ON REDFIELD'S RANGE-CORRESPONDENCES

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1. In an important paper (7), long overlooked, J. H. Redfield dealt with several aspects of enumerative combinatorial analysis. In a previous paper **(1)** I showed the relation between a certain repeated scalar product of a set of permutation characters of a symmetric group and Redfield's composition of his group reduction functions. Here I consider, from a group representational point of view, Redfield's idea of a range-correspondence and its application to enumeration of linear graphs. The details of the application of these ideas to more general enumerations are also given.

2. Range-correspondences. Redfield considers q sets, or *ranges*, each of m symbols. If the symbols in each range are written in any arbitrary order and the q ranges are then arranged as the rows of a rectangular array, any column of the array will determine a correspondence between a symbol in any given range and symbols one from each of the remaining ranges, and so the array is termed a *range-correspondence.*

An equivalence relation is set up between the range-correspondences by associating with each range a permutation group G_r $(r = 1, 2, \ldots, q)$, which is some subgroup of the symmetric group $\mathfrak{S}_m$ on m symbols. G_r is regarded as operating only on the symbols in the rth range, and is termed the *range-group* of this range.

Now let R_r be some fixed initial ordering of the symbols in the rth range, and let $\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}$ denote respectively the range-correspondences

$$\begin{bmatrix} R_1\,\alpha_{11} \\ R_2\,\alpha_{12} \\ \vdots \\ R_q\,\alpha_{1q} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_1\,\alpha_{21} \\ R_2\,\alpha_{22} \\ \vdots \\ R_q\,\alpha_{2q} \end{bmatrix},$$

where $\alpha_{ij} \in \mathfrak{S}_m$. Then $\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}$ are defined to be *equivalent* if and only if there is an $x_r \in G_r$ such that $\alpha_{2r} = x_r\,\alpha_{1r}$ for $r = 1, 2, \ldots, q$. It follows that if an equivalence class consists of $\mathfrak{R}^{(1)}, \mathfrak{R}^{(2)}, \ldots, \mathfrak{R}^{(k)}$, then, for any given r, $\alpha_{1r}, \alpha_{2r}, \ldots, \alpha_{kr}$ all belong to the same right coset of $\mathfrak{S}_m$ with respect to G_r. Hence the class can be characterized by an ordered q-tuple

$$(G_1\,\alpha_1, G_2\,\alpha_2, \ldots, G_q\,\alpha_q),$$

where $\alpha_i \in \mathfrak{S}_m$, of right cosets of $\mathfrak{S}_m$.

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Clearly two arrays will represent the same correspondence between the ranges if they consist of the same columns, regardless of the order in which the columns occur in the arrays. The effect of this further equivalence relation is to regard the two q-tuples

$$(G_1\,\alpha_1,\, G_2\,\alpha_2,\, \ldots,\, G_q\,\alpha_q), \qquad (G_1\,\alpha'_1,\, G_2\,\alpha'_2,\, \ldots,\, G_q\,\alpha'_q)$$

as equivalent if and only if there is an element $y \in \mathfrak{S}_m$ such that

$$(G_1\,\alpha_1,\, G_2\,\alpha_2,\, \ldots,\, G_q\,\alpha_q)y = (G_1\,\alpha'_1,\, G_2\,\alpha'_2,\, \ldots,\, G_q\,\alpha'_q).$$

Each equivalence class consists of all q-tuples that are permuted amongst themselves by elements of $\mathfrak{S}_m$.

The following result is essentially the same as the first theorem in Redfield's paper.

THEOREM 1. *The number of non-equivalent range-correspondences is the scalar product* $(\phi_1, \phi_2, \ldots, \phi_q)$, *where* ϕ_i *is the character of the permutation representation of* $\mathfrak{S}_m$ *induced by* G_i.

Proof. The transitive permutation representation P_i of $\mathfrak{S}_m$ induced by G_i has as its permuted symbols the right cosets of G_i in $\mathfrak{S}_m$. The Kronecker product of $P_1, P_2, \ldots, P_q$ has as its permuted symbols the ordered q-tuples of the form $(G_1\,\alpha_1, G_2\,\alpha_2, \ldots, G_q\,\alpha_q)$. It is well known **(1)** that the number of transitive constituents of this Kronecker product is $(\phi_1, \phi_2, \ldots, \phi_q)$. But this is clearly the number of equivalence classes of the q-tuples of cosets, and hence of the range-correspondences, and so the theorem is proved.

It should be remarked that each transitive constituent of the Kronecker product is induced by some subgroup of $\mathfrak{S}_m$, and so every class of range-correspondences is associated with a subgroup of $\mathfrak{S}_m$, this being a common subgroup of $G_1, G_2, \ldots, G_q$. The determination of the subgroups associated with a given Kronecker product depends in general on a knowledge of the "marks" of the representations of $\mathfrak{S}_m$ **(1)** and in this sense can be regarded as known. Thus the association of a given class of range-correspondences with its appropriate subgroup may be regarded as determinable, though it could be lengthy in all but the simplest cases.

