EXTENSIONS OF SEMILATTICES BY LEFT TYPE-A SEMIGROUPS

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(Received 31 May, 1995)

0. Introduction. On a semigroup S let the relation $\mathcal{R}^*$, sometimes denoted by $\mathcal{R}^*_S$, be defined by $x\mathcal{R}^*y \Leftrightarrow [(\forall s,t \in S^1)sx = tx \Leftrightarrow sy = ty]$. A semigroup S is called *left type-A*, iff the set E_S of idempotents of S forms a semilattice under multiplication, each element x of S is $\mathcal{R}^*$ related to a (necessarily unique) idempotent x^+, and $xe = (xe)^+x$ for all $x \in S$, $e \in E_S$.

Left type-A semigroups are natural generalizations of inverse semigroups and have been the subject of a considerable amount of investigation in recent years (e.g. [**2**], [**3**], [**7**], [**5**], [**4**]).

By a *left type-A congruence* ρ on a left type-A semigroup S we mean a congruence ρ on $(S,\cdot)$, satisfying the implication $x\rho y \Rightarrow x^+\rho y^+$ and making S/ρ into a left type-A semigroup by $(s\rho)^+ := s^+\rho$.

The purpose of this paper is to generalize the results of [**1**] to left type-A semigroups.

In Section 2 we define the notions of *λ-semidirect product* and *full restricted semidirect product*. These notions are not given in full generality but rather as they are needed here (first (second) component a semilattice (left type-A semigroup)).

In Section 3 we prove that, given a left type-A semigroup S and a left type-A congruence ρ on S, satisfying $\rho \cap \mathcal{R}^* = \iota$, the identity relation, then S is isomorphic to a well determined subsemigroup T of a λ-semidirect product $A *_\lambda S/\rho$, with A a semilattice. Moreover $\mathcal{R}^*_T \subseteq \mathcal{R}^*_{A*_\lambda S/\rho}$.

In particular, if S is *proper*, i.e. if $\sigma \cap \mathcal{R}^* = \iota$, where σ is the least right cancellative congruence on S, we obtain that S is isomorphic to an *M-semigroup* T, [**2**], which is embedded into a semidirect product of a semilattice by a right cancellative monoid. Regarding this we generalize Fountain's representation theorem for proper left type-A semigroups, as well as O'Carroll's embedding theorem for *E-unitary inverse* semigroups, [**6**].

A related result, which states that each proper left type-A semigroup is embeddable into a *reverse* semidirect product of a semilattice by a right cancellative monoid, was recently proven using the categorical approach in [**4**].

In Section 4 we apply our methods to a certain class of left type-A semigroups whose intersection with the class of inverse semigroups consists precisely of the E-reflexive ones, in the sense of [**8**].

The notation and terminology of [**8**] will be used throughout the paper, whenever possible. In particular, for any congruence, $\bar{s}$ often denotes the congruence class containing s, and for H, $K \subseteq S$, HK means the set product, i.e. $HK = \{hk \mid h \in H, k \in K\}$.

1. Preliminaries. The following basic results on left type-A semigroups and left type-A congruences are frequently used in the sequel without further reference.

PROPOSITION 1.1. *Let S be a left type-A semigroup. Then:*

(i) $(xy^+)^+ = (xy)^+$,

Glasgow Math. J. **39** (1997) 7–16.

(ii) $x^+(xy)^+ = (xy)^+$,
(iii) $xy^+ = (xy^+)^+x$.

PROPOSITION 1.2. *Let ρ be a left type-A congruence on a left type A-semigroup S. Then the following statements are equivalent:*
(i) $\rho \cap \mathcal{R}^* = \iota$,
(ii) $x\rho y \Rightarrow x^+y = y^+x$.

Proof. (i)$\Rightarrow$(ii). Let $\rho \cap \mathcal{R}^* = \iota$ and $x\rho y$. It follows that $x^+\rho y^+$ which implies $x^+y\rho y^+x$. Moreover, since $\mathcal{R}^*$ is a left congruence on S, we obtain $x^+y\mathcal{R}^*x^+y^+ = y^+x^+\mathcal{R}^*y^+x$. Thus by assumption $x^+y = y^+x$ follows.

(ii) $\Rightarrow$(i). Let (ii) be satisfied and $x\rho \cap \mathcal{R}^*y$. It follows that $x^+ = y^+$ and $x^+y = y^+x$, implying $x = x^+x = y^+x = x^+y = y^+y = y$. □

PROPOSITION 1.3. *Let ρ be a left type-A congruence on a left type-A semigroup S with $\rho \cap \mathcal{R}^* = \iota$. Then ρ is idempotent pure, i.e. $a\rho e = e^2$ implies $a \in E_S$.*

Proof. Let $a\rho e$ with $e \in E_S$. It follows that $a\rho a^2$, implying $a\rho a^+$, since ρ is left type-A, implying $a = a^+$ by assumption. □

Note that the converse of Proposition 1.3 is not true since there are E-unitary left type-A semigroups which are not proper [**2**].

