^{(N)}_{\widetilde{k}_N}$ is not empty.

PROPOSITION 6. *Let $N = p_1 \ldots p_r$ be a square-free positive integer, where $N > 1$. Then the smallest even integer $\widetilde{k}_N$ such that there exists a weight-$\widetilde{k}_N$ cusp form on a genus-zero surface X_N is equal to 8 for $N \in \{2, 3\}$ and equal to 4 for all other N.*

Proof. When $N = 2$, it is immediate that $k = 8$ is the smallest number such that $k \cdot (\mathrm{Vol}(X_2)/4\pi) \geqslant 1$. Since Δ_2 is a weight-eight cusp form, the assertion is proven when $N = 2$.

When $N = 3$, $k = 6$ is the smallest number such that $k \cdot (\mathrm{Vol}(X_3)/4\pi) \geqslant 1$. However, if there exists a weight-six cusp form on X_3, this cusp form also vanishes at order-two elliptic point e_2 of X_3. Therefore, the right-hand side of the formula (11) is at least 3/2, while the left-hand side of the same formula with $k = 6$ is equal to 1, which yields a contradiction. This shows that eight is the smallest possible weight of cusp form on X_3. An example of a weight-eight cusp form on X_3 is $E_8^{(3)} - (E_4^{(3)})^2$, so the case when $N = 3$ is complete.

When $N \geqslant 5$, we can construct the weight-four cusp form on X_N, whether or not the genus is zero, as follows. Let

$$E_{4,N}(z) := (E_{2,N}(z))^2.$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

From Proposition 3, it is immediate that $E_{4,N}$ is a weight-four holomorphic form on $\overline{\Gamma_0(N)^+}$. Recall that for a square-free N with r prime factors, we have the formula

$$\varphi(N) = (-1)^r \sum_{v|N} v\mu(v).$$

The q-expansion (5) implies that $E_{2,N}(z)$ is normalized so that its q-expansion has a leading coefficient equal to one. Therefore, the difference

$$\widetilde{\Delta}_N(z) := E_4^{(N)}(z) - E_{4,N}(z)$$

is a weight-four cusp form. By computing the q-expansion of $E_{4,N}$, we deduce that the term multiplying q in the q-expansion of $E_{4,N}(z)$ is $48/\varphi(N)$, while the term multiplying q in the q-expansion of $E_4^{(N)}(z)$ is equal to $240/(1+N^2)$. In other words, for square-free $N \notin \{2,3\}$, we have the expansion

$$\widetilde{\Delta}_N(z) = 48\left(\frac{1}{\varphi(N)} - \frac{5}{N^2+1}\right)q + \dots.$$

The leading coefficient is non-zero whenever $N \geqslant 5$ and hence $\widetilde{\Delta}_N(z)$ is a weight-four cusp form on X_N. □

4. *Expressing the Hauptmodul in terms of Eisenstein series*

In this section, we discuss the main results of this article.

THEOREM 7. *For any square-free $N \geqslant 1$ such that the surface X_N has genus zero, there exist effectively computable integers M_N and m_N, and explicitly computable polynomials $P_N(x_1, \dots, x_{m_N-1})$ and $Q_N(x_1, \dots, x_{m_N-1})$ in $m_N - 1$ variables with integer coefficients, such that the Hauptmodul j_N can be written as*

$$j_N(z) = \frac{P_N(E_4^{(N)}, \dots, E_{2m_N}^{(N)})}{Q_N(E_4^{(N)}, \dots, E_{2m_N}^{(N)})}$$

and the Kronecker limit function can be written as $\Delta_N^{M_N} = Q_N(E_4^{(N)}, \dots, E_{2m_N}^{(N)})$.

Proof. The result follows, because for each square-free level N, provided that X_N has genus zero, the **JST3** algorithm terminates in finite time. As stated, the computer code as well as the output is available on the web site [8]. In the space below, let us describe in further detail the output of the computational algorithm. We remind the reader that the **JST2** and the **JST3** algorithm are described in § 2.5.

After Gauss elimination, one of the q-expansions has a pole of order one. This is the Hauptmodul; see § 5 for explicit examples. Keeping track of the linear algebra, we have an exact expression for the Hauptmodul as a linear combination of holomorphic modular functions (10) with rational coefficients. In other words,

$$j_N(z) = \frac{1}{(\Delta_N(z))^{M_N}} \sum_b C_b \cdot \left(\prod_\nu (E_{m_\nu}^{(N)}(z))^{b_\nu}\right),$$

where the sum is taken over all $b = (b_1, \dots)$ such that $\sum_\nu b_\nu m_\nu = k_N M_N$, where M_N is given in the right-hand column of Table 1 and $C_b \in \mathbb{Q}$.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

There is also a q-expansion which is equal to the constant 1. Again, by keeping track of the linear algebra, we have an exact expression for the constant 1 as

$$1 = \frac{1}{(\Delta_N(z))^{M_N}} \sum_b D_b \cdot \left(\prod_\nu (E_{m_\nu}^{(N)}(z))^{b_\nu} \right),$$

where the sum is taken over the same set of b as above and $D_b \in \mathbb{Q}$.

By the design of the **JST3** algorithm, this exact expression can easily be solved for the M_Nth power of the Kronecker limit function, showing that

$$j_N(z) = \frac{\sum_b C_b \cdot (\prod_\nu (E_{m_\nu}^{(N)}(z))^{b_\nu})}{\sum_b D_b \cdot (\prod_\nu (E_{m_\nu}^{(N)}(z))^{b_\nu})}.$$

After multiplication of both numerator and denominator with the least common multiple of the denominators of the numbers C_b and D_b, we deduce the statement of the theorem. □

REMARK 8. The polynomials P_N and Q_N appearing in Theorem 7 are weighted homogeneous in the sense that there exists an integer M_N such that the coefficient of the term $(x_1)^{\alpha_1} \ldots (x_{m_N-1})^{\alpha_{m_N-1}}$ is non-zero only if $4\alpha_1 + 6\alpha_2 + \ldots + 2m_N\alpha_{m_N-1} = k_N M_N$, where k_N is the weight of the Kronecker limit function Δ_N.

REMARK 9. Table 1 provides the data regarding the performance of the **JST2** and **JST3** algorithms. More precisely, the first columns of data in Table 1 list, for each level N provided that X_N has genus zero, the weight k_N of the Kronecker limit function and the integer κ_N. To recall, it is shown in [7] that if the q-expansion of a holomorphic modular form has integer coefficients out to q^{κ_N}, then all further coefficients are also integral. The columns of data in Table 1 under the heading **JST2** algorithm list the integer M such that the **JST2** algorithm stops, together with the number of q-expansions which are used in the Gauss elimination algorithm as well as the order of the largest pole at $i\infty$ amongst the rational functions considered. The columns of data in Table 1 under the heading **JST3** algorithm present similar information.

REMARK 10. Table 2 provides a list of the holomorphic Eisenstein series $E_{m_\nu}^{(N)}$ which appear in the expression for the Hauptmodul j_N cited in Theorem 7. For all levels, the highest-weight Eisenstein series has weight 26.

REMARK 11. Expressions that are based on the track record of the linear algebra depend on how the base change is made through Gauss elimination. In particular, there may be linearly dependent functions, some of which survive the Gauss elimination while others get annihilated. We sought to express our results in terms of Eisenstein series whose weights are as small as possible. In other words, in the Gauss elimination we prioritized the holomorphic modular functions accordingly.