3. Application to enumeration of linear graphs. One of the simplest applications of range-correspondences is that of the enumeration of the linear graphs formed by k lines joining some or all of n points in pairs. Two such graphs are defined to be equivalent if and only if there is a one-to-one mapping η of the points of one graph on the points of the other such that whenever two points α, β are joined by a line in the first graph, then $\alpha\eta$ and $\beta\eta$ are joined by a line in the second graph.

The range-correspondences used in this enumeration consist of two rows only, each of $m = \binom{n}{2}$ elements. The range-group G_1 is the representation $\mathfrak{S}_n^{(P)}$ of $\mathfrak{S}_n$ as a transitive group of degree m induced by a certain subgroup P.

$\mathfrak{S}_n^{(P)}$ is the group of permutations of the m lines induced by the permutations of the n points.

If H_1 and H_2 are any permutation groups of degrees d_1, d_2 respectively on disjoint sets of symbols, then their direct product $H_1 \dot{\times} H_2$ is defined to be the set of all products $h_i h_j$, where h_i runs through H_1 and h_j through H_2. Thus $H_1 \dot{\times} H_2$ is of degree $d_1 + d_2$.

LEMMA 1. *$G_1 = \mathfrak{S}_n^{(P)}$ where $P = \mathfrak{S}_2 \dot{\times} \mathfrak{S}_{n-2}$.*

Proof. P is the set of all permutations of n symbols which leave unaltered a given pair of symbols, say 1 and 2, or merely interchange them. We can thus associate P with the undirected line $[1, 2]$ joining the points denoted by 1 and 2, and implicitly with the remaining $n - 2$ isolated points.

If $x \in \mathfrak{S}_n$ and $\notin P$, and $[1, 2]x = [\alpha, \beta]$ (or $[\beta, \alpha]$), then for every $p \in P$ we have $[1, 2]px = [\alpha, \beta]$ (or $[\beta, \alpha]$). Further if z is any element of $\mathfrak{S}_n$ such that $[1, 2]z = [\alpha, \beta]$ (or $[\beta, \alpha]$), then $[1, 2]z = [1, 2]x$ and so $zx^{-1} \in P$ and z lies in the coset Px of P in $\mathfrak{S}_n$. Hence there is a one-to-one correspondence between the lines $[\alpha, \beta]$ and the cosets Px.

If $y \notin P$ or Px, and $[1, 2]y = [\gamma, \delta]$, then Py is mapped on $[\gamma, \delta]$. Let s be any permutation of $\mathfrak{S}_n$ which changes $[\alpha, \beta]$ into $[\gamma, \delta]$. Then for every $p \in P$, $[1, 2]pxs = [1, 2]y$ and so $pxsy^{-1} \in P$, that is $(px)s = p_1 y$, where $p_1 \in P$, and so $(Px)s = Py$.

It follows that the group of permutations of the lines induced by $\mathfrak{S}_n$ acting on the points is isomorphic to the group induced on the cosets by $\mathfrak{S}_n$, which proves the lemma.

We have thus interpreted the first range as the set of m lines joining n points in pairs, and have taken $\mathfrak{S}_n^{(P)}$ as its range-group.

We now interpret the second range as a set of m symbols

$$s_1, s_2, \ldots, s_k, r_{k+1}, r_{k+2}, \ldots, r_m.$$

Any line paired with one of $s_1, s_2, \ldots, s_k$ in a range-correspondence becomes one of the k lines selected for the graph belonging to this range-correspondence, and any line paired with one of $r_{k+1}, \ldots, r_m$ is rejected for the graph. The range-group G_2 is taken as $\mathfrak{S}_k \dot{\times} \mathfrak{S}_{m-k}$, where $\mathfrak{S}_k$ permutes $s_1, s_2, \ldots, s_k$ and $\mathfrak{S}_{m-k}$ permutes $r_{k+1}, r_{k+2}, \ldots, r_m$.

There is thus a one-to-one correspondence between the range-correspondences and the graphs of k lines on n points. The separation of the range-correspondences into classes under the equivalence relation of §2 corresponds exactly to the separation of the k-line graphs into classes under the equivalence relation defined earlier in this section.

To find ϕ_2 we multiply the cycle index of $\mathfrak{S}_k$ by that of $\mathfrak{S}_{m-k}$. This gives a polynomial

$$\frac{1}{m!}\sum_{\rho\vdash m} A_\rho z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_m^{\alpha_m},$$

where $(\rho) = (1^{\alpha_1}2^{\alpha_2} \ldots m^{\alpha_m})$ is a partition of m. If h_ρ is the number of elements in the class of $\mathfrak{S}_m$ associated with (ρ), then ϕ_2 is the set of numbers A_ρ/h_ρ, where ρ ranges over all partitions of m. Alternatively the above symmetric function is the sum $\{m\} + \{m-1, 1\} + \ldots + \{m-k, k\}$ of Schur functions, and ϕ_2 is the sum of the irreducible characters $[m]$, $[m-1, 1], \ldots, [m-k, k]$ of $\mathfrak{S}_m$.