2. λ-Semidirect products and full restricted semidirect products

PROPOSITION 2.1. *Let A be a semilattice and G be a left type-A semigroup acting on A by endomorphisms on the left, i.e. for each $g \in G$, $\alpha \to g\alpha$ is a homomorphism and $g(h\alpha) = (gh)\alpha$, for all $\alpha \in A$, g, $h \in G$. On $T := \{(\alpha, g) \in A \times G \mid g^+\alpha = \alpha\}$ let a multiplication be defined by $(\alpha, g).(\beta, h) := ((gh)^+\alpha \cap g\beta, gh)$. Then T is a left type-A semigroup with $E_T = \{(\alpha, g^+) \in A \times G \mid g^+\alpha = \alpha\}$ and $(\alpha, g)^+ = (\alpha, g^+)$ for $(\alpha, g) \in T$. (T is called a* λ-semidirect product of *A* by *G*, *denoted by $A *_\lambda G$).*

Proof. Let (α, g), (β, h), $(\gamma, k) \in T$. It follows that $(gh)^+((gh)^+\alpha \wedge g\beta) = (gh)^+\alpha \wedge (gh)^+g\beta = (gh)^+\alpha \wedge gh^+\beta = (gh)^+\alpha \wedge g\beta$, proving that $(\alpha, g).(\beta, h) \in T$. Moreover we obtain

$$\begin{aligned}((\alpha, g)(\beta, h)(\gamma, k) &= ((ghk)^+((gh)^+\alpha \wedge g\beta) \wedge gh\gamma, ghk) \\ &= ((ghk)^+(gh)^+\alpha \wedge (ghk)^+g\beta \wedge gh\gamma, ghk) \\ &= ((ghk)^+\alpha \wedge g(hk)^+\beta \wedge gh\gamma, ghk) \\ &= ((ghk)^+\alpha \wedge g((hk)^+\beta \wedge h\gamma), ghk) \\ &= (\alpha, g)((\beta, h)(\gamma, k)).\end{aligned}$$

Thus T is a semigroup.

It remains to show that T is left type-A. Obviously $E_T = \{(\alpha, g^+) \mid (\alpha, g) \in T\}$. E_T is a semilattice, since $(\alpha, g^+)(\beta, h^+) = (g^+h^+(\alpha \wedge \beta), g^+h^+) = (h^+g^+(\beta \wedge \alpha), h^+g^+) = (\beta, h^+)(\alpha, g^+)$.

https://doi.org/10.1017/S0017089500031840 Published online by Cambridge University Press

Further, for (β,h), $(\gamma,k)\in T$ we obtain

$$\begin{aligned}(\beta,h)(\alpha,g)=(\gamma,k)(\alpha,g)&\Leftrightarrow((hg)^{+}\beta\wedge h\alpha,hg)=((kg)^{+}\gamma\wedge k\alpha,kg)\\&\Leftrightarrow(hg)^{+}\beta\wedge h\alpha=(kg^{+})^{+}\gamma\wedge k\alpha\text{ and }hg^{+}=kg^{+}\\&\Leftrightarrow(\beta,h)(\alpha,g^{+})=(\gamma,k)(\alpha,g^{+}).\end{aligned}$$

Consequently $(\alpha,g^{+})\mathcal{R}^{*}(\alpha,g)$, and (α,g^{+}) acts as $(\alpha,g)^{+}$ on T. Finally

$$\begin{aligned}(\alpha,g)(\beta,h^{+})&=((gh)^{+}\alpha\wedge g\beta,gh^{+})\\&=((gh)^{+}\alpha\wedge g\beta\wedge(gh)^{+}\alpha,(gh)^{+}g)\\&=((\alpha,g)(\beta,h^{+}))^{+}(\alpha,g),\end{aligned}$$

completing the proof. □

It should be noted that if, in Proposition 2.1, G is a right cancellative monoid with identity 1, and the action satisfies $1\alpha=\alpha$, for all $\alpha\in A$, then $A*_{\lambda}G$ becomes a semidirect product of A by G.

An important example of the above construction is the following:

Definition 2.2. Let A be a semilattice and G be a left type-A semigroup. Then $F:=A^{G}$ is a semilattice with respect to the multiplication given by $(x)f\wedge g:=(x)f\wedge(x)g$, $x\in G$, f, $g\in F$. For $t\in G$, $f\in F$, let $tf\in F$ be defined by $(x)tf:=(xt)f$, $x\in G$. This defines an action of G on F by endomorphisms on the left. Thus we can form the λ-semidirect product $W:=F*_{\lambda}G$ with respect to this action. W is called the *standard λ-wreath product of A by G*, denoted by $AW_{\lambda}G$.

In general, a semidirect product $A*G$ of a semilattice A by a left type-A semigroup G is not left type-A. However, if the action of G on A satisfies some additional properties, a left type-A subsemigroup T of $A*G$ can be determined with $\mathcal{R}^{*}_{T}\subseteq\mathcal{R}^{*}_{A*G}$.

Proposition 2.3. *Let A be a semilattice and G be a left type-A semigroup acting on A on the left, such that for each $\alpha\in A$, there exists $e_{\alpha}\in E_{G}$, satisfying*

(i) $e_{\alpha}\alpha=\alpha$, for all $\alpha\in A$,

(ii) $e_{\alpha\wedge\beta}=e_{\alpha}e_{\beta}$, for all α, $\beta\in A$,

(iii) $e_{g\alpha}=(ge_{\alpha})^{+}$, for all $\alpha\in A$, $g\in G$.

