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SURJECTIVE LINEAR MAPS BETWEEN ROOT SYSTEMS WITH ZERO

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ABSTRACT. If R_1 and R_2 are root systems and there is a linear map which maps $R_1 \cup \{0\}$ onto $R_2 \cup \{0\}$ we write $R_1 \longrightarrow R_2$. We determine all pairs (R_1, R_2) of irreducible root systems such that $R_1 \longrightarrow R_2$.

1. **Introduction.** Let R_i $(i = 1, 2)$ be a root system in the sense of Bourbaki [Bo, Chapter 6], which is not necessarily reduced, and V_i the vector space spanned by R_i. (Without any loss of generality, we may assume that the field of characteristic 0 used in the definition of root systems in [Bo] is the field $\mathbf{Q}$ of rational numbers.) We say that R_1 *dominates* R_2 if there exists a linear map $u: V_1 \longrightarrow V_2$ such that $u(R_1 \cup \{0\}) = R_2 \cup \{0\}$, and then we write $R_1 \xrightarrow{u} R_2$ or just $R_1 \longrightarrow R_2$. If R_1 does not dominate R_2 we write $R_1 \not\rightarrow R_2$.

This relation between root systems occurs naturally in the study of semisimple subalgebras of complex semisimple Lie algebras. In fact, let $\mathfrak{g}_2$ be a semisimple subalgebra of a semisimple complex Lie algebra $\mathfrak{g}_1$ and choose Cartan subalgebras $\mathfrak{h}_i \subset \mathfrak{g}_i$ such that $\mathfrak{h}_2 \subset \mathfrak{h}_1$. Assume that the weights of $\mathfrak{g}_1$ (considered as a $\mathfrak{g}_2$-module via the adjoint representation of $\mathfrak{g}_1$) are 0 and the roots of $\mathfrak{g}_2$. Then the restriction map $\mathfrak{h}_1^* \longrightarrow \mathfrak{h}_2^*$ maps $R_1 \cup \{0\}$ onto $R_2 \cup \{0\}$, *i.e.* we have $R_1 \longrightarrow R_2$ where R_i is the root system of $\mathfrak{g}_i$ with respect to $\mathfrak{h}_i$. Thus the classification of such pairs $(\mathfrak{g}_1, \mathfrak{g}_2)$ leads to the study of the dominance relation between root systems.

The above relation between root systems is the Lie algebra analog of the following well-known relation between the root system and relative root system of reductive groups. (For all standard notions and notation used below we refer to [B], [Ti].) Namely, let G be a connected reductive group defined over a field k, S a maximal k-split torus contained in a maximal k-torus T of G. Let $\Phi = \Phi(T, G)$ (resp. ${}_k\Phi = \Phi(S, G)$) be the root system of G relative to T (resp. S). Let $\rho: X(T) \longrightarrow X(S)$ be the restriction map of the character groups. Then $\rho(\Phi \cup \{0\}) = {}_k\Phi \cup \{0\}$ and so $\Phi \longrightarrow_k \Phi$.

It is natural to ask

(a) whether or not we obtain all possible relations $R_1 \longrightarrow R_2$ in this way, and if not,

(b) how to find all of them.

It turns out that not all relations $R_1 \longrightarrow R_2$ arise in this way. Our main result (see the Main Theorem) is the determination of all pairs of irreducible root systems (R_1, R_2) such that $R_1 \longrightarrow R_2$.

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The notation concerning root systems, such as their simple roots and Dynkin diagrams are the same as in [Bo, pp. 250–275]. We recall that there are up to isomorphism only five infinite series of irreducible root systems, namely A_n, $n \geq 1$; B_n, $n \geq 2$; C_n, $n \geq 2$; D_n, $n \geq 4$; and BC_n, $n \geq 1$ (not reduced); and five exceptional root systems E_6, E_7, E_8, F_4 and G_2. All these root systems are pairwise non-isomorphic, except for B_2 and C_2. We denote by Σ the set of isomorphism classes of root systems and by Σ^{irr} its subset corresponding to irreducible root systems.

2. **A partial order on Σ.** In this section we relate the dominance relation to orderings of root systems and show that $R_1 \longrightarrow R_2$ and $R_2 \longrightarrow R_1$ imply that R_1 and R_2 are isomorphic. Consequently we obtain a partial order on Σ.

If R is a root system then $\mathbf{Z}R$ will denote the root lattice. We denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a base of R and by R^+ the set of positive roots of R with respect to Π. By $\mathbf{Z}_+\Pi$ we denote the set of all linear combinations $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ with nonnegative integral coefficients a_i. This element will be denoted also by the symbol $a_1 \cdots a_n$. The sum of all coefficients $a_1, \ldots, a_n$ is the *height*, $h(\alpha)$, of α. For $\alpha, \beta \in \mathbf{Z}R$ we write $\alpha \geq \beta$ if $\alpha - \beta \in \mathbf{Z}_+\Pi$.

PROPOSITION 1. *Let (R_1, V_1) and (R_2, V_2) be root systems and $u\colon V_1 \longrightarrow V_2$ a dominant map. If Π_2 is any base of R_2, there exists a base Π_1 of R_1 such that $u(R_1^+ \cup \{0\}) = R_2^+ \cup \{0\}$. In that case $u(\Pi_1) \supset \Pi_2$.*

PROOF. Let $f\colon V_2 \longrightarrow \mathbf{Q}$ be a linear function such that $f(\beta) > 0$ for all $\beta \in \Pi_2$. Then $g := fu$ is a non-zero linear function on V_1. Hence we can choose a base Π_1 of R_1 such that $g(\alpha) \geq 0$ for all $\alpha \in \Pi_1$. Thus if $\alpha \in \Pi_1$ and $\beta = u(\alpha) \in R_2 \cup \{0\}$ then $f(\beta) = g(\alpha) \geq 0$. Consequently $\beta \in R_2^+ \cup \{0\}$ since $f(\gamma) \neq 0$ for all $\gamma \in R_2$. This proves the first assertion.