The character ϕ_1 can be obtained as a special case of a procedure given later in §5(c). This special case has, however, been treated by Slepian (**8**, p. 144) and by Harary (**3**) by constructing a symmetric function $\mathbf{G}_n$ which is "based" on ϕ_1. This symmetric function is a sum of Schur functions, and the number L_{nk} of inequivalent linear graphs of k lines on n points can be expressed in terms of the multiplicities of the S-functions of less than three parts.

THEOREM 2. *If*

$$\mathbf{G}_n = \sum_{r=0}^{[m/2]} \beta_r\{m-r, r\} + \sum \gamma_\nu\{\nu\},$$

where $\{\nu\}$ has more than two parts, then for $k \leqslant [\frac{1}{2}n]$

$$L_{n,m-k}\, L_{nk} = (\phi_1, \phi_2) = \sum_{i=0}^{k} \beta_i.$$

Proof. If $[\lambda]$ is the irreducible character of $\mathfrak{S}_m$ associated with $\{\lambda\}$, then

$$\phi_1 = \sum_{r=0}^{[m/2]} \beta_r[m-r, r] + \sum \gamma_\nu[\nu],$$

where $[m/2]$ is the greatest integer in $m/2$. Also

$$\phi_2 = [m] + [m-1, 1] + \ldots + [m-k, k],$$

and so

$$L_{nk} = (\phi_1, \phi_2) = (\phi_1, [m]) + (\phi_1, [m-1, 1]) + \ldots + (\phi_1, [m-k, k])$$
$$= \sum_{i=0}^{k} \beta_i,$$

since (ξ, ζ), where $[\zeta]$ is absolutely irreducible, is the multiplicity of ζ in ξ.

Hence, if the expression of $\mathbf{G}_n$ as a sum of S-functions is known, L_{nk} is determined at sight from the coefficients of those S-functions with less than three parts. We find

$$\mathbf{G}_2 = \{1\}, \qquad \mathbf{G}_3 = \{3\},$$

$$\mathbf{G}_4 = \{6\} + \{42\} + \{3^2\} + \{2^3\} + \{2^21^2\} + \{1^6\},$$

$$\begin{aligned}\mathbf{G}_5 = {} & \{10\} + \{82\} + 2\{73\} + 2\{64\} + \{721\} + 2\{631\} \\ & + 3\{62^2\} + 2\{621^2\} + \{61^4\} + 2\{541\} + 5\{532\} \\ & + 3\{531^2\} + 6\{52^21\} + 3\{4^22\} + 3\{4^21^2\} + \{43^2\} \\ & + 5\{4321\} + 3\{521^3\} + 2\{3^31\} + 4\{42^3\} + 3\{3^22^2\} \\ & + 5\{431^3\} + 5\{42^21^2\} + 3\{3^221^2\} + 2\{32^31\} + \{2^5\} \\ & + 3\{51^5\} + 3\{421^4\} + 3\{3^21^4\} + 2\{32^21^3\} + \{41^6\} \\ & + \{321^5\} + \{2^31^4\},\end{aligned}$$

from which the part of the table for L_{nk} for $n \leqslant 5$, $k \leqslant 10$ can be read off (**8**, p. 146; **9**, p. 200).

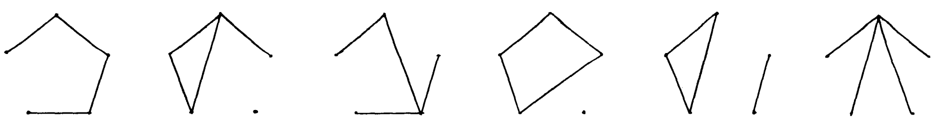
Two points of interest may be noted.

(i) The only significant part of $\mathbf{G}_n$ is the symmetric function

$$\sum_{r=0}^{[m/2]} \beta_r\{m-r, r\},$$

and so the cycle-index $\mathbf{G}_n$ used (**8**, p. 145) in applying Pólya's theorem is not the only symmetric function that will serve the purpose in general. The redundancy of the S-functions of three or more parts is, of course, evident from Theorem 2, but it can also be regarded as due to the fact that the substitution of $1 + x^r$ for every rth power-sum in $\mathbf{G}_n$, as is required in Pólya's theorem, means that we are concerned with S-functions of 1 and x only, and S-functions of more than two parts will be identically zero (**4**, p. 87).

(ii) The non-equivalent graphs on n points with k lines are equi-numerous with certain partitions of m, not in general distinct. Thus for $n = 5$, $k = 4$, it would be of interest to find some correspondence between the graphs and the



partitions (10), (82), (73), (73), (64), (64) in some order.

We now consider more general types of subgroups in place of G_1 and G_2, and show how ϕ_1 and ϕ_2 can be evaluated for these more general types. The results still possess an interpretation in enumerative graph theory.

