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A PROBLEM OF GELFAND ON RINGS OF OPERATORS AND DYNAMICAL SYSTEMS

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0. Introduction. Let G be a separable locally compact group (separable in the sense that the topology of G has a countable base). Let S be a standard Borel space on which G acts on the right such that:

(1) $s \cdot g_1 g_2 = (s \cdot g_1) \cdot g_2$;

(2) $s \cdot e = s$;

(3) $(s, g) \rightarrow s \cdot g$ is a Borel function from $S \times G$ to S.

If μ is a Borel measure on S, let μ_g be the Borel measure on S defined by $\mu_g(E) = \mu(E \cdot g)$.

Let μ be a Borel measure on S which is quasi-invariant under the action of G; i.e., μ_g and μ are absolutely continuous $(g \in G)$. The triple (G, S, μ) is called a dynamical system [**11**; **8**].

Consider the following general problem. Let (G, S, μ) be a dynamical system. G has a natural strongly continuous unitary representation $V(g)$ on $L^2(S, \mu)$ given by

$$(V(g)f)(s) = f(s \cdot g)\left(\frac{d\mu_g}{d\mu}(s)\right)^{1/2}$$

(see [**9**]). For $\varphi \in L^\infty(S, \mu)$, let M_φ be multiplication by φ on $L^2(S, \mu)$. Let $R(g)$ be the right regular representation of G. Form the Hilbert space $\mathscr{H} = L^2(G) \otimes L^2(S, \mu)$ and look at the von Neumann algebra $\mathscr{R}(G, S, \mu)$ on $\mathscr{H}$ generated by $R(g) \otimes V(g)$ and $I \otimes M_\varphi$ $(g \in G, \varphi \in L^\infty(S, \mu))$.

(*) *Problem.* What is the type of this von Neumann algebra?

Many results on this general problem were given by Dixmier in [**3**]. The algebras $\mathscr{R}(G, S, \mu)$, for G discrete, were intensively studied by Murray and von Neumann in their classic papers [**12**; **13**; **14**]; cf. also Dixmier [**4**, pp. 127–137].

The problem (*) is of interest in the classification of dynamical systems. Two dynamical systems (G, S_1, μ_1) and (G, S_2, μ_2) are said to be isomorphic if there exist: (1) G-invariant Borel subsets $S_i' \subseteq S_i$ $(i = 1, 2)$ such that $\mu_i(S_i - S_i') = 0$; (2) a Borel isomorphism ψ: $S_1' \rightarrow S_2'$ such that $\mu_1' \cdot \psi^{-1}$ and μ_1' are absolutely continuous (μ_1' is the restriction of μ_1 to the Borel subsets of S_1'); and (3) $\psi(s \cdot g) = \psi(s) \cdot g$ $(s \in S_1', g \in G)$.

One may easily check that if (G, S_1, μ_1) and (G, S_2, μ_2) are isomorphic dynamical systems, then $\mathscr{R}(G, S_1, \mu_1)$ and $\mathscr{R}(G, S_2, \mu_2)$ are unitarily equi-

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valent. Hence, given a dynamical system (G, S, μ), the Murray-von Neumann type of $\mathscr{R}(G, S, \mu)$ is an isomorphism invariant.

In a survey on functional analysis [5], Gelfand posed the problem (*) for two cases of special interest. In the first case, $G = R^1$ and S = circle group = R^1/Z^1. In the second case, $G = \mathrm{SL}(2, R^1)$ and $S = \mathrm{SL}(2, R^1)/\mathrm{SL}(2, Z^1)$. Notice that in these two special cases $S = G/H$ for some closed subgroup H of G. Denote by (**) the sub-problem of (*) in case S is a quotient space of G. The main result of this note is to reduce the problem (**) to a more tractable problem. This reduction is accomplished in an easy manner using well-known results of Mackey of systems of imprimitivity. Using this reduction, it is then a simple matter to give a complete solution of the problem (**) in the cases of interest to Gelfand.

1. A Problem of Gelfand. In what follows, G is a separable locally compact group, H is a closed subgroup of G, and G/H is the set of right H-cosets with the quotient topology. There is a unique quasi-invariant measure class on G/H [9]. Let μ be an element of this measure class. Then $(G, G/H, \mu)$ is a dynamical system.

If $\mathscr{R}$ is a von Neumann algebra, then $\mathscr{R}'$ is the commutant of $\mathscr{R}$. If K is a locally compact group, $\mathscr{R}(K)$ denotes the von Neumann algebra generated by the right regular representation of K.

The main result of this note is the following theorem.

THEOREM 1.1. *$\mathscr{R}(G, G/H, \mu)'$ is algebraically $*$-isomorphic to $\mathscr{R}(H)$. In particular, if $\mathscr{R}(H)$ has components of type* I, II, *or* III, *then $\mathscr{R}(G, G/H, \mu)$ also has components of type* I, II, *or* III.

Let $\mathscr{R}$ and $\mathscr{S}$ be von Neumann algebras. Recall that if $\mathscr{R}$ and $\mathscr{S}$ are algebraically $*$-isomorphic, then $\mathscr{R}$ and $\mathscr{S}$ have the same type. Recall also that $\mathscr{R}$ and $\mathscr{R}'$ have the same type. Hence, the last statement of Theorem 1.1 follows from the first statement.

The first statement of Theorem 1.1 will be proved in a sequence of lemmas.