Take any $\beta \in \Pi_2$. There is a root $\alpha \in R_1^+$ such that $u(\alpha) = \beta$. Let $\Pi_1 = \{\alpha_1, \ldots, \alpha_n\}$ and $\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n$ with k_i nonnegative integers. Since $u(\Pi_1) \subseteq R_2^+ \cup \{0\}$ and $k_1 h\big(u(\alpha_1)\big) + \cdots + k_n h\big(u(\alpha_n)\big) = h(\beta) = 1$ it follows that $k_i h\big(u(\alpha_i)\big) = 1$ for some i and $k_j h\big(u(\alpha_j)\big) = 0$ for $j \neq i$. Hence $\beta = u(\alpha_i)$ and the second assertion is proved. ∎

PROPOSITION 2. *Let $R_1 \xrightarrow{u} R_2$ with R_1 irreducible. Then R_2 is irreducible. If bases $\Pi_i \subset R_i$ are chosen as in Proposition 1 then $u(\tilde{\alpha}) = \tilde{\beta}$, where $\tilde{\alpha}$ (resp. $\tilde{\beta}$) is the highest root of R_1 (resp. R_2).*

PROOF. Let $\beta \in R_2$ be arbitrary and choose $\alpha \in R_1$ such that $u(\alpha) = \beta$. Then $\tilde{\alpha} - \alpha \in \mathbf{Z}_+\Pi_1$. Since $u(R_1^+ \cup \{0\}) = R_2^+ \cup \{0\}$, it follows that

$$u(\tilde{\alpha}) - \beta = u(\tilde{\alpha} - \alpha) \in \mathbf{Z}_+\Pi_2.$$

Therefore R_2 is irreducible and $u(\tilde{\alpha}) = \tilde{\beta}$. ∎

PROPOSITION 3. *If* $R_1 \xrightarrow{u} R_2$ *and* $R_2 \longrightarrow R_1$, *then* R_1 *and* R_2 *are isomorphic.*

PROOF (DUE TO R. STEINBERG). Clearly u must be an isomorphism of vector spaces. Consequently R_1 and R_2 have the same rank and cardinality. Without any loss of generality we may assume that R_1 and R_2 are irreducible. By Proposition 1 we may assume that bases $\Pi_i \subset R_i$ are chosen so that $u(R_1^+ \cup \{0\}) = R_2^+ \cup \{0\}$, and so $u(\Pi_1) = \Pi_2$. Let $\Pi_1 = \{\alpha_1, \ldots, \alpha_n\}$.

Denote by σ_i the reflection with respect to the root α_i. Since R_1 is invariant under σ_i, and $\sigma_i(\alpha_j) = \alpha_j - 2\alpha_i(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i)$, it follows that, for $i \neq j$, $-2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i)$ is the largest integer m such that $\alpha_j + m\alpha_i$ is a root. If $\beta_k = u(\alpha_k)$ then

$$(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i) = (\beta_j, \beta_i)/(\beta_i, \beta_i).$$

If $(\alpha_j, \alpha_i) \neq 0$, then

$$(\beta_i, \beta_i)/(\alpha_i, \alpha_i) = (\beta_j, \beta_i)/(\alpha_j, \alpha_i) = A,$$

where A is independent of j. Since Π_1 is irreducible, A is also independent of i. In other words, up to a change of scale, u is an isometry and hence an isomorphism in the sense of root systems. ∎

The dominance relation is obviously reflexive and transitive. In view of Proposition 3, the relation that it induces on Σ is also anti-symmetric, and so we obtain a partial order on Σ, which we continue to call dominance.

In the proofs below we often refer to highest roots. For convenience of the reader they are listed in Table 1.

Root system	Highest root
A_n	$111 \cdots 111$
B_n	$122 \cdots 222$
BC_n	$222 \cdots 222$
C_n	$222 \cdots 221$
D_n	$122 \cdots 211$
E_6	122321
E_7	2234321
E_8	23465432
F_4	2342
G_2	32

TABLE 1

The Hasse diagram gives a pictorial representation of a partially ordered set, see [BS, p. 5] for a precise definition. Our main result is a detailed description of the partially ordered set Σ^{irr} introduced in the previous sections.

MAIN THEOREM. *The Hasse diagram of the partially ordered set* Σ^{irr} *is given on Figure 1, except that the arrows* $D_n \longrightarrow A_2$ *have been omitted for the sake of simplicity.*

The proof will be given in the remaining two sections.