By expressing the Hauptmodul in terms of holomorphic Eisenstein series of smallest possible weights, we were able to determine a finite list of holomorphic Eisenstein series which generates the rational function field. Let G denote any modular form of weight k and consider the function

$$F(z) = G(z)(E_6^{(N)}(z))^{n_6}(E_4^{(N)}(z))^{n_4}/(\Delta_N(z))^{nM_N}$$

with non-negative integers n_6, n_4, and n such that $k+6n_6+4n_4 = k_N n M_N$. There is a rational function R in one variable such that $F(z) = R(j_N(z))$. Therefore, we conclude that G can be written as a rational function in terms of the holomorphic Eisenstein series that are listed in Table 2.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

TABLE 1. *Performance of the* **JST2** *and the* **JST3** *algorithms. For all genus-zero groups* $\overline{\Gamma_0(N)^+}$ *we list the level* N*, the weight* k_N *of the Kronecker limit function, and the value of* κ_N *in the proof of integrality* [7] *(left); the level* N*, the number of iterations* M*, the number of equations, and the largest order of pole for the* **JST2** *algorithm (middle) and similar for the* **JST3** *algorithm (right).*

			JST2 algorithm				**JST3** algorithm			
N	k_N	κ_N	N	M	#{eqs}	Pole	N	M_N	#{eqs}	Pole
1	12	19	1	1	5	1	1	1	4	1
2	8	47	2	1	3	1	2	1	2	1
3	12	48	3	1	5	2	3	1	4	2
5	4	19	5	1	2	1	5	3	4	3
6	4	60	6	1	2	1	6	3	4	3
7	12	19	7	1	5	4	7	2	21	8
10	8	75	10	2	10	6	10	2	7	6
11	4	19	11	3	8	6	11	9	88	18
13	12	19	13	2	26	14	13	3	88	21
14	4	47	14	3	8	6	14	6	21	12
15	4	96	15	3	8	6	15	5	12	10
17	4	19	17	4	15	12	17	9	88	27
19	12	19	19	3	114	30	19	4	320	40
21	12	53	21	2	26	16	21	2	21	16
22	4	47	22	4	15	12	22	6	21	18
23	4	19	23	5	27	20	23	15	1 039	60
26	8	47	26	3	31	21	26	4	55	28
29	4	19	29	6	48	30	29	15	1 039	75
30	4	127	30	4	15	12	30	6	21	18
31	12	19	31	4	434	64	31	5	1 039	80
33	4	48	33	5	27	20	33	8	55	32
34	8	47	34	3	31	27	34	4	55	36
35	4	19	35	5	27	20	35	7	34	28
38	4	47	38	5	27	25	38	10	137	50
39	12	48	39	3	114	42	39	3	88	42
41	4	19	41	7	82	49	41	21	8 591	147
42	4	108	42	5	27	20	42	7	34	28
46	4	47	46	6	48	36	46	14	708	84
47	4	19	47	8	137	64	47	27	56 224	216
51	4	48	51	6	48	36	51	11	210	66
55	4	19	55	6	48	36	55	8	55	48
59	4	19	59	9	225	90	59	33	310 962	330
62	4	47	62	7	82	56	62	18	3 094	144
66	4	60	66	6	48	36	66	8	55	48
69	4	48	69	7	82	56	69	14	708	112
70	4	181	70	6	48	36	70	8	55	48
71	4	19	71	10	362	120	71	39	1512 301	468
78	4	81	78	6	48	42	78	9	88	63
87	4	48	87	7	82	70	87	17	2 167	170
94	4	47	94	8	137	96	94	26	41 646	312
95	4	19	95	7	82	70	95	11	210	110
105	4	181	105	7	82	56	105	9	88	72
110	4	89	110	7	82	63	110	9	88	81
119	4	19	119	8	137	96	119	10	137	120

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

TABLE 2. *Finite sets of Eisenstein series which include the generators of the holomorphic Eisenstein series on groups* $\overline{\Gamma_0(N)^+}$ *of genus zero. Listed are level* N *and finite set.*