If W is a strongly continuous unitary representation of $\mathscr{H}$, denote by U^W the representation of G induced by W [9]. Let $\mathscr{H}$ be a separable Hilbert space and (S, μ) a standard Borel measure space. Let $L^2(S, \mu, \mathscr{H})$ denote μ-equivalence classes of all weakly measurable functions $f: S \rightarrow \mathscr{H}$ such that $||f(s)||^2$ is μ-summable. Recall that the map $x \otimes f(s) \rightarrow f(s)x$ extends to a unitary equivalence between $\mathscr{H} \otimes L^2(S, \mu)$ and $L^2(S, \mu, \mathscr{H})$. Of particular interest is the case $S = G/H$. In this case, $L^2(S, \mu, \mathscr{H})$ may be identified with all equivalence classes of weakly measurable functions $f: G \rightarrow \mathscr{H}$ such that f is constant on right H-cosets and $||f(g)||^2$ is summable on G/H.

LEMMA 1.2. *Let $R'(h)$ be the right regular representation of G restricted to H. Then $R(g) \otimes V(g)$ is unitarily equivalent to $U^{R'}(g)$.*

Proof. The proof of this lemma merely consists in checking the proper definitions. $R(g) \otimes V(g)$ acts on the Hilbert space $L^2(G) \otimes L^2(G/H, \mu)$, which, as noted above, is unitarily equivalent in a natural manner to $L^2(G/H, \mu, L^2(G))$. Let f be a typical element of $L^2(G/H, \mu, L^2(G))$, i.e., $f: G \to L^2(G)$ is weakly measurable, is constant on right H-cosets, and $||f(x)||^2$ is μ-summable on G/H. Let $\pi: G \to G/H$ be the canonical quotient mapping. In this representation,

$$(R(g) \otimes V(g))f(x) = R(g)f(x \cdot g)\left(\frac{d\mu_g}{d\mu}(\pi(x))\right)^{1/2}.$$

Recall that $\mathscr{H}(U^{R'}) = [g: G \to L^2(G),\ g$ weakly measurable, and $g(hx) = R'(h)(g(x))\ (h \in H, x \in G)]$.

Define a unitary mapping $T: L^2(G/H, \mu, L^2(G)) \to \mathscr{H}(U^{R'})$ by

$$(Tf)(x) = R(x)(f(x)).$$

T has its range in $\mathscr{H}(U^{R'})$ since

$$(Tf)(hx) = R(hx)(f(hx)) = R(h)R(x)(f(x)) = R(h)(Tf)(x).$$

Hence,

$$\begin{aligned}(T^{-1}U^{R'}(a)Tf)(x) = R(x^{-1})(U^{R'}(a)Tf)(x) &= R(x^{-1})(Tf)(xa)\left(\frac{d\mu_a}{d\mu}(\pi(x))\right)^{1/2}\\ &= R(x^{-1})R(xa)(f(xa))\left(\frac{d\mu_a}{d\mu}(\pi(x))\right)^{1/2}\\ = R(a)(f(xa))\left(\frac{d\mu_a}{d\mu}(\pi(x))\right)^{1/2} &= (R(a) \otimes V(a))(f(x)).\end{aligned}$$

Next, note that $R(g) \otimes V(g)$ and $I \otimes M_\varphi$ form a system of imprimitivity for G based on G/H.

LEMMA 1.3. *$\mathscr{R}(G, G/H, \mu)'$ is algebraically $*$-isomorphic to the von Neumann algebra generated by $[R'(h)|\ h \in H]$.*

Proof. This follows immediately from the imprimitivity theorem [**10** or **1**] and Lemma 1.2.

LEMMA 1.4. *The von Neumann algebra generated by $[R'(h)|\ h \in H]$ is $*$-isomorphic to $\mathscr{R}(H)$.*

Proof. We apply [**9**, Theorem 12.1] with $G_1 = (e)$, $G_2 = H$, and L the one-dimensional representation of (e). Hence, $R'(h)$ is a direct integral of representations of H, each unitarily equivalent to $R(h)$. The lemma now follows from [**4**, p. 173, *théorème* 2].

Theorem 1.1 now follows by combining Lemmas 1.2–1.4.

Now some applications of Theorem 1.1.

Let G be an arbitrary separable locally compact group and $H = (e)$. Hence $\mathscr{R}(G, G, \mu)'$ is $*$-isomorphic to $\mathscr{R}(H)$, which is the scalars. Hence, $\mathscr{R}(G, G, \mu)$ is of type I. This proves [**3**, the last remark on p. 321].

Let G be a vector group and H any discrete subgroup. Since H is abelian, $\mathscr{R}(H)$ is of type I, and hence $\mathscr{R}(G, G/H, \mu)$ is of type I. This solves the first problem posed by Gelfand.

Let $G = \mathrm{SL}(2, R^1)$ and $H = \mathrm{SL}(2, Z^1)$. Claim that $\mathscr{R}(H)$ is a type II_1 von Neumann algebra. Let H_0 be the subgroup of H consisting of those elements with finite conjugacy classes. $\mathscr{R}(H)$ will be of type II_1 if H/H_0 is infinite [**7**, p. 253, Theorem 5]. One may check that H_0, the centre of H, is a two-element group. Since H itself is infinite, H/H_0 is infinite. Hence, $\mathscr{R}(H)$ is of type II_1. Therefore, Theorem 1.1 shows that

$$\mathscr{S} = \mathscr{R}(\mathrm{SL}(2, R^1), \mathrm{SL}(2, R^1)/\mathrm{SL}(2, Z^1), \mu)$$

is a type II von Neumann algebra. Furthermore, $\mathscr{S}$ has no portion of type II_1. For $g \to R(g) \otimes V(g)$ is a faithful unitary representation of the open simple Lie group $\mathrm{SL}(2, R^1)$. But an open simple Lie group has no faithful unitary representations into a finite von Neumann algebra [**6**]. Hence, $\mathscr{S}$ is actually a type II_∞ von Neumann algebra.

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