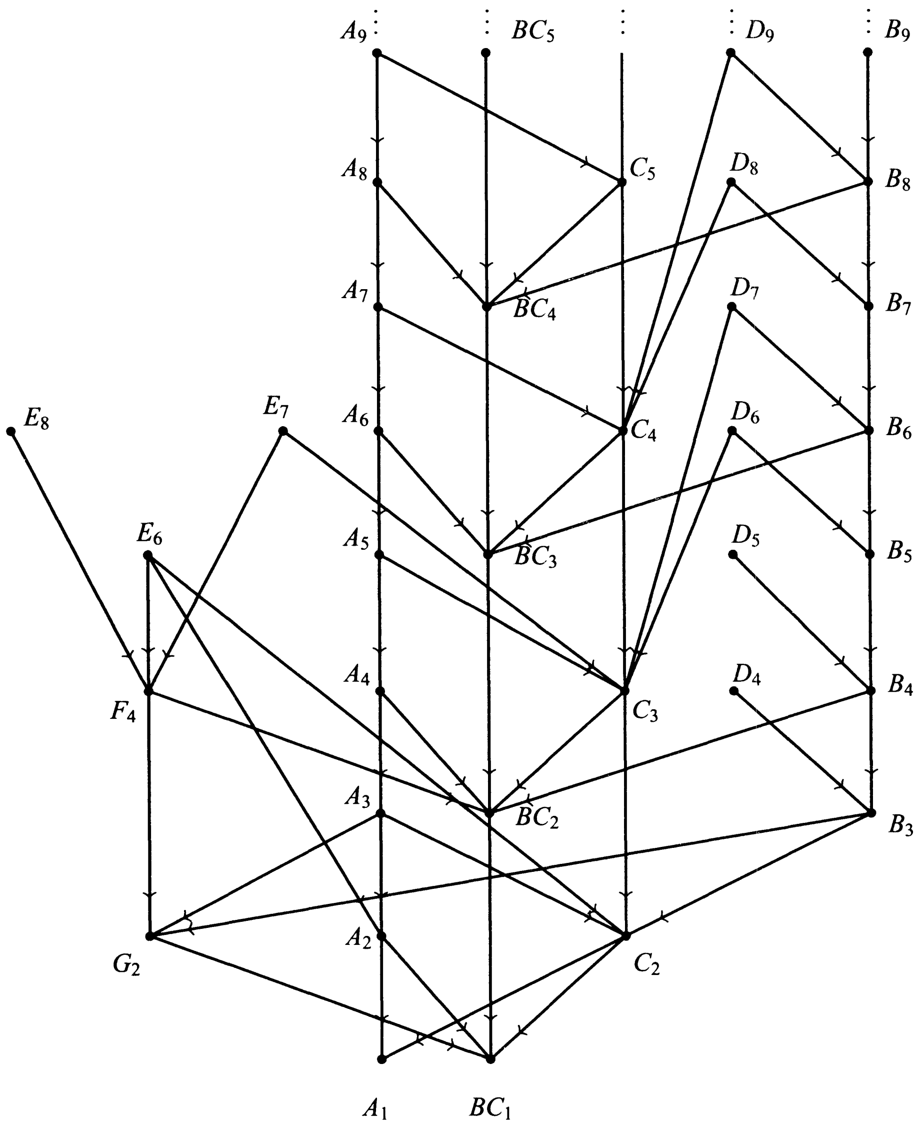

FIGURE 1. DOMINANCE RELATION IN Σ^{irr} ($D_n \longrightarrow A_2$ OMITTED)

3. **The relations** $R_1 \longrightarrow R_2$. In the sequel we shall use the following notation. Assume that $R_1 \longrightarrow R_2$ with R_1 and R_2 irreducible. We shall denote by Π_i a base of R_i, which are chosen so that $u(R_1^+ \cup \{0\}) = R_2^+ \cup \{0\}$. By α_i (resp. β_i) we denote the elements of Π_1 (resp. Π_2) and by $\tilde{\alpha}$ (resp. $\tilde{\beta}$) the highest root of R_1 (resp. R_2). Consequently we have $u(\tilde{\alpha}) = \tilde{\beta}$.

The tables of all relations $R_1 \longrightarrow R_2$ in Σ^{irr}, which can be obtained by using the method described in the Introduction, can be found in many references, *e.g.* [OV, Table 9,

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pp. 314–317], [Se, pp. 129–135], [St, Theorem 32], [W, pp. 30–32]. From these tables we obtain the following lemma.

LEMMA 4. *The following relations hold:*
a) $A_{2n} \longrightarrow BC_n$, $n \geq 1$; $A_{2n-1} \longrightarrow C_n$, $n \geq 2$;
b) $B_n \longrightarrow B_{n-1}$, $n \geq 2$;
c) $D_n \longrightarrow B_{n-1}$, $n \geq 4$; $D_{2n} \longrightarrow C_n$, $n \geq 3$;
d) $E_6 \longrightarrow A_2$, $E_6 \longrightarrow F_4$;
e) $E_7 \longrightarrow C_3$, $E_7 \longrightarrow F_4$;
f) $E_8 \longrightarrow F_4$. ■

This lemma justifies some of the arrows in Figure 1. The remaining arrows in that figure are justified by the next lemma, where $R \longrightarrow (S, \ldots, T)$ means that $R \longrightarrow S, \ldots, R \longrightarrow T$. Similarly $R \not\longrightarrow (S, \ldots, T)$ will mean that $R \not\longrightarrow S, \ldots, R \not\longrightarrow T$.

LEMMA 5. *The following relations hold:*
a) $A_n \longrightarrow A_{n-1}$, $n \geq 2$; $A_3 \longrightarrow G_2$;
b) $B_{2n} \longrightarrow BC_n$, $n \geq 2$; $B_3 \longrightarrow G_2$;
c) $BC_n \longrightarrow BC_{n-1}$, $n \geq 2$;
d) $C_2 \longrightarrow A_1$; $C_n \longrightarrow BC_{n-1}$, $n \geq 2$; $C_n \longrightarrow C_{n-1}$, $n \geq 3$;
e) $D_n \longrightarrow A_2$, $n \geq 4$; $D_{2n+1} \longrightarrow C_n$, $n \geq 3$;
f) $E_6 \longrightarrow C_2$;
g) $F_4 \longrightarrow (BC_2, G_2)$;
h) $G_2 \longrightarrow BC_1$.