N	Finite set
1	$E_4^{(1)}, E_6^{(1)}$
2	$E_4^{(2)}, E_6^{(2)}, E_8^{(2)}$
3	$E_4^{(3)}, E_6^{(3)}, E_{12}^{(3)}$
5	$E_4^{(5)}, E_6^{(5)}, E_8^{(5)}, E_{12}^{(5)}$
6	$E_4^{(6)}, E_6^{(6)}, E_8^{(6)}, E_{12}^{(6)}$
7	$E_4^{(7)}, E_6^{(7)}, E_8^{(7)}, E_{10}^{(7)}, E_{12}^{(7)}$
10	$E_4^{(10)}, E_6^{(10)}, E_8^{(10)}, E_{10}^{(10)}, E_{12}^{(10)}, E_{16}^{(10)}$
11	$E_4^{(11)}, E_6^{(11)}, E_8^{(11)}, E_{10}^{(11)}, E_{12}^{(11)}$
13	$E_4^{(13)}, E_6^{(13)}, E_8^{(13)}, E_{10}^{(13)}, E_{12}^{(13)}$
14	$E_4^{(14)}, E_6^{(14)}, E_8^{(14)}, E_{10}^{(14)}, E_{12}^{(14)}$
15	$E_4^{(15)}, E_6^{(15)}, E_8^{(15)}, E_{10}^{(15)}, E_{12}^{(15)}, E_{14}^{(15)}, E_{16}^{(15)}$
17	$E_4^{(17)}, E_6^{(17)}, E_8^{(17)}, E_{10}^{(17)}, E_{12}^{(17)}$
19	$E_4^{(19)}, E_6^{(19)}, E_8^{(19)}, E_{10}^{(19)}, E_{12}^{(19)}$
21	$E_4^{(21)}, E_6^{(21)}, E_8^{(21)}, E_{10}^{(21)}, E_{12}^{(21)}, E_{14}^{(21)}, E_{16}^{(21)}$
22	$E_4^{(22)}, E_6^{(22)}, E_8^{(22)}, E_{10}^{(22)}, E_{12}^{(22)}, E_{14}^{(22)}, E_{16}^{(22)}, E_{18}^{(22)}$
23	$E_4^{(23)}, E_6^{(23)}, E_8^{(23)}, E_{10}^{(23)}, E_{12}^{(23)}$
26	$E_4^{(26)}, E_6^{(26)}, E_8^{(26)}, E_{10}^{(26)}, E_{12}^{(26)}, E_{14}^{(26)}$
29	$E_4^{(29)}, E_6^{(29)}, E_8^{(29)}, E_{10}^{(29)}, E_{12}^{(29)}$
30	$E_4^{(30)}, E_6^{(30)}, E_8^{(30)}, E_{10}^{(30)}, E_{12}^{(30)}, E_{14}^{(30)}, E_{16}^{(30)}, E_{18}^{(30)}$
31	$E_4^{(31)}, E_6^{(31)}, E_8^{(31)}, E_{10}^{(31)}, E_{12}^{(31)}$
33	$E_4^{(33)}, E_6^{(33)}, E_8^{(33)}, E_{10}^{(33)}, E_{12}^{(33)}, E_{14}^{(33)}$
34	$E_4^{(34)}, E_6^{(34)}, E_8^{(34)}, E_{10}^{(34)}, E_{12}^{(34)}, E_{14}^{(34)}, E_{16}^{(34)}$
35	$E_4^{(35)}, E_6^{(35)}, E_8^{(35)}, E_{10}^{(35)}, E_{12}^{(35)}, E_{14}^{(35)}, E_{16}^{(35)}, E_{18}^{(35)}$
38	$E_4^{(38)}, E_6^{(38)}, E_8^{(38)}, E_{10}^{(38)}, E_{12}^{(38)}, E_{14}^{(38)}$
39	$E_4^{(39)}, E_6^{(39)}, E_8^{(39)}, E_{10}^{(39)}, E_{12}^{(39)}, E_{14}^{(39)}$
41	$E_4^{(41)}, E_6^{(41)}, E_8^{(41)}, E_{10}^{(41)}, E_{12}^{(41)}$
42	$E_4^{(42)}, E_6^{(42)}, E_8^{(42)}, E_{10}^{(42)}, E_{12}^{(42)}, E_{14}^{(42)}, E_{16}^{(42)}, E_{18}^{(42)}$
46	$E_4^{(46)}, E_6^{(46)}, E_8^{(46)}, E_{10}^{(46)}, E_{12}^{(46)}$
47	$E_4^{(47)}, E_6^{(47)}, E_8^{(47)}, E_{10}^{(47)}, E_{12}^{(47)}$
51	$E_4^{(51)}, E_6^{(51)}, E_8^{(51)}, E_{10}^{(51)}, E_{12}^{(51)}, E_{14}^{(51)}$
55	$E_4^{(55)}, E_6^{(55)}, E_8^{(55)}, E_{10}^{(55)}, E_{12}^{(55)}, E_{14}^{(55)}, E_{16}^{(55)}, E_{18}^{(55)}, E_{20}^{(55)}, E_{22}^{(55)}$
59	$E_4^{(59)}, E_6^{(59)}, E_8^{(59)}, E_{10}^{(59)}, E_{12}^{(59)}$
62	$E_4^{(62)}, E_6^{(62)}, E_8^{(62)}, E_{10}^{(62)}, E_{12}^{(62)}$
66	$E_4^{(66)}, E_6^{(66)}, E_8^{(66)}, E_{10}^{(66)}, E_{12}^{(66)}, E_{14}^{(66)}, E_{16}^{(66)}, E_{18}^{(66)}, E_{20}^{(66)}, E_{22}^{(66)}$
69	$E_4^{(69)}, E_6^{(69)}, E_8^{(69)}, E_{10}^{(69)}, E_{12}^{(69)}$
70	$E_4^{(70)}, E_6^{(70)}, E_8^{(70)}, E_{10}^{(70)}, E_{12}^{(70)}, E_{14}^{(70)}, E_{16}^{(70)}, E_{18}^{(70)}, E_{20}^{(70)}, E_{22}^{(70)}$
71	$E_4^{(71)}, E_6^{(71)}, E_8^{(71)}, E_{10}^{(71)}, E_{12}^{(71)}$
78	$E_4^{(78)}, E_6^{(78)}, E_8^{(78)}, E_{10}^{(78)}, E_{12}^{(78)}, E_{14}^{(78)}, E_{16}^{(78)}, E_{18}^{(78)}$
87	$E_4^{(87)}, E_6^{(87)}, E_8^{(87)}, E_{10}^{(87)}, E_{12}^{(87)}$
94	$E_4^{(94)}, E_6^{(94)}, E_8^{(94)}, E_{10}^{(94)}, E_{12}^{(94)}$
95	$E_4^{(95)}, E_6^{(95)}, E_8^{(95)}, E_{10}^{(95)}, E_{12}^{(95)}, E_{14}^{(95)}, E_{16}^{(95)}$
105	$E_4^{(105)}, E_6^{(105)}, E_8^{(105)}, E_{10}^{(105)}, E_{12}^{(105)}, E_{14}^{(105)}, E_{16}^{(105)}, E_{18}^{(105)}, E_{20}^{(105)}$
110	$E_4^{(110)}, E_6^{(110)}, E_8^{(110)}, E_{10}^{(110)}, E_{12}^{(110)}, E_{14}^{(110)}, E_{16}^{(110)}, E_{18}^{(110)}, E_{20}^{(110)}, E_{22}^{(110)}, E_{24}^{(110)}, E_{26}^{(110)}$
119	$E_4^{(119)}, E_6^{(119)}, E_8^{(119)}, E_{10}^{(119)}, E_{12}^{(119)}, E_{14}^{(119)}, E_{16}^{(119)}, E_{18}^{(119)}, E_{20}^{(119)}, E_{22}^{(119)}, E_{24}^{(119)}$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

REMARK 12. We note that the sets in Table 2 are not necessarily minimal sets of generators. A specific example in the case $N=2$ is discussed below. As stated in the introduction, our goal was to determine a set of generators of the function field. Indeed, it seems to be a difficult problem to determine the structure of the ring of modular forms in any setting when $M_N > 1$, meaning when there is an expression for the M_Nth power of the Kronecker limit function in terms of holomorphic Eisenstein series yet no apparent expression for any smaller power of the Kronecker limit function.

5. *Examples*

In this section, we will present a number of specific formulae for various levels. It seems as if each level has its own idiosyncratic characteristics, so we choose various examples, which, in our opinion, depict some of the most comprehensible and quantifiable nuances.

5.1. $N = 2$

We will cite specific results here, referring the reader to the article [9] for additional information and proofs. The Kronecker limit function can be written as

$$\Delta_2(z) = \tfrac{17}{1152}(E_4^{(2)}(z))^2 - \tfrac{17}{1152}E_8^{(2)}(z). \tag{16}$$

In addition, one has that

$$j_2(z)\Delta_2(z) = -\tfrac{77}{144}(E_4^{(2)}(z))^2 + \tfrac{221}{144}E_8^{(2)}(z).$$

By arguing as in [10], one can prove a dimension formula for the space of automorphic forms of weight $2k$, namely that

$$\dim \mathcal{T}_{2k}^{(2)} = \begin{cases} \left\lfloor \dfrac{k}{4} \right\rfloor & \text{if } k \text{ is congruent to 1 modulo 4, } k \geqslant 0, \\ \left\lfloor \dfrac{k}{4} \right\rfloor + 1 & \text{if } k \text{ is not congruent to 1 modulo 4, } k \geqslant 0. \end{cases} \tag{17}$$

The space $\mathcal{T}_{2k}^{(2)}$ is generated by the set of monomials $(E_4^{(2)}(z))^l(E_6^{(2)}(z))^m(E_8^{(2)}(z))^n$, where l, m, n are non-negative integers such that $4l + 6m + 8n = 2k$. The dimension formula (17) yields some interesting number-theoretical formulae. For example, since $\dim \mathcal{T}_{10}^{(2)} = 1$, we see that $E_{10}^{(2)}(z) = E_6^{(2)}(z)E_4^{(2)}(z)$. By equating the q-expansions (8) for $k \in \{2, 3, 5\}$, one obtains the following summation formula for the generalized sum of divisors:

$$A_9^{(2)}(n) = 336\sum_{j=1}^{n-1} A_3^{(2)}(j)A_5^{(2)}(n-j) + 7A_5^{(2)}(n) - 6A_3^{(2)}(n),$$

where $A_{2k-1}^{(2)}(n) = \sigma_{2k-1}(n) + 2^k\delta(n)\sigma_{2k-1}(n/2)$ for $k = 1, 2, \ldots$ and $\delta(n) = 1$ for even positive integers n and $\delta(n) = 0$, otherwise.