4. Generalization of G_1. If we select any unordered pair from a set of n nodes we are in an obvious sense partitioning the nodes in accordance with the partition $(1^{n-2}2)$. The structural unit from which a linear graph is constructed is the line, regarded here as a line together with $n - 2$ isolated points, and so associated with the partition $(1^{n-2}2)$. We now consider a structural unit more complicated than the line and $n - 2$ isolated points, namely an undirected graph associated with any partition of n.

Take any undirected graph X on n nodes and let its decomposition into connected components be

$$\begin{aligned} X = X_{11} &+ X_{12} + \ldots + X_{1\alpha_1} \\ &+ X_{21} + X_{22} + \ldots + X_{2\alpha_2} \\ &+ \ldots\ldots\ldots \\ &+ X_{n1} + X_{n2} + \ldots + X_{n\alpha_n}, \end{aligned}$$

where X_{ij} is a connected graph on i nodes. This decomposition determines a partition $(\lambda) = (1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n})$ of the n nodes. If we apply a permutation η of $\mathfrak{S}_n$ to the nodes and require that if (α, β) is an edge in X, then $(\alpha\eta, \beta\eta)$ is an edge in the permuted graph, then the permuted graph will not necessarily be the same as X, though, of course, it will be equivalent to it. The number of different graphs

$$X^{(1)} = X, X^{(2)}, \ldots, X^{(m)}$$

obtained by applying $\mathfrak{S}_n$ to the nodes depends on the automorphism group of X.

Suppose that the α_i graphs $X_{i1}, X_{i2}, \ldots, X_{i\alpha_i}$ consist of α_{i1} graphs equivalent to X_{ia_1}, α_{i2} graphs equivalent to X_{ia_2}, and so on, where $X_{ia_1}, X_{ia_2}, \ldots$ are inequivalent, and

$$\alpha_i = \sum_{k=1}^{s} \alpha_{ik}.$$

If $G_{a_j}{}^{(i)}$ is the automorphism group of X_{ia_j}, then the automorphism group of $X_{i1} + X_{i2} + \ldots + X_{i\alpha_i}$ is the direct product

$$\mathfrak{G}_i = (G_{a_1}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{i1}}) \dot{\times} (G_{a_2}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{i2}}) \dot{\times} \ldots \dot{\times} (G_{a_s}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{is}}),$$

where $\backsim$ denotes a wreath product **(2**, p. 81**)**.

The automorphism group of X is $\mathfrak{G}_\lambda = \mathfrak{G}_1 \dot{\times} \mathfrak{G}_2 \dot{\times} \ldots \dot{\times} \mathfrak{G}_n$. This is a subgroup of $\mathfrak{S}_n$ and so induces a transitive permutation representation $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$ of $\mathfrak{S}_n$ of degree $n!/n_\lambda$, where n_λ is the order of $\mathfrak{G}_\lambda$. The symbols on which $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$ operates can be taken as the cosets $\mathfrak{G}_\lambda\, x$ of $\mathfrak{S}_n$ with respect to $\mathfrak{G}_\lambda$.

LEMMA 2. (i) *There is a one-to-one correspondence between the graphs* $X^{(1)}, X^{(2)}, \ldots, X^{(m)}$ *and the cosets* $\mathfrak{G}_\lambda\, x$, *and* (ii) *the permutation group induced on the graphs by applying* $\mathfrak{S}_n$ *to the nodes is isomorphic with* $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$.

Proof. (i) If u_i is an element of $\mathfrak{S}_n$ such that $X^{(1)}u_i = X^{(i)}$, then for every $g \in \mathfrak{G}_\lambda$ we have $X^{(1)}gu_i = X^{(i)}$ and so every element of a given coset changes $X^{(1)}$ into the same graph.

Further, if z is any element of $\mathfrak{S}_n$ such that $X^{(1)}z = X^{(i)}$, then $X^{(1)}z = X^{(1)}u_i$ and so $zu_i^{-1} \in \mathfrak{G}_\lambda$ and z belongs to the coset $\mathfrak{G}_\lambda\, u_i$. Hence the correspondence $\mathfrak{G}_\lambda\, u_i \to X^{(i)}$ is one-to-one, where $i = 1, 2, \ldots, m$.

(ii) If s is any permutation of $\mathfrak{S}_n$ which changes $X^{(i)}$ into $X^{(j)}$, then for every $g \in \mathfrak{G}_\lambda$ we have

$$X^{(1)}gu_i\, s = X^{(i)}s = X^{(j)} = X^{(1)}u_j$$

and so $gu_i\, su_j^{-1} \in \mathfrak{G}_\lambda$ and $\mathfrak{G}_\lambda\, u_i\, s = \mathfrak{G}_\lambda\, u_j$. Conversely if s is such that $\mathfrak{G}_\lambda\, u_i\, s = \mathfrak{G}_\lambda\, u_j$, then $X^{(i)}s = X^{(j)}$. It follows that the group induced on the $X^{(i)}$ by $\mathfrak{S}_n$ is isomorphic with the group induced on the cosets by $\mathfrak{S}_n$, and so the lemma is proved.