Then $T:=\{(\alpha,g)\in A\times G\mid e_{\alpha}=g^{+}\}$ is a subsemigroup of $A*G$, which is left type-A, with $(\alpha,g)^{+}:=(\alpha,e_{\alpha})$. T is called a full restricted semidirect product of A by G, *denoted by $A\otimes G$.*

Proof. Let (α,g), $(\beta,h)\in T$. It follows $e_{\alpha}=g^{+}$, $e_{\beta}=h^{+}$, and we obtain by (ii) and (iii) $e_{\alpha\wedge g\beta}=e_{\alpha}(ge_{\beta})^{+}=g^{+}(gh)^{+}=(gh)^{+}$, implying $(\alpha,g)(\beta,h)\in T$. Further, $E_{T}=\{(\alpha,e_{\alpha})\mid\alpha\in A\}$, and for (α,e_{α}), $(\beta,e_{\beta})\in E_{T}$, we have $(\alpha,e_{\alpha})(\beta,e_{\beta})=(\alpha\wedge e_{\alpha}\beta,e_{\alpha}e_{\beta})\overset{(i)}{=}(e_{\alpha}(\alpha\wedge\beta),\ e_{\alpha}e_{\beta})\overset{(ii)}{=}(e_{\alpha}e_{\beta}(\alpha\wedge\beta),\ e_{\alpha}e_{\beta})\overset{(i)}{=}(\alpha\wedge\beta,e_{\alpha}e_{\beta})=(\beta\wedge\alpha,e_{\beta}e_{\alpha})=(\beta,e_{\beta})(\alpha,e_{\alpha})$, by symmetry. Thus E_{T} is a semilattice. Moreover $(\alpha,g)\mathcal{R}^{*}(\alpha,e_{\alpha})$ follows directly from the fact that G is left type-A. Finally we get $(\alpha,g)(\beta,h)^{+}=(\alpha\wedge g\beta,ge_{\beta})\overset{(i)}{=}(\alpha\wedge g\beta\wedge(ge_{\beta})^{+}(\alpha\wedge g\beta),\quad(ge_{\beta})^{+}g)=(\alpha\wedge g\beta\wedge(ge_{\beta})^{+}\alpha,\quad(ge_{\beta})^{+}g)=((\alpha,g)(\beta,h))^{+}(\alpha,g)$, completing the proof.

https://doi.org/10.1017/S0017089500031840 Published online by Cambridge University Press

The notions of λ-semidirect product and full restricted semidirect product are closely related as the following theorem shows.

THEOREM 2.4. *Let $S := A *_\lambda G$ be a λ-semidirect product of a semilattice A by a left type-A semigroup G. Then $A *_\lambda G$ can be embedded into a full restricted semidirect product $A' \otimes G$ of a semilattice A' by G.*

Proof. Consider the semilattice $A' := E_S = \{(\alpha, e) \in S \mid e \in E_G\}$. For $g \in G$, $(\alpha, e) \in A'$ let $g(\alpha, e) := (g\alpha, (ge)^+)$. A straightforward verification shows that this defines an action of G on A' by endomorphisms on the left. For $(\alpha, f) \in A'$ let $e_{(\alpha,f)} := f$. Then the conditions (i), (ii), (iii) of Proposition 2.3 are satisfied. Hence we may construct $A' \otimes G$ in which S is embedded via $(\alpha, g) \to ((\alpha, g^+), g)$. □

In the following proposition we determine a certain subsemigroup $A *_\lambda G$ which will be useful in view of the representation theorem in Section 3.

PROPOSITION 2.5. *Let $A *_\lambda G$ be as in Theorem 2.4. Let C be a nonempty subset of A, and $B := C \cup \{g\alpha \mid g \in G, \alpha \in C\}$. Let ε be a fixed element of A and assume further that for each $\mu \in B$ there is an $e_\mu \in E_G$ such that*
(i) $e_\mu \mu = \mu$, *for all* $\mu \in B$,
(ii) $\mu, \nu, \mu \wedge \nu \in B \Rightarrow e_{\mu\wedge\nu} = e_\mu e_\nu$,
(iii) $e_{g\mu} = (ge_\mu)^+$, *for all* $\mu \in B$, $g \in G$,
(iv) $\alpha \wedge e_\alpha \varepsilon = \alpha$, *for all* $\alpha \in C$,
(v) $\alpha, \beta \in C$, $e_\alpha = g^+$ *and* $\alpha \wedge g\varepsilon = \alpha$ *imply* $(ge_\beta)^+\alpha \wedge g\beta \in C$,
(vi) *for each* $g \in G$, *there is* $\alpha \in C$ *such that* $e_\alpha = g^+$ *and* $\alpha \wedge g\varepsilon = \alpha$.
Then $T := \{(\alpha, g) \in C \times G \mid e_\alpha = g^+ \text{ and } \alpha \wedge g\varepsilon = \alpha\}$ *is a left type-A subsemigroup of* $A *_\lambda G$ *with* $\mathcal{R}^*_T \subseteq \mathcal{R}^*_{A*_\lambda G}$. *Moreover,* ρ_T, *defined by* $(\alpha, g)\rho_T(\beta, h) \Leftrightarrow g = h$, *is a left type-A congruence on* T, *satisfying* $\rho_T \cap \mathcal{R}^*_T = \iota$, *and* $T/\rho_T \cong G$.

Proof. Let (α, g), $(\beta, h) \in T$. It follows that $e_\alpha = g^+$, $e_\beta = h^+$, $\alpha \wedge g\varepsilon = \alpha$, and $\beta \wedge h\varepsilon = \beta$. Thus $(\alpha, g)(\beta, h) = ((gh)^+\alpha \wedge g\beta, gh) = ((ge_\beta)^+\alpha \wedge g\beta, gh) \in C \times G$, by (v), and $e_{(ge_\beta)^+\alpha\wedge g\beta} \overset{\text{(ii)}}{=} e_{(ge_\beta)^+\alpha} e_{g\beta} \overset{\text{(iii)}}{=} ((ge_\beta)^+e_\alpha)^+(ge_\beta)^+ = (ge_\beta)^+ = (gh)^+$. Further, $(ge_\beta)^+\alpha \wedge g\beta \wedge gh\varepsilon = (ge_\beta)^+\alpha \wedge g(\beta \wedge h\varepsilon) = (ge_\beta)^+\alpha \wedge g\beta$. Consequently, T is a subsemigroup of $A *_\lambda G$. Moreover, $(\alpha, e_\alpha) \in T$ by (iv), which implies that T is left type-A with $(\alpha, g)^+ = (\alpha, e_\alpha)$. From this we directly obtain that ρ_T is left type-A, $\rho_T \cap \mathcal{R}^*_T = \iota$, and $R^*_T \subseteq \mathcal{R}^*_{A*_\lambda G}$. Finally T/ρ_T is isomorphic to G via $(\alpha, g) \to g$, by (vi). □

In connection with Theorem 2.4, Proposition 2.5 leads to a certain subsemigroup of a full restricted semidirect product, which will be described below.