PROOF. a) To obtain $A_n \longrightarrow A_{n-1}$ we just map α_1 to zero and $\alpha_{i+1} \longrightarrow \beta_i$ for all i. More generally for any j, $1 \leq j \leq n$, we have a dominant map $A_n \longrightarrow A_{n-1}$ such that $\alpha_i \longrightarrow \beta_i$ if $i < j$; $\alpha_j \longrightarrow 0$; and $\alpha_i \longrightarrow \beta_{i-1}$ if $i > j$.

For $A_3 \longrightarrow G_2$, we map $\alpha_1 \longrightarrow \beta_2$, $\alpha_2 \longrightarrow \beta_1$, $\alpha_3 \longrightarrow \beta_2 + 2\beta_1$.

b) For $B_{2n} \longrightarrow BC_n$, we map $\alpha_{2i-1} \longrightarrow 0$ and $\alpha_{2i} \longrightarrow \beta_i$ for $1 \leq i \leq n$.

For $B_3 \longrightarrow G_2$, we map α_1 and $\alpha_3 \longrightarrow \beta_1$ and $\alpha_2 \longrightarrow \beta_2$.

c) For $BC_n \longrightarrow BC_{n-1}$, we map $\alpha_n \longrightarrow 0$ and $\alpha_i \longrightarrow \beta_i$ for $i < n$.

d) For $C_n \longrightarrow BC_{n-1}$, we map $\alpha_n \longrightarrow 0$ and $\alpha_i \longrightarrow \beta_i$ for $i < n$.

For $C_n \longrightarrow C_{n-1}$, we map $\alpha_1 \longrightarrow 0$ and $\alpha_{i+1} \longrightarrow \beta_i$ for all i.

For $C_2 \longrightarrow A_1$, we map $\alpha_1 \longrightarrow 0$ and $\alpha_2 \longrightarrow \beta_1$.

e) For $D_n \longrightarrow A_2$, we map $\alpha_{n-1} \longrightarrow \beta_1$, $\alpha_n \longrightarrow \beta_2$ and $\alpha_i \longrightarrow 0$ for $i < n-1$.

For $D_{2n+1} \longrightarrow C_n$, we map $\alpha_1 \longrightarrow 0$, $\alpha_{2i} \longrightarrow 0$ and $\alpha_{2i+1} \longrightarrow \beta_i$ for $1 \leq i \leq n$.

For the remaining cases we map the simple roots as follows:

$E_6 \longrightarrow C_2$: $\alpha_5 \longrightarrow \beta_1$, $\alpha_6 \longrightarrow \beta_2$;
$F_4 \longrightarrow BC_2$: $\alpha_1 \longrightarrow \beta_1$, $\alpha_4 \longrightarrow \beta_2$;
$F_4 \longrightarrow G_2$: $\alpha_1 \longrightarrow \beta_2$, $\alpha_2 \longrightarrow \beta_1$;
$G_2 \longrightarrow BC_1$: $\alpha_2 \longrightarrow \beta_1$;

and map all other simple roots to zero. ■

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4. **The relations** $R_1 \not\to R_2$. We prove here the non-existence of dominant relations between various irreducible root systems. The proofs are more difficult than the existence proofs given in the previous section.

LEMMA 6. *The following relations hold:*
a) $A_n \not\to (B_3, F_4)$;
b) $B_n \not\to (A_2, C_3, D_k)$;
c) $BC_n \not\to (A_1, G_2)$;
d) $C_n \not\to (A_2, B_3, G_2)$;
e) $D_n \not\to (A_3, D_k)$, $n > k$.

PROOF. Each of the assertions above has the form $R_1 \not\to R_2$. We shall assume that $R_1 \xrightarrow{u} R_2$ and obtain a contradiction. We choose bases $\Pi_i \subset R_i$ such that $u(R_1^+ \cup \{0\}) = R_2^+ \cup \{0\}$.

a) Assume that $A_n \longrightarrow B_3$ for some n and let n be minimal. The minimality of n implies that $u(\Pi_1) \subset B_3^+$ (see the proof of Lemma 5, part a)). Let $\alpha_i \longrightarrow \beta_2$. Since $\tilde{\beta} = 122$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, there exists a unique $j \neq i$ such that $u(\alpha_j) \geq \beta_2$. Let, say, $i < j$ and let $\alpha = \alpha_i + \cdots + \alpha_j$. As $\alpha \in A_n^+$, we have $u(\alpha) \in B_3^+$. Since $u(\alpha) \geq 2\beta_2$ and $\tilde{\beta}$ is the only root of B_3 which is $\geq 2\beta_2$, we conclude that $u(\alpha) = \tilde{\beta}$. Hence $u(\tilde{\alpha} - \alpha) = 0$ and so $i = 1$ and $j = n$. Since $u(\Pi_1) \supset \Pi_2$ and $\alpha' := \tilde{\alpha} - \alpha_1 - \alpha_n \in A_n^+$, we have $u(\alpha') = p\beta_1 + q\beta_3$ with $p, q > 0$. As $p\beta_1 + q\beta_3 \notin B_3$, we have a contradiction.