5. Permutation character of $\mathfrak{S}_m$ induced by $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$. $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$ is a subgroup of $\mathfrak{S}_m$ and so induces a transitive representation of $\mathfrak{S}_m$. We determine the character ϕ_λ of $\mathfrak{S}_m$ corresponding to this representation. The procedure may be considered in three stages.

(a) *Cycle index of* $G_{a_r}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{ir}}$. Let the cycle index of a permutation group G of order $|G|$ and degree k be

$$F = \frac{1}{|G|}\sum_{\sigma} g_\sigma\, t_1^{x_1} t_2^{x_2} \ldots t_k^{x_k},$$

the summation being over all partitions $(\sigma) = 1^{x_1}\,2^{x_2} \ldots k^{x_k}$ of k. The cycle index of $\mathfrak{S}_p$ is

$$\{p\} = \frac{1}{p!}\sum_{\rho} h_\rho s_1^{y_1} s_2^{y_2} \ldots s_p^{y_p},$$

summed over all partitions $(\rho) = 1^{y_1}\,2^{y_2} \ldots p^{y_p}$ of p. It is known **(5)** that the cycle index of $G \backsim \mathfrak{S}_p$ is

$$F \otimes \{p\} = \frac{1}{p!}\sum_{\rho} h_\rho (F^{(1)})^{y_1} (F^{(2)})^{y_2} \ldots (F^{(p)})^{y_p}$$

where $F^{(r)}$ is the symmetric function obtained from F by replacing every t_i by t_{ir}, and $\otimes$ denotes the operation of "plethysm."

(b) *Permutation character of* $\mathfrak{S}_n$ *induced by* $\mathfrak{G}_\lambda$. The cycle index P_λ of $\mathfrak{G}_\lambda$ is the product of the cycle indices of $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n$, and since the cycle index of $\mathfrak{G}_i$ is the product of the cycle indices of

$$G_{a_1}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{i1}}, G_{a_2}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{i2}}, \ldots, G_{a_s}{}^{(i)} \backsim \mathfrak{S}_{\alpha_{is}},$$

then P_λ is known from (a) above.

If

$$P_\lambda = \frac{1}{n!}\sum_{\nu} B_\nu s_1^{\nu_1} s_2^{\nu_2} \ldots s_n^{\nu_n},$$

where $(\nu) = 1^{\nu_1}\,2^{\nu_2} \ldots n^{\nu_n}$ is a partition of n, then the character of $\mathfrak{S}_n$ induced by the unit representation of $\mathfrak{G}_\lambda$ is found by evaluating B_ν/n_ν for each (ν), where n_ν is the number of elements of $\mathfrak{S}_n$ in the class C_ν. Alternatively, if the required plethysms are known, the character could be found by multiplication of S-functions.

(c) *Permutation character of* $\mathfrak{S}_m$ *induced by* $\mathfrak{S}_n{}^{(\mathfrak{G}_\lambda)}$. We consider the transitive permutation representation R, of character η, of $\mathfrak{S}_n$ induced by any subgroup $\mathfrak{G}$, of order g. Denote the conjugate classes of $\mathfrak{S}_n$ by C_ν as in (b). It is assumed that η is known; when $\mathfrak{G}$ is $\mathfrak{G}_\lambda$, η is found as in (b), but the procedure described here is valid for any $\mathfrak{G}$. The characteristic η_ν of C_ν is, of course, known if the number g_ν of elements of $\mathfrak{G}$ lying in C_ν is known, for

$$\eta_\nu = \frac{g_\nu}{g}\cdot\frac{n!}{h_\nu} = \frac{g_\nu}{g}\cdot 1^{\nu_1}\cdot 2^{\nu_2} \ldots n^{\nu_n}\cdot \nu_1!\,\nu_2! \ldots \nu_n!.$$

If $\mathfrak{G}$ is of index ω in $\mathfrak{S}_n$, then η_ν is the sum of the units in the leading diagonal of any of the permutation $\omega \times \omega$ matrices representing those elements of $\mathfrak{S}_n$ lying in C_ν. We wish to associate every (ν) with a partition

$$(\Gamma) = (1^{\gamma_1} 2^{\gamma_2} \ldots \omega^{\gamma_\omega})$$

of ω, and so obtain the cycles which occur in the representation R of degree ω.

Clearly $\gamma_1 = \eta_\nu$ for every ν. To determine γ_r for $r > 1$ we use the rth power of a partition (**7**, p. 450). If R_x is a permutation whose cycles are given by (ν), then the cycles of $R_x{}^r$ give a partition of n called the rth power of (ν) and which we denote by $\nu^{(r)}$.