PROPOSITION 2.6. *Let $A \otimes G$ be as in Proposition 2.3. Let B be a subsemilattice of A and ε an element of A, such that the following holds:*
(i) $\alpha \wedge \varepsilon = \alpha$, *for all* $\alpha \in B$,
(ii) $\alpha, \beta \in A$, $e_\alpha = g^+$ *and* $\alpha \wedge g\varepsilon = \alpha$ *imply* $\alpha \wedge g\beta \in B$,
(iii) *for each* $g \in G$, *there is* $\alpha \in B$, *such that* $e_\alpha = g^+$ *and* $\alpha \wedge g\varepsilon = \alpha$.
Then $T := \{(\alpha, g) \in B \times G \mid e_\alpha = g^+ \text{ and } \alpha \wedge g\varepsilon = \alpha\}$ *is a left type-A subsemigroup of* $A \otimes G$ *with* $\mathcal{R}^*_T \subseteq \mathcal{R}^*_{A\otimes G}$. *Moreover,* ρ_T, *defined by* $(\alpha, g)\rho_T(\beta, h) \Leftrightarrow g = h$, *is a left type-A congruence on* T, *satisfying* $\rho_T \cap \mathcal{R}^*_T = \iota$, *and* $T/\rho_T \cong G$.

https://doi.org/10.1017/S0017089500031840 Published online by Cambridge University Press

Proof. Let (α, g), $(\beta, h) \in T$. It follows that $e_\alpha = g^+$, $e_\beta = h^+$, $\alpha \wedge g\varepsilon = \alpha$, and $\beta \wedge h\varepsilon = \beta$. We obtain $e_{\alpha \wedge g\beta} = e_\alpha e_{g\beta} = g^+(gh^+)^+ = (gh)^+$, and $\alpha \wedge g\beta \wedge gh\varepsilon = \alpha \wedge g(\beta \wedge h\varepsilon) = \alpha \wedge g\beta$. Further, by (ii), $\alpha \wedge g\beta \in B$. Consequently, T is a subsemigroup of $A \otimes G$. Moreover, $\alpha \wedge e_\alpha \varepsilon = \alpha \wedge \varepsilon = \alpha$ implies $(\alpha, e_\alpha) \in T$, and T is left type-A with $(\alpha, g)^+ = (\alpha, e_\alpha)$. It follows directly that ρ_T is left type-A, $\rho_T \cap \mathcal{R}^*_T = \iota$, and $\mathcal{R}^*_T \subseteq \mathcal{R}^*_{A \otimes G}$. Finally T/ρ_T is isomorphic to G via $(\alpha, g) \to g$, by (iii). □

According to [2], [7] we note the following results.

COROLLARY 2.7. *If G is a right cancellative monoid in Proposition 2.6, then (G, GB, B) is a left admissible triple and $T = M(G, GB, B)$.*

COROLLARY 2.8. *Let T be a semigroup constructed by Proposition 2.5. Assume further that G is a monoid with identity 1 and $1\varepsilon = \varepsilon$. Then T is isomorphic to a semigroup constructed by Proposition 2.6.*

Proof. Consider $A' \otimes G$ as in Theorem 2.4. Note that by assumption $(\varepsilon, 1)$ belongs to A'. Then T is embedded into $A' \otimes G$ via ψ, defined by $\psi : (\alpha, g) \to ((\alpha, e_\alpha), g)$. Moreover $B' := \{(\alpha, e_\alpha) \mid \alpha \in C\}$ is a subsemilattice of A', and $T\psi$ is built up by B' and $\varepsilon' := (\varepsilon, 1)$, as T is by B and ε in Proposition 2.6. □

3. The representation theorem. Let S be a left type-A semigroup and ρ be a left type-A congruence on S satisfying $\rho \cap \mathcal{R}^* = \iota$. The aim of this section is to show that S is isomorphic to a semigroup constructed by Proposition 2.5 with $G = S/\rho$.

The following proposition is a generalization of Theorem 4 of [1].

PROPOSITION 3.1. *Let S be a left type-A semigroup and ρ be a left type-A congruence on S such that $\rho \cap \mathcal{R}^* = \iota$. Consider $C'(S) := \{H \subseteq S \mid H \neq \phi,\ H \subseteq \bar{s},\ \overline{s^+}H = H$, for some $\bar{s} \in S/\rho\}$.*

(i) *$C'(S)$ is a left type-A semigroup with multiplication given by $H \circ K := \overline{(st)^+}HK$, for $H, K \in C'(S)$, with $H \subseteq \bar{s}$, $K \subseteq \bar{t}$, and $H^+ := \{h^+ \mid h \in H\}$.*

(ii) *$S/\rho \subseteq C'(S)$.*

(iii) *S is embedded into $C'(S)$ via $s \to \hat{s}$, where $\hat{s} := \overline{s^+}\{s\}$, and the embedding respects $\mathcal{R}^*$, i.e. $\mathcal{R}^*_S \subseteq \mathcal{R}^*_{C'(S)}$.*

Proof. (i): Let H, K, $L \in C'(S)$, with $H \subseteq \bar{s}$, $K \subseteq \bar{t}$, $L \subseteq \bar{u}$. We prove that $H \circ K$ is in $C'(S)$. Obviously $H \circ K$ is uniquely defined and $H \circ K \subseteq \overline{st}$. Moreover $\overline{(st)^+}(H \circ K) = \overline{(st)^+}\,\overline{(st)^+}HK = H \circ K$.