Analogously, using formula (16), the q-expansion (8), and the q-expansion for the delta function, $\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n$, where $\tau(n)$ is the Ramanujan function, one obtains relations involving τ, σ_3, and σ_7.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

5.2. $N = 3$

As with the case $N = 2$, we refer the reader to [9] for additional information and proofs. The Kronecker limit function vanishes to order two at $i\infty$ and has weight twelve. The smallest-weight cusp form has weight eight, but it vanishes to order one at $i\infty$, and, consequently, it vanishes elsewhere. The Kronecker limit function can be written as

$$\Delta_3(z) = -\tfrac{25}{3456}(E_4^{(3)}(z))^3 - \tfrac{1049}{72900}(E_6^{(3)}(z))^2 + \tfrac{50443}{2332800}E_{12}^{(3)}(z). \tag{18}$$

and the Hauptmodul is given by

$$j_3(z)\Delta_3(z) = \tfrac{541}{1728}(E_4^{(3)}(z))^3 + \tfrac{14461}{24300}(E_6^{(3)}(z))^2 - \tfrac{353101}{388800}E_{12}^{(3)}(z) \tag{19}$$

The dimension formula for the space of automorphic forms of weight $2k$ is

$$\dim \mathcal{T}_{2k}^{(3)} = \begin{cases} \left\lfloor \dfrac{k}{3} \right\rfloor & \text{if } k \text{ is congruent to 1 or 3 modulo 6, } k \geqslant 0, \\[2ex] \left\lfloor \dfrac{k}{3} \right\rfloor + 1 & \text{if } k \text{ is not congruent to 1 or 3 modulo 6, } k \geqslant 0. \end{cases}$$

We note that the forms $E_8^{(3)}(z) - (E_4^{(3)}(z))^2$ and $E_{10}^{(3)}(z) - E_4^{(3)}(z)E_6^{(3)}(z)$ are cusp forms which vanish at elliptic points on X_3; see Appendix B of [9]. In other words, there are cusp forms of weight smaller than the weight of the Kronecker limit function, but these forms necessarily vanish at some point in the interior of X_3, whereas the Kronecker limit function vanishes at $i\infty$ only.

Finally, let us explain why $E_8^{(3)}$ does not appear in Table 2. The information in Appendix B of [9] describes the zeros of small-weight holomorphic forms. In particular, we conclude from the information provided that

$$\frac{E_8^{(3)}(z)}{E_8^{(3)}(z) - (E_4^{(3)}(z))^2} = c_1 j_3(z) + c_2$$

for some explicitly computable constants c_1 and c_2. From this, we get that

$$E_8^{(3)}(z) = (E_4^{(3)}(z))^2 \frac{c_1 j_3(z) + c_2}{c_1 j_3(z) + c_2 - 1}. \tag{20}$$

When combining (19), (18), and (20), we get a formula which expresses $E_8^{(3)}$ as a rational function involving $E_4^{(3)}$, $E_6^{(3)}$, and $E_{12}^{(3)}$, as asserted by Table 2.

5.3. $N = 5$

In the case $N = 5$, the surface X_5 has genus zero, three order-two elliptic elements $e_1 = i/\sqrt{5}$, $e_2 = 2/5 + i/5$, $e_3 = 1/2 + i/(2\sqrt{5})$, and one cusp; hence, $\mathrm{Vol}_{\mathrm{hyp}}(X_5) = \pi$. Its Kronecker limit function has weight four, which is minimal, and the function vanishes at $i\infty$ to order one, which is also minimal. As a result, we have that the mapping $f \mapsto \Delta_5 f$ is an isometry between the spaces $\mathcal{T}_{2k-4}^{(5)}$ and $S_{2k}^{(5)}$; therefore, we arrive at the dimension formula

$$\dim \mathcal{T}_{2k}^{(2)} = \begin{cases} \left\lfloor \dfrac{k}{2} \right\rfloor & \text{if } k \text{ is congruent to 1 modulo 2, } k \geqslant 0, \\[2ex] \left\lfloor \dfrac{k}{2} \right\rfloor + 1 & \text{if } k \text{ is not congruent to 0 modulo 2, } k \geqslant 0. \end{cases}$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

The space $\mathcal{T}_{2k}^{(5)}$ is generated by the set of monomials $(E_4^{(5)}(z))^l(\Delta_5(z))^m(E_6^{(5)}(z))^n$, where l, m, n are non-negative integers such that $4l + 4m + 6n = 2k$. From the output of the **JST2** algorithm, we have that

$$j_5(z)\Delta_5(z) = E_4^{(5)}(z) - \tfrac{172}{13}\Delta_5(z).$$

The analysis of Δ_5^3 differs between the **JST2** and the **JST3** algorithms. From **JST2**, we have that Δ_5^3 is a rational function in the holomorphic Eisenstein series of weights four, six, eight, and twelve. From **JST3**, we have that Δ_5^3 is a *polynomial* in the holomorphic Eisenstein series of weights four, six, eight, and twelve. Namely, from the output of the **JST3** algorithm, we have that

$$\begin{aligned} j_5(z)(\Delta_5(z))^3 = &\tfrac{10330419229}{11016000000}(E_4^{(5)}(z))^3 + \tfrac{36659}{2448000}(E_6^{(5)}(z))^2 \\ &- \tfrac{28493266087}{11016000000}E_8^{(5)}(z)E_4^{(5)}(z) + \tfrac{2999646893}{1836000000}E_{12}^{(5)}(z) \end{aligned}$$

and

$$\begin{aligned} (\Delta_5(z))^3 = &-\tfrac{9383387}{162000000}(E_4^{(5)}(z))^3 - \tfrac{13}{9000}(E_6^{(5)}(z))^2 \\ &+ \tfrac{3226717}{20250000}E_8^{(5)}(z)E_4^{(5)}(z) - \tfrac{5398783}{54000000}E_{12}^{(5)}(z). \end{aligned}$$

5.4. $N = 6$

Topologically, X_5 and X_6 are identical, with the same number of cusps, elliptic points of order two, and, consequently, the same hyperbolic volume. The **JST2** and **JST3** algorithms performed similarly in both cases, as one can see from Tables 1 and 2. All comments above regarding the holomorphic function theory for X_5 hold for X_6. However, as shown in [**6**], the analytic function theories of X_5 and X_6 are different. Specifically, the counting functions for the analytic Maass forms, when ordered by their Laplacian eigenvalues, are shown to be equal in their leading term but unequal in lower order terms.

5.5. $N = 17$

As we stated in the introduction, as N becomes larger, the formulae become massive. In our last example for $N = 17$, the Kronecker limit function has weight four and vanishes at $i\infty$ to order four. From the **JST3** algorithm, we have the following formulae:

$$\begin{aligned} j_{17}(z)\big(\Delta_{17}(z)\big)^9 = &\tfrac{8168280188935682000179022497005847191761310812736219246161322053 3}{32698468557734924209442422997054319013259751265789326049746616 32}\big(E_4^{(17)}(z)\big)^9 \\
&- \tfrac{57998022455299820152689336251300228068357304045275286805301}{11974575211909486263275041423879967918942902295200247316 48}\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^6 \\
&+ \tfrac{404974365153387984085320455232254890259654575613305563162 91}{185277423919485732646665270671327635368475202513849548800 0}\big(E_6^{(17)}(z)\big)^4\big(E_4^{(17)}(z)\big)^3 \\
&- \tfrac{375848025769022506123369320872979359492445357431516 3}{550341367907988517797569501748755788899740024832000}\big(E_6^{(17)}(z)\big)^6 \\
&- \tfrac{19414695740146736017085565287911573267947533788487530546931336997391}{23502024275871976775536741529132791790780446222286078098255380480 0}E_8^{(17)}(z)\big(E_4^{(17)}(z)\big)^7 \\
&+ \tfrac{8739242957393066251361784976687179313105384005680862643673981125 7}{4292136802518806482492647660371976000946096541435838647500800 00}E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^4 \\
&- \tfrac{520329180900272292342072705904267052967833829968110057234849 7}{159222786180808051493227966983172186644783377160339456000000}E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^4E_4^{(17)}(z) \\
&- \tfrac{3408021881707620602850044141317857445104537752516243513916285865231}{546558704090045971524110268119367250948382470285722746471055360000}\big(E_8^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^5 \\
&- \tfrac{1661350353470581362919888851808493749469628498780806945010292117 1}{953808178337512551665032813415994666876910342541297477222400 00}\big(E_8^{(17)}(z)\big)^2\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^2 \\
&+ \tfrac{20843107640694642663754521813793021239436718966306147304104902827089 41}{244812752873666424745174390928466581153962981482146646856826880000 00}\big(E_8^{(17)}(z)\big)^3\big(E_4^{(17)}(z)\big)^3 \\
&- \tfrac{992213652247899202105908914854404059954617423680834206114938 9}{600269903901646354129469435526559143650833331894479749120000}\big(E_8^{(17)}(z)\big)^3\big(E_6^{(17)}(z)\big)^2 \\
&+ \tfrac{39971724261482388723963548784518805970985444209456555554807081177551}{580296895700542636433005963682291155327912252402125385142108160000}\big(E_8^{(17)}(z)\big)^4E_4^{(17)}(z) \\
&- \tfrac{1217981359488142573172153095487639500606482756486571223723170900 7}{111595556865488968544808839169671376024598510077331804835020800}E_{10}^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^5 \end{aligned}$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

$$
\begin{aligned}
&- \tfrac{16252586301480981588446088610594287626798674107605236541884 93}{305707749467151458866997696607690598357984084147851755520000} E_{10}^{(17)}(z)\big(E_6^{(17)}(z)\big)^3\big(E_4^{(17)}(z)\big)^2 \\
&+ \tfrac{2019561651698249364467964532149121989222375466484480429312473506 43}{697472230409306053405055244810446100153740687983323780218880000} E_{10}^{(17)}(z)E_8^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^3 \\
&+ \tfrac{281349209280450977701955048416616425916637167247000417986598 7}{110819059181842403839286665020287841904769230503596261376000} E_{10}^{(17)}(z)E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^3 \\
&- \tfrac{298150566267220507427939657298379519874877536521040468383984890437}{278988892163722421362022097924178440061496275193329512087552 0000} E_{10}^{(17)}(z)\big(E_8^{(17)}(z)\big)^2E_6^{(17)}(z)E_4^{(17)}(z) \\
&+ \tfrac{270142921637107712433606444209937936587102913556256973385136064101}{133914668238586762253770607003605651229518212092798165802024960 0} \big(E_{10}^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^4 \\
&- \tfrac{11858124470124365462549270194638667187513160035092174536623818 1}{13247335810243229884236566852999925928845976979740242739200 00} \big(E_{10}^{(17)}(z)\big)^2\big(E_6^{(17)}(z)\big)^2E_4^{(17)}(z) \\
&- \tfrac{622123658423176240663440801764445330920253599234807877389715511211}{128764104075564194474779429811159280028382896243075159425024000 0} \big(E_{10}^{(17)}(z)\big)^2E_8^{(17)}(z)\big(E_4^{(17)}(z)\big)^2 \\
&- \tfrac{41865928593946018666000515901728601665582738769261235242020892533923}{9708813447297540263398369007761409714140070376727867020646809600 00} \big(E_{10}^{(17)}(z)\big)^2\big(E_8^{(17)}(z)\big)^2 \\
&+ \tfrac{22629302311806563160495642791579770576036281698199408592953 7}{267622945661479391600738724303028806643353070297782681600 0} \big(E_{10}^{(17)}(z)\big)^3E_6^{(17)}(z) \\
&+ \tfrac{659399884410960186638853342599954696 77}{193474038051232521643544669816094720 0} E_{12}^{(17)}(z)\big(E_4^{(17)}(z)\big)^6 \\
&- \tfrac{47508694350054116027131578232203571279280588142292406574968917}{496775092884121120658871256987497222331724136740259102720000} E_{12}^{(17)}(z)\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^3 \\
&+ \tfrac{8323913080043949355404898975830414265620742437956645743 1}{281812734945987223768755417274793912591763426508800000 0} E_{12}^{(17)}(z)\big(E_6^{(17)}(z)\big)^4 \\
&+ \tfrac{18888098569562683617871650219704377045625972851708713833077441}{993550185768242224131774251397499444466344827348051820544000} E_{12}^{(17)}(z)E_{10}^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^2 \\
&- \tfrac{645810074718509651315791862946327105235 9}{483685095128081304108861674540236800 00} \big(E_{12}^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^3 \\
&+ \tfrac{134561818140838229841965297051998190746 19}{771837917757576549109885650862080000 00} \big(E_{12}^{(17)}(z)\big)^3
\end{aligned}
$$

and

$$
\begin{aligned}
\big(\Delta_{17}(z)\big)^9 = &- \tfrac{44101751522668636304970170952875737991081012873201 69}{51326978514980667300250472864486930901152788434124 8} \big(E_4^{(17)}(z)\big)^9 \\
&+ \tfrac{198652153282810789192198685818306731161162812798 61}{107798278347994315500197318639345406049247035392 0} \big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^6 \\
&- \tfrac{114799409985064266227585720155413610825193224 3}{116332425049635581779544719243012827040972800} \big(E_6^{(17)}(z)\big)^4\big(E_4^{(17)}(z)\big)^3 \\
&+ \tfrac{25716334809915305740593757059321357640 1}{8638740837859942220704050954726604800 0} \big(E_6^{(17)}(z)\big)^6 \\
&+ \tfrac{4110275602561195487616512760454051197916582070385787933}{1475650632305694184882201094853999263408142667481088 00} E_8^{(17)}(z)\big(E_4^{(17)}(z)\big)^7 \\
&- \tfrac{24557837523110861701789177775227814265868927006948 51}{33686961983748223593811662074795439390389698560000} E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^4 \\
&+ \tfrac{158177525583834772874576577815717906884476544180 1}{99973177777030578091796243099464148238336000000} E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^4E_4^{(17)}(z) \\
&+ \tfrac{65181620271972259986469145603312742073005041211329298 47}{18445632903821177311027513685674990792601783343513600 00} \big(E_8^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^5 \\
&+ \tfrac{87742847504094687050591248057267387716589923374210 3}{14971983103888099375027405366575750840173199360000} \big(E_8^{(17)}(z)\big)^2\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^2 \\
&- \tfrac{905386954382815429576749294608296584568282296576830392199}{307427215063686288517125228094583179876696389058560000 00} \big(E_8^{(17)}(z)\big)^3\big(E_4^{(17)}(z)\big)^3 \\
&+ \tfrac{20745283346018953837213070761977891074307877102545 7}{36182292501062906822982896302558064530418565120000} \big(E_8^{(17)}(z)\big)^3\big(E_6^{(17)}(z)\big)^2 \\
&- \tfrac{67657080512199038283988883908678585471939230808112806 31}{2732686356121655897930002027507406043348412347187200 00} \big(E_8^{(17)}(z)\big)^4E_4^{(17)}(z) \\
&+ \tfrac{70258768040043562400557906214691148079678388226 91}{19409662306425569272888713882430613277565255680 0} E_{10}^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^5 \\
&+ \tfrac{20394432665320755176107617426132569177967253 7}{26585665004418103869286535561076338589696000 0} E_{10}^{(17)}(z)\big(E_6^{(17)}(z)\big)^3\big(E_4^{(17)}(z)\big)^2 \\
&- \tfrac{880500666078403629835430898324252543787571068757867 33}{875861011577453813439103213944681424150132162560000} E_{10}^{(17)}(z)E_8^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^3 \\
&- \tfrac{11475489120090534120361109729772934561151288858 9}{13916266346562656470378037039445409434776371200} E_{10}^{(17)}(z)E_8^{(17)}(z)\big(E_6^{(17)}(z)\big)^3 \\
&+ \tfrac{663944495719159389020699923076942268844390571603249 51}{175172202315490762687820642788936284830026432512000 0} E_{10}^{(17)}(z)\big(E_8^{(17)}(z)\big)^2E_6^{(17)}(z)E_4^{(17)}(z) \\
&- \tfrac{36648232278677928809901022846165062701534415892036 01}{52551660694647228806346192836680885449007929753600} \big(E_{10}^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^4 \\
&+ \tfrac{14427551517079169370308214911137713786234600256489}{41588841955244720486187237129377085667147776000 0} \big(E_{10}^{(17)}(z)\big)^2\big(E_6^{(17)}(z)\big)^2E_4^{(17)}(z) \\
&+ \tfrac{34114607946828890598140117698005174033430842408608771}{20212177190248934156286997244877263634233819136000 0} \big(E_{10}^{(17)}(z)\big)^2E_8^{(17)}(z)\big(E_4^{(17)}(z)\big)^2 \\
&+ \tfrac{11862779401388611355016855416333672453433423994813953 43}{76199908007238481769201979613187283901061498142720000} \big(E_{10}^{(17)}(z)\big)^2\big(E_8^{(17)}(z)\big)^2 \\
&- \tfrac{22400746720053456919360945826730212236677495587 47}{73095540406187690551480598591026392990744576000} \big(E_{10}^{(17)}(z)\big)^3E_6^{(17)}(z)
\end{aligned}
$$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