If $x \in C_\nu$, then $x^p \in C_{\nu(p)}$. If p is prime, $\eta_{\nu(p)}$ will be the same as η_ν unless R_x has cycles of length p. If R_x has γ_p p-cycles, then $\eta_{\nu(p)} = \eta_\nu + p\gamma_p$. For a composite integer d, with divisors $d_0 = 1, d_1, d_2, \ldots, d_t, d$, we have in the same way

$$\eta_{\nu(d)} = \eta_\nu + \gamma_{d_1}\cdot d_1 + \gamma_{d_2}\cdot d_2 + \ldots + \gamma_{d_t}\cdot d_t + \gamma_d\cdot d,$$

which leads to the following result:

THEOREM 3. *If, in a permutation representation R, of degree ω and character η, of $\mathfrak{S}_n$ the class C_ν of $\mathfrak{S}_n$ is mapped on the class C'_Γ of $\mathfrak{S}_\omega$, then*

$$\gamma_d = \frac{1}{d}\left(\eta_{\nu(d)} - \sum_{i=0}^{t} \gamma_{d_i}\cdot d_i\right),$$

for $d = 1, 2, \ldots, \omega$, where $d_i < d$ are the divisors of d.

This theorem gives a convenient and rapid procedure for the construction of the classes of R whenever η is known. In essence the theorem is equivalent to a theorem of Redfield (**7**, p. 451). Formulae for special cases have been given by Slepian (**8**, p. 145) and Harary (**3**, pp. 451, 452).

As a numerical illustration we take $n = 7$ and $\mathfrak{G} = \mathfrak{S}_5 \dot{\times} \mathfrak{S}_2$. η is then the sum of the irreducible characters [7], [61], [52] of $\mathfrak{S}_7$, giving

$$\eta = 21, 11, 6, 3, 5, 2, 1, 0, 1, 3, 0, 1, 2, 0, 0$$

where the classes of $\mathfrak{S}_7$ are arranged in the order given by D. E. Littlewood in his table of irreducible characters of $\mathfrak{S}_7$ (**4**). Alternatively η can be found from the product of the cycle indices of $\mathfrak{S}_5$ and $\mathfrak{S}_2$. The mapping of each ν on the corresponding Γ is given in Table I.

Taking the columns in order, we find that the partitions of 21 arising are 1^{21}, $1^{11}2^5$, 1^63^5, 1^324^4, 1^52^8, $1^22^23^36$, 15^4, 36^3, 12^24^4, 1^32^9, 3^7, $1.5^2.10$, $1^22^236^2$, 2.3.4.12, and 7^3. We observe (i) η_d need be computed only for those values of d which are least common multiples of cycle lengths of some ν, (ii) for a given d we can ignore columns for which the least common multiple of the cycles of ν is not divisible by d, (iii) entries in any column terminate when $\nu^{(d)} = (1^7)$, (iv) if the least common multiple of the cycles in any column is composite, then other columns can be derived from it. Thus since $(34) \to (2.3.4.12)$, we have $(2^23) \to (1^22^236^2)$, $(1^34) \to (1^324^4)$, $(1^43) \to (1^63^5)$, $(1^32^2) \to (1^52^8)$.

TABLE I

ν	1^7	1^52	1^43	1^34	1^32^2	1^223	1^25	16	124	12^3	13^2	25	2^23	34	7
l.c.m. of cycles	1	2	3	4	2	6	5	6	4	2	3	10	6	12	7
η_1	21	11	6	3	5	2	1	0	1	3	0	1	2	0	0
$\nu^{(2)}$		1^7		1^32^2	1^7	1^43		13^2	1^32^2	1^7		1^25	1^43	2^23	
η_2		5		1	8	2		0	2	9		0	2	1	
$\nu^{(3)}$			1^7			1^52		12^3			1^7		1^32^2	1^34	
η_3			5			3		1			7		1	1	
$\nu^{(4)}$				1^7					1^7					1^43	
η_4				4					4					1	
$\nu^{(5)}$							1^7					1^52			
η_5							4					2			
$\nu^{(6)}$						1^7		1^7					1^7	1^32^2	
η_6						1		3					2	0	
$\nu^{(7)}$															1^7
η_7															3
$\nu^{(10)}$												1^7			
η_{10}												1			
$\nu^{(12)}$														1^7	
η_{12}														1	

Having related every (ν) to its appropriate (Γ), the cycle index of R can be written as

$$\frac{1}{n!}\sum_{\nu \nmid n} h_\nu\, s_1^{\gamma_1} s_2^{\gamma_2} \ldots s_\omega^{\gamma_\omega},$$

where h_ν is the order of the class C_ν of $\mathfrak{S}_n$. In general two or more partitions (ν) may be mapped on the same (Γ). Collecting like terms, the cycle index becomes

$$\frac{1}{\omega!}\sum_{\Gamma \nmid \omega} \frac{\omega!}{n!} H_\Gamma\, s_1^{\gamma_1} s_2^{\gamma_2} \ldots s_\omega^{\gamma_\omega},$$

where $H_\Gamma = 0$ if (Γ) is not the image of any (ν), and $H_\Gamma = h_\nu + h_{\nu'} + \ldots$, where (ν), (ν'), ... are the partitions mapped on (Γ). The character of the permutation representation of $\mathfrak{S}_\omega$ induced by R is now $\omega!\,H_\Gamma/n!\,K_\Gamma$, where K_Γ is the order of the class C'_Γ of $\mathfrak{S}_\omega$, and (Γ) ranges over all partitions of ω.