In what follows, we will use Proposition 1.1 several times without further reference. Next we show that $\circ$ is associative. We have

$$(H \circ K) \circ L = \overline{(stu)^+}\,\overline{(st)^+}HKL \subseteq \overline{(stu)^+}HKL,$$

and

$$\overline{(stu)^+}HKL \subseteq \overline{(stu)^+}\,\overline{(st)^+}HKL \text{ imply } (H \circ K) \circ L = \overline{(stu)^+}HKL.$$

On the other hand we have

$$H \circ (K \circ L) = \overline{(stu)^+}H\overline{(tu)^+}KL \subseteq \overline{(stu)^+}HKL,$$

and

$$\overline{(stu)^+}HKL \subseteq \overline{(stu)^+}H\overline{(tu)^+}KL, \text{ implying } H \circ (K \circ L) = \overline{(stu)^+}HKL.$$

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Consequently $\circ$ is associative. Note that $E_{C'(S)} = \{H \subseteq E_S \mid H \subseteq \bar{e},\ \bar{e}H = H$ for some $e \in E_S\}$ since ρ is left type-A and idempotent pure by Proposition 1.3. Hence $E_{C'(S)}$ is a semilattice.

We continue proving that $C'(S)$ is left type-A. For this, let $H \in C'(S)$ with $H \subseteq \bar{s}$. Let $H^+ := \{h^+ \mid h \in H\}$. Then clearly $H^+ \subseteq \overline{s^+}$. Further, for $e \in \overline{s^+}$, $h^+ \in H^+$, we obtain $eh^+ \mathcal{R}^* eh \mathcal{R}^* (eh)^+$ implying $eh^+ = (eh)^+ \in H^+$ since $eh \in H$. Consequently $\overline{s^+}H^+ \subseteq H^+$ and $H^+ \in E_{C'(S)}$. Next we show that H^+ is $\mathcal{R}^*$-related to H: Consider $K, L \in C'(S)$ with $K \subseteq \bar{t}$, $L \subseteq \bar{u}$ and $K \circ H = L \circ H$, i.e. $\overline{(ts)^+}KH = \overline{(us)^+}LH$.

Note that this implies $\overline{ts} = \overline{us}$ and $\overline{ts^+} = \overline{us^+}$ since ρ is left type-A. Let now $x \in K \circ H^+ = \overline{(ts)^+}KH^+$. Then $x = ekh^+$ with $e \in \overline{(ts^+)}$, $k \in K$, $h^+ \in H^+$. We obtain $x = e(kh^+)^+k = e(flh_1^+)^+k = f(lh_1^+)^+ek$, for suitable $f \in \overline{(us^+)}$, $l \in L$, $h_1 \in H$. Since $\overline{lh_1^+} = \overline{us^+} = \overline{ts^+} = \overline{ek}$, we get by Proposition 1.2 that $x = f(ek)^+lh_1^+ \in \overline{(us)^+}LH^+ = L \circ H^+$. Analogously, it follows that $L \circ H^+ \subseteq K \circ H^+$, whence $K \circ H^+ = L \circ H^+$. Finally, since for arbitrary $M, N \in C'(S)$, $M \circ H = H$ (rsp. $H = N \circ H$) implies $M \circ H = H^+ \circ H$ (rsp. $H^+ \circ N = N \circ H$), we obtain by the above $M \circ H^+ = H^+$ (rsp. $H^+ = N \circ H^+$). Summarizing, we have proven that H^+ is $\mathcal{R}^*$-related to H.

To complete the proof we have to show that for arbitrary $H, K \in C'(S)$ the equation $H \circ K^+ = (H \circ K^+)^+ \circ H$ is valid. Let $H \subseteq \bar{s}$, $K \subseteq \bar{t}$. We must verify the equality $A := \overline{(st)^+}HK^+ = \overline{(st)^+}((\overline{st})^+HK)^+H =: B$. Let $x \in A$. Then $x = ehk^+$ with $e \in \overline{(st)^+}$, $h \in H$, $k^+ \in K^+$. It follows that $x = e(hk)^+h \in B$. On the other hand, let $x \in B$. Then $x = e(fh_1k)^+h_2$, with $e, f \in \overline{(st)^+}$, $h_1, h_2 \in H$, $k \in K$. We get $x = ef(h_1k)^+h_2 = ef(h_1k)^+h_1^+h_2 = ef(h_1k)^+h_2^+h_1$, by Proposition 1.2. Thus $x = efh_2^+(h_1k)^+h_1 = efh_2^+h_1k^+ \in A$, and $A = B$ follows.

(ii): The assertion is obvious from the fact that $\bar{s} = \overline{s^+}\bar{s}$ for each $\bar{s} \in S/\rho$.

(iii): By definition of $C'(S)$ each $\hat{s}$ lies in $C'(S)$. Further, $\hat{s} = \hat{t}$ implies $s = et$ and $t = fs$, for some $e, f \in E_S$, which implies $es = s$. Thus $s = et = efs = fes = fs = t$ follows.