Assume that $A_n \longrightarrow F_4$ with n minimal. Note that $\tilde{\beta} = 2342 \in F_4$ is the only root of F_4 which is $\geq 2\beta_1$. As above we may assume that $\alpha_1 \longrightarrow \beta_1$, and $u(\alpha_n) \geq \beta_1$. Then $u(\alpha_i) \not\geq \beta_1$ for $1 < i < n$ and consequently $u(A_{n-2}) = C_3$ where A_{n-2} respectively C_3 are root systems with bases $\{\alpha_2, \ldots, \alpha_{n-1}\}$ respectively $\{\beta_2, \beta_3, \beta_4\}$. This implies that u maps the highest root $\alpha = \alpha_2 + \cdots + \alpha_{n-1}$ of A_{n-2} to the one of C_3, *i.e.*, $\alpha \longrightarrow \beta = 0122$, and consequently $\alpha_n \longrightarrow 1220$. Since $h(\beta) = 5$, we have $n - 2 \leq 5$, *i.e.*, $n \leq 7$. As A_6 has 42 roots and F_4 has 48, we must have $n = 7$. It follows that $u(\alpha_i) \in \Pi_2$ for $i < 7$. As $\alpha_1 + \alpha_2 \in A_n$ is mapped to $\beta_1 + u(\alpha_2) \in F_4$, we have $u(\alpha_2) = \beta_2$. As $\alpha' = \alpha - \alpha_2 \in A_n$ and $\alpha' \longrightarrow 0022 \notin F_4$, we have a contradiction.

b) Assume that $B_n \longrightarrow A_2$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta}$, it follows that $\alpha_i \longrightarrow 0$ for $i > 1$. As u is surjective, we have a contradiction.

Assume that $B_n \longrightarrow C_3$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 221$, we have $\alpha_1 \longrightarrow \beta_3$. Let $\alpha_i \longrightarrow \beta_1$ and $\alpha_j \longrightarrow \beta_2$. All other simple roots of B_n are mapped to 0. Since $\alpha_1 + \cdots + \alpha_i \in B_n$ and $101 \notin C_3$, we must have $i > j$. As $021 \in C_3$ but $021 \notin u(B_n)$, we have a contradiction.

Assume that $B_n \longrightarrow D_k$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta}$ we conclude that $u(\alpha_i) \not\geq \beta_1, \beta_{k-1}, \beta_k$ for $i \neq 1$. As $u(\Pi_1) \supset \{\beta_1, \beta_{k-1}, \beta_k\}$ we have a contradiction.

c) Assume that $BC_n \longrightarrow A_1$ or G_2. If α and 2α are in BC_n then $\alpha \longrightarrow 0$. Since such α span the ambient space, we have a contradiction.

d) Assume that $C_n \longrightarrow A_2$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 11$ we have $\alpha_i \longrightarrow 0$ for $i \neq n$. As u is surjective, we have a contradiction.

Assume that $C_n \longrightarrow B_3$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 122$, we have $\alpha_n \longrightarrow \beta_1$. Let $\alpha_i \longrightarrow \beta_2$ and $\alpha_j \longrightarrow \beta_3$. All other $n - 3$ simple roots of C_n are mapped to 0. Since $\alpha_j + \cdots + \alpha_n$ is a root

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of C_n and $101 \notin B_3$, we have $i > j$. As $\alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{n-1}) + \alpha_n$ is a root of C_n, we have $121 \in u(C_n)$ but $121 \notin B_3$, a contradiction.

Assume that $C_n \longrightarrow G_2$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 32$, we have $\alpha_n \longrightarrow \beta_1$. There exist two indices $i,j < n$ such that $\alpha_i \longrightarrow \beta_1$ and $\alpha_j \longrightarrow \beta_2$, while the other $n-3$ simple roots of C_n are mapped to 0. Since $\alpha_i + \cdots + \alpha_n$ is a root of C_n and $20 \notin G_2$, we have $i < j$. Since $2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n$ is a root of C_n, we have $12 \in u(C_n)$. On the other hand $12 \notin G_2$, and so we have a contradiction.

e) Assume that $D_n \longrightarrow A_3$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 111$ we conclude that $\alpha_i \longrightarrow 0$ for $1 < i < n-1$ and that u maps $\{\alpha_1, \alpha_{n-1}, \alpha_n\}$ onto Π_2. Since $\alpha_1 + \cdots + \alpha_{n-1}$, $\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$, and $\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ are roots of D_n, it follows that $101 \in u(D_n)$. Since $101 \notin A_3$, we have a contradiction.

Assume that $D_n \longrightarrow D_k$, $n > k$. As $\tilde{\alpha} \longrightarrow \tilde{\beta}$, it follows that u maps $\{\alpha_1, \alpha_{n-1}, \alpha_n\}$ onto $\{\beta_1, \beta_{k-1}, \beta_k\}$. Also u maps $k-3$ of the roots $\alpha_2, \ldots, \alpha_{n-2}$ onto $\beta_2, \ldots, \beta_{k-2}$ and the others to 0. Let i be the largest index such that $\alpha_i \longrightarrow 0$, which exists because $n > k$. Then $\alpha = \alpha_i + 2(\alpha_{i+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ is a root of D_n while $u(\alpha) \notin D_k$. Hence we have a contradiction. ∎

LEMMA 7. $D_n \not\rightarrow F_4$.

PROOF. Assume that $D_n \xrightarrow{u} F_4$. Suppose that $u(\alpha_1) \geq \beta_1$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 2342$ and $\beta_1 \in u(\Pi_1)$, we infer that $u(\alpha_{n-1}) \geq \beta_1$ or $u(\alpha_n) \geq \beta_1$. By symmetry of the Dynkin diagram of D_n, we may assume that $u(\alpha_{n-1}) \geq \beta_1$. If $\alpha = \alpha_1 + \cdots + \alpha_{n-1}$ then $u(\alpha) \geq 2\beta_1$. Since $\tilde{\beta}$ is the only root of F_4 which is $\geq 2\beta_1$, we infer that $u(\alpha) = \tilde{\beta} = u(\tilde{\alpha})$. Thus $u(\tilde{\alpha} - \alpha) = 0$, *i.e.*, $\alpha_i \longrightarrow 0$ for $i \neq 1, n-1$. As u is surjective, we have a contradiction.