$$
\begin{aligned}
&- \tfrac{473713406463236803998887}{40792493008974879129600} E_{12}^{(17)}(z)\big(E_4^{(17)}(z)\big)^6 \\
&+ \tfrac{18762718194456394455327896537653270422382541 0987}{503090830103766780074845610436013133070336 0000} E_{12}^{(17)}(z)\big(E_6^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^3 \\
&- \tfrac{31764327300039630476106994378339106645526 31}{22118155421072438271975027013394841600 0000} E_{12}^{(17)}(z)\big(E_6^{(17)}(z)\big)^4 \\
&- \tfrac{214478793382351384078429507243661160957806584 8101}{311916314664335403646404278470328142503608 32000} E_{12}^{(17)}(z)E_{10}^{(17)}(z)E_6^{(17)}(z)\big(E_4^{(17)}(z)\big)^2 \\
&+ \tfrac{46882982116711758510391631}{1019812325224371978240000} \big(E_{12}^{(17)}(z)\big)^2\big(E_4^{(17)}(z)\big)^3 \\
&- \tfrac{463996581512517217133820050 3}{764859243918278983680000 00} \big(E_{12}^{(17)}(z)\big)^3.
\end{aligned}
$$

Using the exact identity of the Hauptmodul in terms of Eisenstein series, we can read off that $E_4^{(17)}(z)$, $E_6^{(17)}(z)$, $E_8^{(17)}(z)$, $E_{10}^{(17)}(z)$, and $E_{12}^{(17)}(z)$ generate the holomorphic Eisenstein series $E_k^{(17)}(z)$ for all even $k \geqslant 4$.

6. *Concluding remarks*

6.1. *Known relations for Hauptmoduli*

In [2], the authors computed expressions for j_N, up to an additive constant. The authors call their function t_N. The data from [2] relates to genus-zero groups $\Gamma_0(N)^+$ with square-free level N as given in Table 3, using the Dedekind eta function together with $\theta(a,b,c)$, which is the theta function defined by the series

$$\theta(a,b,c) = \sum_{(x,y)\in\mathbb{Z}^2} q^{(ax^2+bxy+cy^2)/2}.$$

Additionally, one has, in the notation of [2], the functions $\theta_x(a,b,c)$ and $\theta_y(a,b,c)$, which are defined by the same series which defines $\theta(a,b,c)$ except that one restricts the sum to odd values of x and y, respectively. By combining our results with the relations for the Hauptmoduli in Table 3, it is possible to deduce many potentially interesting relations between classical Eisenstein series $E_k(z)$, eta functions, and theta functions.

For example, let us take $N = 17$. In the notation of Theorem 7, one has $M_{17} = 9$ and the Hauptmodul $j_{17}(z)$ is given as a rational function of the form

$$j_{17}(z) = \frac{P_{17}(E_4^{(17)}, E_6^{(17)}, E_8^{(17)}, E_{10}^{(17)}, E_{12}^{(17)})}{Q_{17}(E_4^{(17)}, E_6^{(17)}, E_8^{(17)}, E_{10}^{(17)}, E_{12}^{(17)})},$$

where P_{17} and Q_{17} denote polynomials of degree nine in five variables with integer coefficients, where coefficients are non-zero only if the sum of products of weights and corresponding degrees is equal to 36.

In a sense, this result is a direct analogue of formula (2) expressing the classical j-invariant for $\mathrm{PSL}(2,\mathbb{Z})$ in terms of classical holomorphic Eisenstein series.

Furthermore, formula (7) implies that the Eisenstein series $E_{2k}^{(17)}$ for $k = 2,3,4,5,6$ may be expressed as a linear combination of dilations of the series E_{2k}; hence, the function

$$\left(\frac{\theta_x(\frac{1}{2},0,\frac{17}{2}) - \theta_y(\frac{1}{2},0,\frac{17}{2})}{2\eta(z)\eta(17z)}\right)^2$$

is a rational function in the Eisenstein series $E_4(z)$, $E_4(17z)$, $E_6(z)$, $E_6(17z)$, $E_8(z)$, $E_8(17z)$, $E_{10}(z)$, $E_{10}(17z)$, $E_{12}(z)$, and $E_{12}(17z)$ with integer coefficients.

Proceeding in a similar manner, for example when $N = 29$ or $N = 47$, we obtain other relations between theta functions, eta functions, and holomorphic Eisenstein series E_{2k}.