We prove $\hat{s} \circ \hat{t} = \widehat{st}$. Let $x \in \hat{s} \circ \hat{t}$. Then $x = cdset$ with $\bar{c} = \overline{(st)^+}$, $\bar{d} = \overline{s^+}$ and $\bar{e} = \overline{t^+}$. It follows that $x = cd(se)^+st \in \widehat{st}$ since $\overline{cd(se)^+} = \overline{(st)^+s^+(st^+)^+} = \overline{(st)^+}$. Consequently, $\hat{s} \circ \hat{t} \subseteq \widehat{st}$. Since $\widehat{st} \subseteq \hat{s} \circ \hat{t}$ evidently holds, the assertion is proven.

Finally, the embedding respects $\mathcal{R}^*$, since $s^+ = t^+$ implies $\hat{s}^+ = (\overline{s^+}\{s\})^+ = (\overline{t^+}\{t\})^+ = \hat{t}^+$. □

Note that S/ρ is a subset of $C'(S)$ but not a subsemigroup in general. To avoid confusion, $\bar{s}^+$ always means the element of $E_{C'(S)}$ which is $\mathcal{R}^*$-related to $\bar{s}$ in $C'(S)$, whereas for the idempotent which is $\mathcal{R}^*$-related to $\bar{s}$ in S/ρ we use exclusively the term $\overline{s^+}$.

Lemma 3.2. *Let $H, K \in C'(S)$ such that $H \subseteq \bar{s}$, $K \subseteq \bar{t}$ and $\overline{(st)^+} = \overline{s^+}$. Then: $H \circ K = HK$.*

Proof. By definition of $\circ$ and the assumption we obtain

$$H \circ K = \overline{(st)^+}HK = \overline{s^+}HK = HK. \qquad \square$$

Proposition 3.1 enables us to embed S into a standard λ-wreath product of a semilattice by S/ρ.

Theorem 3.3. *Let S be a left type-A semigroup and ρ be a left type-A congruence on*

S such that $\rho \cap \mathcal{R}^* = \iota$. Then the mapping $\varphi: S \to E_{C'(S)}W_\lambda S/\rho$ defined by $s \to (f_s, \bar{s})$, where $(\bar{u})f_s := (\overline{us^+} \circ \hat{s})^+$, $\bar{u} \in S/\rho$, is an $\mathcal{R}^*$-respecting embedding.

Proof. Obviously ϕ is uniquely defined. Note that by Lemma 3.2 we have $(\bar{u})f_s = (\overline{us^+} \circ \overline{s^+}\{s\})^+ = (\overline{us^+}\ \overline{s^+}\{s\})^+$, since $(us^+s)^+ = (us^+)^+$.

We prove that ϕ is injective. Let $(f_s, \bar{s}) = (f_t, \bar{t})$ for some $s, t \in S$. Then $f_s = f_t$ and $\bar{s} = \bar{t}$. We obtain $(\overline{s^+})f_s = (\overline{s^+})f_t$, implying $(\overline{s^+}\{s\})^+ = (\overline{s^+t^+}\ \overline{t^+}\{t\})^+$, implying $s^+ = (dt)^+ = dt^+$, for some $d \in E_S$. On the other hand from $(\overline{t^+})f_s = (\overline{t^+})f_t$ we have $t^+ = es^+$, for some $e \in E_S$. Thus we get $s^+ = t^+$, implying $s\rho \cap \mathcal{R}^* t$, implying $s = t$.

Next we prove that ϕ is a homomorphism. All we have to show is that $L := (\overline{u(st)^+})f_s \circ (\overline{us})f_t = (\bar{u})f_{st} =: R$ holds for all $s, t, u \in S$.

From Lemma 3.2 we obtain $L = (\overline{u(st)^+}\ \overline{s^+}\{s\})^+(\overline{ust^+}\ \overline{t^+}\{t\})^+$, $R = (\overline{u(st)^+}\ \overline{(st)^+}\{st\})^+$. Let $z \in L$. Then $z = (xes)^+(yft)^+$, with $x \in \overline{u(st)^+}$, $e \in \overline{s^+}$, $y \in \overline{ust^+}$, $f \in \overline{t^+}$. We conclude that $z\mathcal{R}^*(xes)^+yft = (yf)^+xest$, by Lemma 1.2, since $xes\rho yf$, implying $z\mathcal{R}^*(yf)^+(xest)^+ = (((yf)^+xe)(st)^+st)^+ \in R$ since $(yf)^+xe\rho u(st)^+$. Consequently $z \in R$ since $\mathcal{R}^*$ is the identity relation on E_S. Now let $z \in R$. Then $z = (xest)^+$ with $x \in \overline{u(st)^+}$, $e \in \overline{(st)^+}$. We may write $z = (xes)^+(xest)^+ = ((xe)s^+s)^+(xest^+)t^+t)^+$, implying $z \in L$ since $xe\rho u(st)^+$ and $xest^+\rho ust^+$.

Finally ϕ respects $\mathcal{R}^*$ since $(f_s, \bar{s})^+ = (f_s, \overline{s^+})$ in $E_{C'(S)}W_\lambda S/\rho$. □

Corollary 3.4. *Let S be a left type-A semigroup and ρ be a left type-A congruence on S such that $\rho \cap \mathcal{R}^* = \iota$. Then S is embeddable into some $A *_\lambda G$, where A is a semilattice, $G = S/\rho$, and $\mathcal{R}^*$ is respected.*

Recall that a left type-A semigroup S is called proper, if $\sigma \cap \mathcal{R}^* = \iota$, where σ is the least right cancellative congruence on S, (see [**2**]). In view of the remark following Proposition 2.1, Theorem 3.3 yields:

Corollary 3.5. *Each proper left type-A semigroup is embeddable into a semidirect product of a semilattice by a right cancellative monoid such that $\mathcal{R}^*$ is respected.*

Now we are ready to formulate the main result of this section.