Now suppose that $u(\alpha_n) \geq \beta_1$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta}$ and $\beta_1 \in u(\Pi_1)$ we must have $u(\alpha_{n-1}) \geq \beta_1$. If $\alpha = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ then $u(\alpha) \geq 2\beta_1$ and so $u(\tilde{\alpha} - \alpha) = 0$, *i.e.*, $\alpha_i \longrightarrow 0$ for $i < n-1$. As u is surjective, we have a contradiction.

It follows that $\alpha_i \longrightarrow \beta_1$ for some i with $1 < i < n-1$, and consequently $u(\alpha_j) \not\geq \beta_1$ for $j \neq i$. The elements $\alpha = \alpha_{i+1} + \cdots + \alpha_n$ and $\alpha' = \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ are roots of D_n. Since $u(\alpha) \not\geq \beta_1$, by inspecting the list of positive roots of F_4, we conclude that $u(\alpha) \not\geq 2\beta_2$. If $u(\alpha_k) \not\geq \beta_2$ for all $k < i$ then $\tilde{\beta} \geq 3\beta_2$ implies that $u(\alpha_k) \geq \beta_2$ for at least two indices $k > i$. But this is impossible since $u(\alpha) \not\geq 2\beta_2$. Hence we can fix a $k < i$ such that $u(\alpha_k) \geq \beta_2$. Since $u(\alpha') \in F_4$ and $u(\alpha') \geq 2\beta_1$, it follows that $u(\alpha') = \tilde{\beta} = u(\tilde{\alpha})$. Since $u(\alpha_k) \geq \beta_2$ and $u(\tilde{\alpha} - \alpha') = 0$, we infer that $\tilde{\alpha} = \alpha'$, $i = 2$, $k = 1$, and so $u(\alpha_1) \geq \beta_2$. Now let j be the smallest index such that $j > 2$ and $u(\alpha_j) \neq 0$. Since u is surjective, $u(\Pi_1) \supset \Pi_2$, and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, we have $j < n-1$. As $\alpha_2 + \cdots + \alpha_j \in D_n$, we have $\beta_1 + u(\alpha_j) \in F_4$ and so $u(\alpha_j) \geq \beta_2$. It follows that $u(\alpha_1) \not\geq 2\beta_2$ and $u(\alpha_s) \not\geq \beta_2$ for $s \neq 1, j$.

Suppose that $\alpha_1 \longrightarrow \beta_2$. Then $u(\alpha_j) + \beta_1$ and $u(\alpha_j) + \beta_1 + \beta_2$ are in F_4 and $u(\alpha_j) + \beta_1 + \beta_2 \geq 2\beta_2 + \beta_1$. This implies that $u(\alpha_j)$ is 0120, 0121 or 0122. Since $u(\Pi_1) \supset \Pi_2$ and $u(2\alpha_j) \geq 4\beta_3$ we have a contradiction.

Since $u(\alpha_1) \neq \beta_2$, we must have $\alpha_j \longrightarrow \beta_2$. Since $u(\alpha_1) + \beta_1$ and $u(\alpha_1) + \beta_1 + \beta_2$ are in F_4 and $u(\alpha_1) + \beta_1 + \beta_2 \geq \beta_1 + 2\beta_2$, we must have $u(\alpha_1) = 0120, 0121$ or 0122. As $\tilde{\alpha} \longrightarrow \tilde{\beta}$ we infer that $\alpha_1 \longrightarrow 0120$.

Let $l > j$ be the smallest index such that $u(\alpha_l) \neq 0$. Since $u(\Pi_1) \supset \Pi_2$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, we have $l < n-1$. Since $1230 \notin F_4$ and $\alpha_1 + \cdots + \alpha_l \in D_n$, we have $u(\alpha_l) \neq \beta_3$. As $\alpha_j + \cdots + \alpha_l \in D_n$, we have $\beta_2 + u(\alpha_l) \in F_4$ and so $\alpha_l \longrightarrow \beta_3 + \beta_4$. Since $u(\Pi_1) \supset \Pi_2$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, we have a contradiction. ∎

LEMMA 8. *The following relations hold:*
a) $E_6 \not\rightarrow (A_3, B_3, C_3, BC_3)$,
b) $E_7 \not\rightarrow (A_2, B_3, BC_3)$,
c) $E_8 \not\rightarrow (A_1, BC_3)$.

PROOF. a) Assume that $E_6 \longrightarrow A_3$. Since $\tilde{\alpha} = 122321$ and $\tilde{\beta} = 111$ all the roots $\alpha_2, \ldots, \alpha_5 \longrightarrow 0$. This is impossible since u is surjective.

Assume that $E_6 \longrightarrow B_3$. Since $\tilde{\alpha} = 122321$ and $\tilde{\beta} = 122$, we must have $\alpha_4 \longrightarrow 0$ and α_1 or α_6 is mapped to β_1. By using symmetry of the Dynkin diagram of E_6 we may assume that $\alpha_1 \longrightarrow \beta_1$. Since $h(\tilde{\beta}) = 5$ is odd, $h\big(u(\alpha_6)\big)$ must be even, and so $u(\alpha_6) \neq \beta_2, \beta_3$. Consequently two of the roots α_2, α_3, α_5 must be mapped to β_2 and β_3, while the third and α_6 must be mapped to 0. Since $101 \notin B_3$ we conclude first that $u(\alpha_3) \neq 0$ and then that $\alpha_3 \longrightarrow \beta_2$. This leads to a contradiction because $121 \in u(E_6) \setminus B_3$.