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

TABLE 3. *Known expressions of the Hauptmoduli j_N for the genus-zero groups $\Gamma_0(N)^+$.*

N	Formula for $t_N = j_N + \text{const.}$
2	$t_2 = \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 4096\left(\frac{\eta(2z)}{\eta(z)}\right)^{24}$
3	$t_3 = \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} + 729\left(\frac{\eta(3z)}{\eta(z)}\right)^{12}$
5	$t_5 = \left(\frac{\eta(z)}{\eta(5z)}\right)^{6} + 125\left(\frac{\eta(5z)}{\eta(z)}\right)^{6}$
6	$t_6 = \left(\frac{\eta(z)\eta(2z)}{\eta(3z)\eta(6z)}\right)^4 + 81\left(\frac{\eta(3z)\eta(6z)}{\eta(z)\eta(2z)}\right)^4 = \left(\frac{\eta(z)\eta(3z)}{\eta(2z)\eta(6z)}\right)^6 + 64\left(\frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)}\right)^6 + c_1$
	$= \left(\frac{\eta(2z)\eta(3z)}{\eta(z)\eta(6z)}\right)^{12} + \left(\frac{\eta(z)\eta(6z)}{\eta(2z)\eta(3z)}\right)^{12} + c_2$
7	$t_7 = \left(\frac{\eta(z)}{\eta(7z)}\right)^4 + 49\left(\frac{\eta(7z)}{\eta(z)}\right)^4$
10	$t_{10} = \left(\frac{\eta(z)\eta(2z)}{\eta(5z)\eta(10z)}\right)^2 + 25\left(\frac{\eta(5z)\eta(10z)}{\eta(z)\eta(2z)}\right)^2 = \left(\frac{\eta(z)\eta(5z)}{\eta(2z)\eta(10z)}\right)^4 + 16\left(\frac{\eta(2z)\eta(10z)}{\eta(z)\eta(5z)}\right)^4 + c_1$
	$= \left(\frac{\eta(2z)\eta(5z)}{\eta(z)\eta(10z)}\right)^6 + \left(\frac{\eta(z)\eta(10z)}{\eta(2z)\eta(5z)}\right)^6 + c_2$
11	$t_{11} = \left(\frac{\theta(2,2,6)}{\eta(z)\eta(11z)}\right)^2 = \left(\frac{\eta(z)\eta(11z)}{\eta(2z)\eta(22z)}\right)^2 + 16\left(\frac{\eta(2z)\eta(22z)}{\eta(z)\eta(11z)}\right)^2 + 16\left(\frac{\eta(2z)\eta(22z)}{\eta(z)\eta(11z)}\right)^4 + c_1$
13	$t_{13} = \left(\frac{\eta(z)}{\eta(13z)}\right)^2 + 13\left(\frac{\eta(13z)}{\eta(z)}\right)^2$
14	$t_{14} = \left(\frac{\eta(z)\eta(7z)}{\eta(2z)\eta(14z)}\right)^3 + 8\left(\frac{\eta(2z)\eta(14z)}{\eta(z)\eta(7z)}\right)^3 = \left(\frac{\eta(2z)\eta(7z)}{\eta(z)\eta(14z)}\right)^4 + \left(\frac{\eta(z)\eta(14z)}{\eta(2z)\eta(7z)}\right)^4 + c_1$
15	$t_{15} = \left(\frac{\eta(z)\eta(5z)}{\eta(3z)\eta(15z)}\right)^2 + 9\left(\frac{\eta(3z)\eta(15z)}{\eta(z)\eta(5z)}\right)^2 = \left(\frac{\eta(3z)\eta(5z)}{\eta(z)\eta(15z)}\right)^3 + \left(\frac{\eta(z)\eta(15z)}{\eta(3z)\eta(5z)}\right)^3 + c_1$
17	$t_{17} = \left(\frac{\theta_x(\frac{1}{2},0,\frac{17}{2}) - \theta_y(\frac{1}{2},0,\frac{17}{2})}{2\eta(z)\eta(17z)}\right)^2$
19	$t_{19} = \left(\frac{2\theta(2,2,10)}{\theta(1,2,20) - \theta(4,2,5)}\right)^2$
21	$t_{21} = \left(\frac{\eta(z)\eta(3z)}{\eta(7z)\eta(21z)}\right) + 7\left(\frac{\eta(7z)\eta(21z)}{\eta(z)\eta(3z)}\right) = \left(\frac{\eta(3z)\eta(7z)}{\eta(z)\eta(21z)}\right)^2 + \left(\frac{\eta(z)\eta(21z)}{\eta(3z)\eta(7z)}\right)^2 + c_1$
22	$t_{22} = \left(\frac{\eta(z)\eta(11z)}{\eta(2z)\eta(22z)}\right)^2 + 4\left(\frac{\eta(2z)\eta(22)}{\eta(z)\eta(11z)}\right)^2$
23	$t_{23} = \left(\frac{\theta(2,2,12)}{\eta(z)\eta(23z)}\right) = \left(\frac{\eta(z)\eta(23z)}{\eta(2z)\eta(46z)}\right) + 4\left(\frac{\eta(2z)\eta(46z)}{\eta(z)\eta(23z)}\right) + 4\left(\frac{\eta(2z)\eta(46z)}{\eta(z)\eta(23z)}\right)^2 + c_1$
26	$t_{26} = \left(\frac{\eta(2z)\eta(13z)}{\eta(z)\eta(26z)}\right)^2 + \left(\frac{\eta(z)\eta(26z)}{\eta(2z)\eta(13z)}\right)^2$
29	$t_{29} = \frac{\theta_x(\frac{1}{2},0,\frac{29}{2}) - \theta_y(\frac{1}{2},0,\frac{29}{2})}{2\eta(z)\eta(29z)}$
30	$t_{30} = \left(\frac{\eta(z)\eta(6z)\eta(10z)\eta(15z)}{\eta(2z)\eta(3z)\eta(5z)\eta(30z)}\right)^3 + \left(\frac{\eta(z)\eta(6z)\eta(10z)\eta(15z)}{\eta(2z)\eta(3z)\eta(5z)\eta(30z)}\right)^{-3}$
	$= \left(\frac{\eta(z)\eta(3z)\eta(5z)\eta(15z)}{\eta(2z)\eta(6z)\eta(10z)\eta(30z)}\right) + 4\left(\frac{\eta(z)\eta(3z)\eta(5z)\eta(15z)}{\eta(2z)\eta(6z)\eta(10z)\eta(30z)}\right)^{-1} + c_1$
30	$t_{30} = \left(\frac{\eta(3z)\eta(5z)\eta(6z)\eta(10z)}{\eta(z)\eta(2z)\eta(15z)\eta(30z)}\right) + \left(\frac{\eta(3z)\eta(5z)\eta(6z)\eta(10z)}{\eta(z)\eta(2z)\eta(15z)\eta(30z)}\right)^{-1} + c_2$
	$= \left(\frac{\eta(2z)\eta(3z)\eta(10z)\eta(15z)}{\eta(z)\eta(5z)\eta(6z)\eta(30z)}\right)^2 + \left(\frac{\eta(2z)\eta(3z)\eta(10z)\eta(15z)}{\eta(z)\eta(5z)\eta(6z)\eta(30z)}\right)^{-2} + c_3$
31	$t_{31} = \left(\frac{\theta(2,2,16) - \theta(4,2,8)}{2\eta(z)\eta(31z)}\right)^3$
33	$t_{33} = \left(\frac{\eta(z)\eta(11z)}{\eta(3z)\eta(33z)}\right) + 3\left(\frac{\eta(z)\eta(11z)}{\eta(3z)\eta(33z)}\right)^{-1}$
34	t_{34} is deduced from the formula $t_{34}^2(z) + t_{34}(z) - 6 = j_{17}(z) + j_{17}(2z)$
35	$t_{35} = \left(\frac{\eta(5z)\eta(7z)}{\eta(z)\eta(35z)}\right) - \left(\frac{\eta(5z)\eta(7z)}{\eta(z)\eta(35z)}\right)^{-1}$
38	t_{38} is deduced from the formula $t_{38}^2(z) + t_{38}(z) - 4 = j_{19}(z) + j_{19}(2z)$
39	$t_{39} = \left(\frac{\eta(3z)\eta(13z)}{\eta(z)\eta(39z)}\right) + \left(\frac{\eta(3z)\eta(13z)}{\eta(z)\eta(39z)}\right)^{-1}$
41	$t_{41} = \frac{\theta_x(\frac{3}{2},2,\frac{15}{2}) - \theta_y(\frac{3}{2},2,\frac{15}{2})}{2\eta(z)\eta(41z)}$
42	$t_{42} = \left(\frac{\eta(z)\eta(6z)\eta(14z)\eta(21z)}{\eta(2z)\eta(3z)\eta(7z)\eta(42z)}\right)^2 + \left(\frac{\eta(z)\eta(6z)\eta(14z)\eta(21z)}{\eta(2z)\eta(3z)\eta(7z)\eta(42z)}\right)^{-2}$
	$= \left(\frac{\eta(2z)\eta(6z)\eta(7z)\eta(21z)}{\eta(z)\eta(3z)\eta(14z)\eta(42z)}\right) + \left(\frac{\eta(2z)\eta(6z)\eta(7z)\eta(21z)}{\eta(z)\eta(3z)\eta(14z)\eta(42z)}\right)^{-1} + c_1$