When $\mathfrak{G} = \mathfrak{G}_\lambda$ and $\omega = m = n!/n_\lambda$ we have the character ϕ_λ required at the start of §5.

6. Generalization of G_2. We now consider a set T of m elements, which are completely arbitrary, but which for convenience of description we refer to as "colours." Repetitions of the same colour are allowable and a "colour" is allowed which when paired with an element of another set D of elements obliterates, or rejects, the element of D. Let $c_{11}, c_{12}, \ldots, c_{1\beta_1}$ be the colours that appear once only, $c_{21}, c_{22}, \ldots, c_{2\beta_2}$ be the colours that appear twice only, $\ldots, c_{m1}, c_{m2}, \ldots, c_{m\beta_m}$ be the colours that appear m times. Then $(1^{\beta_1} 2^{\beta_2} \ldots m^{\beta_m})$ is a partition (μ) of m. For fixed values of s and t, denote the s elements of T having the colour c_{st} by $c_{st}^{(1)}, c_{st}^{(2)}, \ldots, c_{st}^{(s)}$. The set T thus consists of the m elements $c_{ij}^{(r)}$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \beta_i$ and $r = 1, 2, \ldots, i$.

An automorphism group A_μ of T can be set up in various ways, depending on the way in which we require the colours to be invariant under the permutations of A_μ.

Consider the group A_s, of degree $s\beta_s$, which (i) permutes the elements of each row of

$$\begin{array}{c} c_{s1}^{(1)}, c_{s1}^{(2)}, \ldots, c_{s1}^{(s)}, \\ c_{s2}^{(1)}, c_{s2}^{(2)}, \ldots, c_{s2}^{(s)}, \\ \ldots \\ c_{s\beta_s}^{(1)}, c_{s\beta_s}^{(2)}, \ldots, c_{s\beta_s}^{(s)} \end{array}$$

in all possible ways, and (ii) induces a permutation group A_{s1} on β_{s1} of the rows, A_{s2} on β_{s2} of the rows, and so on, where $\beta_{s1} + \beta_{s2} + \ldots + \beta_{sf} = \beta_s$. Then

$$A_s = (\mathfrak{S}_s \sim A_{s1}) \dot{\times} (\mathfrak{S}_s \sim A_{s2}) \dot{\times} \ldots \dot{\times} (\mathfrak{S}_s \sim A_{sf}),$$

and we may define the automorphism group of T to be

$$A_\mu = A_1 \dot{\times} A_2 \dot{\times} \ldots \dot{\times} A_m.$$

In the notation of §5(a), the cycle index of $\mathfrak{S}_p \sim G$ is

$$\{p\} \otimes F = \frac{1}{|G|} \sum_\sigma g_\sigma(\{p\}^{(1)})^{x_1}(\{p\}^{(2)})^{x_2} \ldots (\{p\}^{(k)})^{x_k},$$

and so the cycle index of A_μ is known when those of $A_{s1}, A_{s2}, \ldots, A_{sf}$ are known. If the cycle index of A_μ is

$$\frac{1}{m!} \sum_{w \vdash m} F_w\, s_1^{w_1} s_2^{w_2} \ldots s_m^{w_m},$$

then the permutation character ϕ_μ of $\mathfrak{S}_m$ induced by A_μ is found by evaluating F_w/h_w for each w. Alternatively, if the necessary plethysms are known, then ϕ_μ could be found by the multiplication of S-functions by the Littlewood–Richardson rule.

7. Evaluation of (ϕ_λ, ϕ_μ). A simplification, arithmetical in nature, may be noted. Instead of evaluating (ϕ_λ, ϕ_μ) as a scalar product of characters of $\mathfrak{S}_m$, we reach the same result by evaluating the scalar product $(1_S, \phi_{\mu(S)})$ of characters of $\mathfrak{S}_n$, where 1_S is the unit representation of $S = \mathfrak{S}_n^{(\mathfrak{G}\lambda)}$ and $\phi_{\mu(S)}$

denotes ϕ_μ restricted to S. This follows from Frobenius' Reciprocity Theorem (**1**, p. 277). Hence the computation of ϕ_λ is superfluous. All that is required is to relate each (ν) of $\mathfrak{S}_n$ with its appropriate (Γ) in $\mathfrak{S}_m$, as in §5, and then compute (ϕ_λ, ϕ_μ) as

$$\frac{1}{n!}\sum_{\nu \nmid n} h_\nu\, \phi_{\mu(\Gamma)},$$

where $\phi_{\mu(\Gamma)}$ is the character of the class C'_Γ of $\mathfrak{S}_m$ obtained by restricting ϕ_μ to S.