Assume that $E_6 \longrightarrow C_3$. As $\tilde{\beta} = 221$, we must have $\alpha_4 \longrightarrow 0$ and α_1 or $\alpha_6 \longrightarrow \beta_3$. By symmetry of E_6, we may assume that $\alpha_1 \longrightarrow \beta_3$. Since $h(\tilde{\beta}) = 5$, $h\big(u(\alpha_6)\big)$ must be even, and so $u(\alpha_6) \neq \beta_1, \beta_2$. Consequently two of the roots α_2, α_3, α_5 must be mapped to β_1 and β_2, while the third and α_6 must be mapped to 0. Since $101 \notin C_3$, we conclude first that $u(\alpha_3) \neq 0$ and then that $\alpha_3 \longrightarrow \beta_2$. This leads to a contradiction because $122 \in u(E_6) \setminus C_3$.

Assume that $E_6 \longrightarrow BC_3$. As $\tilde{\alpha} = 122321$ and $\tilde{\beta} = 222$, we must have $\alpha_4 \longrightarrow 0$. If $u(\{\alpha_2, \alpha_3, \alpha_4\}) = \Pi_2$ then α_1 and $\alpha_6 \longrightarrow 0$ and $101 \in u(E_6) \setminus BC_3$, a contradiction. By symmetry of E_6, we may assume that $u(\alpha_1) \in \Pi_2$. Then $u(\alpha_6) \geq u(\alpha_1)$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$ implies $u(\alpha_6) = u(\alpha_1)$. Clearly one of α_2, α_3, α_5 is mapped to 0 and the other two to simple roots. If $\alpha_3 \longrightarrow 0$ or $\alpha_5 \longrightarrow 0$ then $101 \in u(E_6) \setminus BC_3$, a contradiction. Hence $\alpha_2 \longrightarrow 0$. Since $\alpha_1 + \alpha_3$, $\alpha_3 + \alpha_4 + \alpha_5$, $\alpha_5 + \alpha_6 \in E_6$, we obtain $101 \in u(E_6)$, while $101 \notin BC_3$, a contradiction.

b) Assume that $E_7 \longrightarrow A_2$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 11$, we see that $\alpha_i \longrightarrow 0$ for $i \neq 7$. As u is surjective, we have a contradiction.

Assume that $E_7 \longrightarrow B_3$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 122$, we conclude that $\alpha_3, \alpha_4, \alpha_5 \longrightarrow 0$ and $\alpha_7 \longrightarrow \beta_1$. If $\alpha_6 \longrightarrow 0$ then $u(\{\alpha_1, \alpha_2\}) = \{\beta_2, \beta_3\}$. Since 1011111, 0101111 are in E_7, we have $101 \in u(E_7)$, while $101 \notin B_3$, a contradiction. Hence $u(\alpha_6)$ must be a simple root. As $\alpha_6 + \alpha_7 \in E_7$, we have $\alpha_6 \longrightarrow \beta_2$. There are two cases to consider: $\alpha_1 \longrightarrow \beta_3$, $\alpha_2 \longrightarrow 0$ and $\alpha_1 \longrightarrow 0$, $\alpha_2 \longrightarrow \beta_3$. In both cases we find that $121 \in u(E_7)$ while $121 \notin B_3$, a contradiction.

Assume that $E_7 \longrightarrow BC_3$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta} = 222$, we see that $\alpha_3, \alpha_4, \alpha_5, \alpha_7 \longrightarrow 0$ and u induces a bijection $\{\alpha_1, \alpha_2, \alpha_6\} \longrightarrow \Pi_2$. Since 1111000, 1011110, 0101110 are in E_7, we see that $101 \in u(E_7)$ while $101 \notin BC_3$, a contradiction.

c) For E_8 each of the coefficients of $\tilde{\alpha}$ is > 1 and the sum of any three of them is > 6, and thus $E_8 \not\rightarrow A_1$ and $E_8 \not\rightarrow BC_3$. ∎

The following lemma finishes the proof of the Main Theorem.

LEMMA 9. *The following relations hold:*

a) $A_{2n-2} \not\rightarrow C_n$, $n \geq 2$;

b) $A_{2n-1} \not\rightarrow BC_n$, $n \geq 1$;

c) $D_n \not\rightarrow BC_k$, $n < 2k+1$;

d) $D_n \not\rightarrow C_k$, $n < 2k$.

PROOF. a) Assume that $A_n \xrightarrow{u} C_k$. We shall prove that $n \geq 2k-1$ by induction on k. If $k = 2$ this follows from the fact that A_2 has 6 roots while C_2 has 8. There are exactly two indices i and j, $i < j$, such that $u(\alpha_i) \geq \beta_1$ and $u(\alpha_j) \geq \beta_1$. Furthermore $u(\alpha_s) \not\geq \beta_1$ for $s \neq i, j$. Then $u(\alpha_i + \cdots + \alpha_j) = \tilde{\beta}$, since $\tilde{\beta}$ is the only root of C_k which is $\geq 2\beta_1$. Hence $\alpha_s \longrightarrow 0$ for $s < i$ or $s > j$. Since $u(A_n^+) = C_k^+$, it follows that $u(A_{j-i-1}) = C_{k-1}$ where A_{j-i-1} resp. C_{k-1} has base $\{\alpha_{i+1}, \ldots, \alpha_{j-1}\}$ resp. $\Pi_2 \setminus \{\beta_1\}$. By induction hypothesis $j - i - 1 \geq 2k - 3$ and so $n \geq 2k - 1$.

b) If $A_n \longrightarrow BC_k$ we shall prove that $n \geq 2k$. This is obvious if $k = 1$. By using the same argument as in a) we obtain $A_{j-i-1} \longrightarrow BC_{k-1}$ and we can use the induction on k.