Theorem 3.6. *Let S and ρ be as in Theorem 3.3. Then S is isomorphic to a subsemigroup of some $A *_\lambda G$, constructed by Proposition 2.5, where $A = E^{S/\rho}_{C'(S)}$ and $G = S/\rho$.*

Proof. Consider the embedding $\varphi: s \to (f_s, \bar{s})$ of Theorem 3.3. Let $A := E^{S/\rho}_{C'(S)}$, $B := \{\bar{t}f_s \mid s, t \in S\}$, $C := \{f_s \mid s \in S\}$, and $G := S/\rho$. Note that $C \subseteq B$ since $\overline{s^+}f_s = f_s$ for each $s \in S$. Let $\varepsilon \in A$ be defined by $(\bar{u})\varepsilon := \overline{u^+}$, $\bar{u} \in G$, and for each $\bar{t}f_s \in B$ let $e_{\bar{t}f_s} := (\overline{ts})^+$. Then $e_{\bar{t}f_s}$ is uniquely defined since $\bar{t}f_s = \bar{q}f_p$ implies $(\bar{t})f_s = (\overline{t^+q})f_p$ and $(\bar{q})f_p = (\overline{q^+t})f_s$, implying $(\overline{ts})^+ = (\overline{t^+qp})^+$ and $(\overline{qp})^+ = (\overline{q^+ts})^+$, implying $(\overline{ts})^+ = (\overline{qp})^+$.

We show that the conditions (i) to (vi) of Proposition 2.5 are satisfied with respect to C, B, G, ε, and that $S\phi$ is equal to $T := \{(f_s, \bar{t}) \in C \times G \mid \overline{s^+} = \overline{t^+}$ and $f_s \wedge \bar{t}\varepsilon = f_s\}$, where $\wedge$ denotes the operation in A.

(i): holds, since $(\overline{u(ts)^+t})f_s = (\overline{uts^+})f_s = (\overline{ut})f_s$, for all $s, t, u \in S$.

(ii): Let $\bar{t}f_s \wedge \bar{q}f_p = \bar{y}f_x$, for some $p, q, s, t, x, y \in S$. It follows that $(\overline{y^+t})f_s \circ (\overline{y^+q})f_p =$

$(\bar{y})f_x$ implying $y^+(ts)^+(qp)^+ = (y^+ts)^+(y^+qp)^+p(yz)^+$. On the other hand we have $\bar{t}f_s \circ (\overline{t^+q})f_p = (\overline{t^+y})f_x$, implying $(ts)^+(qp)^+ = (ts)^+(t^+qp)^+\rho(t^+yx)^+ = t^+(yx)^+$. Summarizing, we obtain $(ts)^+(qp)^+\rho(yx)^+$ and (ii) is established.

(iii): straightforward, by definition.

Before establishing (iv) to (vi) we show that $S\varphi$ equals T.

Let $(f_s, \bar{s}) \in S\varphi$. By a direct calculation we get $(\bar{u})f_s \circ (\overline{us})\varepsilon = (\overline{us^+}\,\overline{s^+}\{s\})^+ \circ \overline{us^+} = (\overline{us^+}\,\overline{s^+}\{s\})^+\overline{us^+} = (\overline{us^+}\,\overline{s^+}\{s\})^+ = (\bar{u})f_s$, implying $(f_s, \bar{s}) \in T$. Note that obviously $f_s = f_{s^+}$, for all $s \in S$.

Now let $(f_s, \bar{t}) \in T$. It follows that $\overline{s^+} = \overline{t^+}$ and $f_s \wedge \bar{t}\varepsilon = f_s$. We obtain $(\overline{s^+})f_s \circ (\overline{s^+t})\varepsilon = (\overline{s^+})f_s$, implying $(\overline{s^+}\{s\})^+ \circ \overline{s^+t^+} = (\overline{s^+}\{s\})^+$, implying $\overline{s^+t^+}(\overline{s^+}\{s\})^+\overline{s^+t^+} = (\overline{s^+}\{s\})^+$. Thus we have $s^+ = e(fs)^+x^+$, for some $e \in \overline{s^+t^+}$, $f \in \overline{s^+}$, $x \in \overline{s^+t} = \bar{t}$. Let $p := s^+x$. Then $\bar{p} = \bar{t}$ and $p^+ = (s^+x)^+ = s^+x^+ = s^+$. Consequently $f_p = f_{p^+} = f_{s^+} = f_s$, and we obtain $(f_s, \bar{t}) = (f_p, \bar{p}) \in S\varphi$.

(iv): follows from the fact that $(f_s, \overline{s^+}) \in T$ for all $s \in S$.

(v): follows from the embedding, since $f_s, f_p \in C$, $\bar{t} \in G$ with $\overline{s^+} = \overline{t^+}$, and $f_s \wedge \bar{t}\varepsilon = f_s$ imply $(f_s, \bar{t})(f_p, \overline{p^+}) = ((\overline{tp^+})^+ f_s \wedge \bar{t}f_p, \overline{tp^+}) \in T$, which implies $(\overline{tp^+})^+ f_s \wedge \bar{t}f_p = f_{sp^+} \in C$.

(vi): is clear. □

According to Corollary 2.8 we get:

Corollary 3.7. *Let S and ρ be as in Theorem* 3.3. *Assume further that S/ρ is a monoid. Then S is isomorphic to a semigroup constructed by Proposition* 2.6.