8. Interpretation of (ϕ_λ, ϕ_μ)**.** Many complicated graphical structures can be enumerated by appropriate choice of (λ) and (μ), and still further complications can be dealt with by taking more than two rows in the range-correspondences and evaluating the corresponding multiple scalar product. Even for small values of n it is not always a simple matter to enumerate these structures by trial.

As an illustration take $n = 6$ and $(\lambda) = (3^2)$. Then $\mathfrak{G}_\lambda = \mathfrak{S}_3 \sim \mathfrak{S}_2$ and is of degree 6 and order 72. $\mathfrak{S}_6^{(\mathfrak{G}_\lambda)}$ is a transitive representation of $\mathfrak{S}_6$ of degree 10. The character η of this representation is the sum of the irreducible characters [6] and [42] since $\{3\} \otimes \{2\} = \{6\} + \{42\}$, and so

$$\eta = (10, 4, 1, 0, 2, 1, 0, 1, 2, 4, 1)$$

where the arrangement of the classes is

$$1^6,\ 1^42,\ 1^33,\ 1^24,\ 1^22^2,\ 123,\ 15,\ 6,\ 24,\ 2^3,\ 3^2.$$

The procedure of §5 maps these partitions, in the above order, on

$$1^{10},\ 1^42^3,\ 13^3,\ 24^2,\ 1^22^4,\ 136,\ 5^2,\ 136,\ 1^24^2,\ 1^42^3,\ 13^3.$$

The structural units that we now use to construct our generalized graphs are the 10 different pairs of disjoint triangles which can be drawn using six points in general position.

Various colourings and automorphism groups for the colourings may be chosen.

(a) Let there be three colours, green (once), red (twice), and an obliterating colour (seven times). The automorphism group of the colours is $\mathfrak{S}_1 \dot{\times} \mathfrak{S}_2 \dot{\times} \mathfrak{S}_7$ and since

$$\{1\}\{2\}\{7\} = \{10\} + 2\{91\} + 2\{82\} + \{81^2\} + \{73\} + \{721\}$$

the character ϕ_μ, as far as required, is 360, 24, 0, 0, 8, 0, 0, 0, 0, 24, 0, where the arrangement of the classes is as given above. When restricted to $\mathfrak{S}_6$, the only non-zero terms are 360, 24, 8, 24 for the classes 1^6, 1^42, 1^22^2, 2^3 respectively, and so the number of inequivalent generalized graphs constructed on six points from one pair of disjoint green triangles and two pairs of disjoint red triangles is

$$\frac{1}{720}(360.1 + 24.15 + 45.8 + 24.15) = 2.$$

(b) Let there be three colours each taken twice, and an obliterating colour taken four times. If the three colours are interchangeable, the automorphism group of the colours is $(\mathfrak{S}_2 \sim \mathfrak{S}_3) \dot{\times} \mathfrak{S}_4$ of cycle index

$$\begin{aligned}(\{2\} \otimes \{3\})\{4\} &= (\{6\} + \{42\} + \{2^3\})\{4\} \\ &= \{10\} + \{91\} + 2\{82\} + 2\{73\} + 2\{64\} \\ &+ \{721\} + \{631\} + 2\{62^2\} + \{541\} + \{532\} + \{52^21\} \\ &\qquad + \{4^22\} + \{42^3\},\end{aligned}$$

and so ϕ_μ, as far as required, is 3150, 70, 9, 2, 46, 1, 0, 19, 10, 70, 9, from which

$$(\phi_\lambda, \phi_\mu) = \frac{1}{720}(3150 + 70.15 + 9.40 + 2.90 + 46.45 + 19.120 + 1.120 + 10.90 + 70.15 + 9.40) = 16.$$

Finally it should be noted that the set of generalized graphs obtained here for a given $\mathfrak{G}_\lambda$ and $\mu = (k, m - k)$ forms a subset of a set of superposed graphs formed by superposing k graphs of automorphism group $\mathfrak{G}_\lambda$ as defined by Read **(6)**.

Thus if $n = 5$ and the structural unit is a line and three isolated points, the automorphism group is $\mathfrak{G}_{21^3} = \mathfrak{S}_2 \dot{\times} \mathfrak{S}_3$, giving rise to the character

$$[5] + [41] + [32] = 10, 4, 1, 0, 2, 1, 0$$

of $\mathfrak{S}_5$, where the classes are 1^5, 1^32, 1^23, 14, 12^2, 23, 5 respectively. From this, Read's scalar product for enumerating the superposed graphs obtained by superposing three graphs each with automorphism group $\mathfrak{G}_{21^3}$ is

$$\frac{1}{120}(1000 + 640 + 20 + 0 + 120 + 20 + 0) = 15,$$

whereas the number of graphs without multiple edges formed by three lines on five points is four.

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