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

TABLE 3. (*Continued.*)

N	Formula for $t_N = j_N + \text{const.}$
46	$t_{46} = \left(\frac{\eta(z)\eta(23z)}{\eta(2z)\eta(46z)}\right) + 2\left(\frac{\eta(z)\eta(23z)}{\eta(2z)\eta(46z)}\right)^{-1}$
47	$t_{47} = \frac{\theta(2,2,24)-\theta(4,2,12)}{2\eta(z)\eta(47z)}$
51	t_{51} is deduced from the formula $t_{51}^3(z) - 2t_{51}(z) - 6 = j_{17}(z) + j_{17}(3z)$
55	t_{55} is deduced from the formula $t_{55}^5(z) - 10t_{55}^3(z) - 5t_{55}^2(z) + 16t_{55}(z) = j_{11}(z) + j_{11}(5z)$
59	$t_{59} = \frac{2\theta(6,2,10)}{\theta(2,2,30)-\theta(6,2,10)}$
62	t_{62} is deduced from the formula $t_{62}^2(z) + t_{62}(z) - 2 = j_{31}(z) + j_{31}(2z)$
66	$t_{66} = \left(\frac{\eta(2z)\eta(3z)\eta(22z)\eta(33z)}{\eta(z)\eta(6z)\eta(11z)\eta(66z)}\right) + \left(\frac{\eta(2z)\eta(3z)\eta(22z)\eta(33z)}{\eta(z)\eta(6z)\eta(11z)\eta(66z)}\right)^{-1}$
69	t_{69} is deduced from the formula $t_{69}^3(z) - 2t_{69}(z) - 3 = j_{23}(z) + j_{23}(3z)$
70	$t_{70} = \left(\frac{\eta(z)\eta(10z)\eta(14z)\eta(35z)}{\eta(2z)\eta(5z)\eta(7z)\eta(70z)}\right) + \left(\frac{\eta(z)\eta(10z)\eta(14z)\eta(35z)}{\eta(2z)\eta(5z)\eta(7z)\eta(70z)}\right)^{-1}$
71	$t_{71} = \frac{\theta(4,2,18)-\theta(6,2,12)}{2\eta(z)\eta(71z)}$
78	$t_{78} = \left(\frac{\eta(z)\eta(6z)\eta(26z)\eta(39z)}{\eta(2z)\eta(3z)\eta(13z)\eta(78z)}\right) + \left(\frac{\eta(z)\eta(6z)\eta(26z)\eta(39z)}{\eta(2z)\eta(3z)\eta(13z)\eta(78z)}\right)^{-1}$
87	t_{87} is deduced from the formula $t_{87}^3(z) + t_{87}(z) - 3 = j_{29}(z) + j_{29}(3z)$
94	t_{94} is deduced from the formula $t_{94}^2(z) + t_{94}(z) - 2 = j_{47}(z) + j_{47}(2z)$
95	t_{95} is deduced from the formula $t_{95}^5(z) - 3t_{95}^3(z) + t_{95}(z) - 2 = j_{19}(z) + j_{19}(5z)$
105	t_{105} is deduced from the formula $t_{105}^3(z) - 2t_{105}(z) - 3 = j_{35}(z) + j_{35}(3z)$
110	t_{110} is deduced from the formula $t_{110}^2(z) + t_{110}(z) = j_{55}(z) + j_{55}(2z)$
119	t_{119} is deduced from the formula $t_{119}^7(z) - 7t_{119}^3(z) - 7t_{119}^2(z) - 6t_{119}(z) - 7 = j_{17}(z) + j_{17}(7z)$

6.2. *Groups* $\Gamma_0(N)^+$ *of higher genus*

There are 38 different square-free levels N such that X_N has genus one. Similarly, there are 39 and 31 different square-free N such that X_N has genus two and three, respectively. In [7], the authors studied the q-expansions for the corresponding function fields, proving that each function field admits two generators with various properties, such as minimal pole at infinity and integer coefficients. In particular, a polynomial relation was computed for each pair of generators, thus giving an algebraic equation for the corresponding projective curve. In future studies, we plan to investigate the various properties of these elliptic (genus-one) and hyperelliptic (genus-two) curves. There are a vast number of problems, both arithmetic and analytic, to be considered given that one knows the uniformizing group, a projective equation, q-expansions, and relations to holomorphic Eisenstein series.

References

1. A. BOREL, S. CHOWLA, C. S. HERZ, K. IWASAWA and J. P. SERRE (eds), *Seminar on complex multiplication*, Lecture Notes in Mathematics 21 (Springer, Berlin–New York, 1966).
2. J. H. CONWAY and S. P. NORTON, 'Monstrous moonshine', *Bull. Lond. Math. Soc.* 11 (1979) 308–339.
3. C. J. CUMMINS, 'Congruence subgroups of groups commensurable with PSL$(2,\mathbb{Z})$ of genus 0 and 1', *Exp. Math.* 13 (2004) 361–382.
4. T. GANNON, 'Monstrous moonshine: the first twenty-five years', *Bull. Lond. Math. Soc.* 38 (2006) 1–33.
5. T. GANNON, *Moonshine beyond the monster. The bridge connecting algebra, modular forms and physics*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2006).

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press

6. J. Jorgenson, L. Smajlović and H. Then, 'On the distribution of eigenvalues of Maass forms on certain moonshine groups', *Math. Comp.* 83 (2014) 3039–3070.

7. J. Jorgenson, L. Smajlović and H. Then, 'Kronecker's limit formula, holomorphic modular functions and q-expansions on certain arithmetic groups', *Exp. Math.* 25 (2016) 295–320.

8. J. Jorgenson, L. Smajlović and H. Then, Data page, http://www.efsa.unsa.ba/~lejla.smajlovic/.

9. T. Miezaki, H. Nozaki and J. Shigezumi, 'On the zeros of Eisenstein series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$', *J. Math. Soc. Japan* 59 (2007) 693–706.

10. J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics 7 (Springer, New York, 1973).

Jay Jorgenson
Department of Mathematics
The City College of New York
Convent Avenue at 138th Street
New York
NY 10031
USA

jjorgenson@mindspring.com

Lejla Smajlović
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35
71 000 Sarajevo
Bosnia and Herzegovina

lejlas@pmf.unsa.ba

Holger Then
Department of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW
United Kingdom

holger.then@bristol.ac.uk

https://doi.org/10.1112/S1461157016000425 Published online by Cambridge University Press