c) Assume that $D_n \longrightarrow BC_k$ for some n and k with $4 \leq n < 2k+1$. We may assume that n is minimal. Let $u(\alpha_i) \geq \beta_1$ with i minimal. First assume that $i \in \{1, n-1, n\}$. There is a unique $j > i$ such that $u(\alpha_j) \geq \beta_1$ and it is clear that $j \in \{n-1, n\}$. If $i = n-1$ then $j = n$ and $u(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) \geq 2\beta_1$. This implies that $u(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = u(\tilde{\alpha})$ and so $\alpha_s \longrightarrow 0$ for $s < n-1$. This forces $k = 1$, a contradiction. Hence $i = 1$ and by using the symmetry of D_n, we may assume that $j = n-1$. Since $\alpha = \alpha_1 + \cdots + \alpha_{n-1}$ is a root of D_n and $u(\alpha) \geq 2\beta_1$, we have $u(\alpha) = \tilde{\beta} = u(\tilde{\alpha})$. This implies that $\alpha_s \longrightarrow 0$ for $s \neq 1, n-1$. Since $u(\alpha_1) \geq \beta_1$, $u(\alpha_{n-1}) \geq \beta_1$, and $u(\Pi_1) \supset \Pi_2$, we obtain $k = 1$, $n < 3$, a contradiction. Hence $1 < i < n-1$ and $\alpha_i \longrightarrow \beta_1$ while $u(\alpha_s) \not\geq \beta_1$ for $s \neq i$.

Since $\alpha = \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ is a root of D_n and $u(\alpha) \geq 2\beta_1$, we have $u(\alpha) = \tilde{\beta} = u(\tilde{\alpha})$. If $i > 2$ then we obtain $\alpha_s \longrightarrow 0$ for $s < i$. By restricting u to the subsystem D_{n-1} with base $\Pi_1 \setminus \{\alpha_1\}$ we obtain $D_{n-1} \longrightarrow BC_k$, contradicting the minimality of n. Hence we must have $i = 2$.

Assume that $u(\alpha_1) \neq 0$. Since $u(\alpha_1 + \alpha_2) = u(\alpha_1) + \beta_1$ is a root of BC_k, we have $u(\alpha_1) \geq \beta_2$. Let j be the minimal index such that $j > 2$ and $u(\alpha_j) \neq 0$. Such j exists because $k \geq 2$. By using the symmetry of D_n, we may assume that $j \neq n$. Since $u(\alpha_2 + \cdots + \alpha_j)$ is a root of BC_k, we have $u(\alpha_j) \geq \beta_2$. Since $\tilde{\alpha} \longrightarrow \tilde{\beta}$, we must have $j = n-1$ and $u(\alpha_n) \not\geq \beta_2$. Since $u(\alpha_2 + \cdots + \alpha_{n-2} + \alpha_n)$ is also a root of BC_k, we conclude that $\alpha_n \longrightarrow 0$. Now $u(\alpha_1) \geq \beta_2$, $u(\alpha_{n-1}) \geq \beta_2$, $\tilde{\alpha} \longrightarrow \tilde{\beta}$ and $u(\Pi_1) \supset \Pi_2$ imply that $k = 2$, $n = 4$, $\alpha_1 \longrightarrow \beta_2$ and $\alpha_{n-1} \longrightarrow \beta_2$. As $2\beta_2 \notin u(D_4)$, we have a contradiction. This shows that $\alpha_1 \longrightarrow 0$.

If $n = 4$ then by symmetry of D_4, we also have $\alpha_3 \longrightarrow 0$ and $\alpha_4 \longrightarrow 0$, a contradiction. If $n = 5$ then we may assume that $k = 3$. As $u(\Pi_1) \supset \Pi_2$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, $u(\alpha_3) \neq 0$ and in fact $u(\alpha_3) \in \Pi_2$. As $\alpha_2 \longrightarrow \beta_1$ and $\alpha_2 + \alpha_3 \in D_5$, we infer that $\alpha_3 \longrightarrow \beta_2$. Therefore $\alpha_4, \alpha_5 \longrightarrow \beta_3$ since $\tilde{\alpha} \longrightarrow \tilde{\beta}$. Then $022 \in BC_3 \setminus u(D_5)$, a contradiction. If $n > 5$ let D_{n-2} be the subsystem of D_n with base $\Pi_1 \setminus \{\alpha_1, \alpha_2\}$ and BC_{k-1} the subsystem of BC_k with base $\Pi_2 \setminus \{\beta_1\}$. If $\alpha \in D_{n-2}$ then $u(\alpha) \not\geq \beta_1$ and so $u(\alpha) \in BC_{k-1}$. Conversely, if $\beta \in BC_{k-1}$

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and $\alpha \in D_n$ such that $\alpha \longrightarrow \beta$ then we must have $\alpha \in D_{n-2}$. Hence the restriction of u gives $D_{n-2} \longrightarrow BC_{k-1}$ which contradicts the minimality of n.

d) Assume that $D_n \longrightarrow C_k$ for some n and k with $n < 2k$ and n minimal. We can use the same argument as in c) to show that $\alpha_1 \longrightarrow 0$, $\alpha_2 \longrightarrow \beta_1$, and to reduce the proof to the case $n = 5$.

Since $u(\alpha_3) \neq 0$ and $\tilde{\alpha} \longrightarrow \tilde{\beta}$, we have $u(\alpha_3) \in \Pi_2$. Since $\alpha_2 + \alpha_3 \in D_5$, we have $\alpha_3 \longrightarrow \beta_2$. One of the simple roots α_4, α_5 is mapped to β_3 and the other to 0. Now $021 \notin u(D_5)$ while $021 \in C_3$, a contradiction. ∎

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