Corollary 3.8. [**2**]. *Each proper left type-A semigroup is isomorphic to an M-semigroup $M(G, GB, B)$.*

If S and ρ are as in Theorem 3.3 and $G := S/\rho$ is not a monoid, then S is not a monoid too and we may consider S^1 and $\bar{\rho} := \rho \cup \{(1,1)\}$. Obviously S^1 is a left type-A semigroup and $\bar{\rho}$ is a left type-A congruence on S^1 satisfying $\bar{\rho} \cap \mathcal{R}^*_{S^1} = \iota$ and $S^1/\bar{\rho} = G^1$. By the above, S^1 is isomorphic to a submonoid T of some $A \otimes G^1$, constructed by Proposition 2.6, and $T \setminus \{1\}$ is a representation for S.

4. E-Reflexive left type-A semigroups. In this section we consider left type-A semigroups, which admit a left type-A congruence ρ, satisfying $\rho \cap \mathcal{R}^* = \iota$ and $xe\rho ex$, for all $x \in S$, $e \in E_S$. In this case S/ρ is a left type-A semigroup with central idempotents. Such semigroups where investigated by Fountain [**3**] who proved that they are precisely the strong semilattices of right cancellative monoids.

The following concept seems to be appropriate:

Definition 4.1. Let s be a left type-A semigroup. S is called *E-reflexive* if

$$(u(xy)^+v)^+z = z^+u(xy)^+v \Leftrightarrow (ux^+y^+v)^+z = z^+ux^+y^+v,\ u, v, x, y, z \in S. \qquad \text{(ER)}$$

Proposition 4.2. *Let S be a left type-A semigroup. Let τ be a binary relation on S, defined by*

$$x\tau y :\Leftrightarrow [(uxv)^+z = z^+(uxv) \Leftrightarrow (uyv)^+z = z^+(uyv)], \text{ for all } u, v \in S^1,\ z \in S.$$

Then τ is a congruence on $(S, \cdot)$ satisfying $\tau \cap \mathcal{R}^* = \iota$.

Proof. That τ is a congruence on $(S, \cdot)$ is straightforward. Let $x\tau \cap \mathcal{R}^* y$. By definition of τ and $(1\cdot x\cdot 1)^+x = x^+(1\cdot x\cdot 1)$ we get $y^+x = x^+y$. Moreover $x\mathcal{R}^*y$ implies $x^+ = y^+$. Thus $x = y^+x = x^+y = y$. □

THEOREM 4.3. *Let S be an E-reflexive left type-A semigroup. Then ρ defined by $x\rho y :\Leftrightarrow x\tau y$ and $x^+\tau y^+$, is a left type-A congruence on S, which satisfies $\rho\cap\mathcal{R}^* = \iota$ and $xe\rho ex$, for all $x\in S$, $e\in E_S$.*

Proof. Obviously ρ is an equivalence relation on S. Let $x\rho y$, $a\in S$. It follows that $x\tau y$ and $x^+\tau y^+$, implying $ax\tau ay$. Further $(u(ax)^+v)^+z = z^+(u(ax)^+v)$ iff $(ua^+x^+v)^+z = z^+(ua^+x^+v)$, by (ER), iff $(ua^+y^+v)^+z = z^+(ua^+y^+v)$, since $x^+\tau y^+$, iff $(u(ay)^+v)^+z = z^+(u(ay)^+v)$, by (ER). Consequently, $(ax)^+\tau(ay)^+$ and $ax\rho ay$ follows. By duality we obtain that ρ is a congruence on $(S,\cdot)$.

We show next that $xe\rho ex$, for all $x\in S$, $e\in E_S$. Note first that by (ER) and the definition of ρ, $x\rho y$ implies $x^+\rho y^+$ and $(xw)^+\rho x^+w^+$, for all $w, x, y\in S$. We infer

$$xe = (xe)^+xe\rho x^+exe = exe = e(xe)^+x\rho ex^+ex = ex.$$

We continue, showing that ρ is left type-A. Let $xw\rho yw$. It follows that $x^+w^+\rho y^+w^+$ and by the proof of Proposition 4.2 $(xw)^+yw = (yw)^+xw$, implying $(xw)^+yw^+ = (yw)^+xw^+$. We conclude that

$$xw^+\rho xx^+w^+\rho xy^+w^+\rho(yw)^+xw^+ = (xw)^+yw^+\rho yx^+w^+\rho yy^+w^+\rho yw^+,$$

proving the assertion.

Finally from Proposition 4.2 we get $\iota\subseteq\rho\cap\mathcal{R}^*\subseteq\tau\cap\mathcal{R}^* = \iota$. □

Taking into account Theorem 3.3 we immediately obtain the following equivalences:

COROLLARY 4.4. *Let S be a left type-A semigroup. Then the following statements are equivalent:*

(i) *S is E-reflexive.*
(ii) *S has a left type-A congruence ρ, satisfying $\rho\cap\mathcal{R}^* = \iota$ and $xe\rho ex$, $x\in S$, $e\in E_S$.*
(iii) *S admits an $\mathcal{R}^*$-respecting embedding into some $A *_\lambda G$ with A a semilattice and G a strong semilattice of right cancellative monoids.*

Proof. (i)⇒(ii) by Theorem 4.3.
(ii)⇒(iii) by Theorem 3.3.
(iii)⇒(i) is straightforward. □

Further, (if we replace the expressions gg^{-1} in the proof by g^+), [**1**, Corollary 14] yields the following.

COROLLARY 4.5. *Each E-reflexive left type-A semigroup admits an $\mathcal{R}^*$-respecting embedding into a strong semilattice of proper left type-A semigroups, which are semidirect products of semilattices by right cancellative monoids.*

For inverse semigroups our concept of E-reflexivity coincides with that of [**8**, III.8.1], since ρ is an idempotent pure Clifford congruence in this case, and an inverse semigroup is E-reflexive iff it has an idempotent pure Clifford congruence [**8**, III, 8.3. Theorem].

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