A NOTE ON d-IDEALS IN SOME NEAR-ALGEBRAS

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Let E be a real Banach space. The set of all continuous linear mappings of E into E is a Banach algebra under the usual algebraic operations and the operator bound as norm. We denote this Banach algebra by $\mathscr{L}$, *if E is a separate Hilbert space.*

It has been proved by Calkin [1] that *the set of all compact linear mappings of E into E is the only closed* (2-*sided*) *ideal of $\mathscr{L}$.*

The purpose of this paper is to make a study of some ideals of some near-algebras and to obtain similar results as that of Calkin.

Near-algebras

A set $\mathscr{A}$ is said to be *a near-algebra* if it satisfies all axioms for algebras except for the left distributive law: $f(g+h) = fg+fh$. Therefore, a near-algebra is a near-ring which has first been defined in [5]. (cf. [2])

In this paper we consider near-algebras of mappings of a Banach space E into itself. Let f and g be mappings of E into E. The linear combination $\alpha f+\beta g$ for real numbers α and β is defined by

$$(\alpha f+\beta g)(x) = \alpha f(x)+\beta g(x) \text{ for all } x \in E,$$

and the product fg is defined by

$$(fg)(x) = f(g(x)) \text{ for all } x \in E.$$

Let us consider some examples.

1. *The near-algebra $I(E)$*. A mapping f of E into E is said to be *constant* if there exists an element $a \in E$ such that $f(x) = a$ for all $x \in E$. This constant mapping is denoted by c_a. It is easy to see that

$$\alpha c_a+\beta c_b = c_{\alpha a+\beta b} \text{ and } c_a c_b = c_a.$$

Therefore, the set $I(E)$ of all constant mappings of E into E is a near-algebra. (cf. [3] and [4])

2. *The near-algebra $\mathscr{C}$*. A mapping f is said to be *compact* if, for any bounded set M of E, $f(M)$ is contained in a compact set. The set of all compact and continuous mappings of E into E is denoted by $\mathscr{C}$. It is easy to see that $\mathscr{C}$ is a near-algebra.

3. *The near-algebra* $\mathscr{D}$. A mapping f is said to be *differentiable* if, for any $a \in E$, there exists $l \in L$ such that

$$f(a+x)-f(a) = l(x)+r(a, x) \quad \text{for every } x \in E,$$

where

$$\lim_{||x|| \to 0} \frac{||r(a, x)||}{||x||} = O.$$

This linear mapping l depends on a and is denoted by $f'(a)$. It is well-known that, if f and g are differentiable, $\alpha f+\beta g$ and fg are differentiable and

$$(\alpha f+\beta g)'(a) = \alpha f'(a)+\beta g'(a)$$
$$(fg)'(a) = f'(g(a))g'(a)$$

for every $a \in E$. Therefore, the set $\mathscr{D}$ of all differentiable mappings of E into E is a near-algebra.

Ideals

Let $\mathscr{A}$ be a near-algebra. A non-empty subset I of $\mathscr{A}$ is said to be *an ideal* if it is a linear subset and $fg, gf \in I$ whenever $f \in I$ and $g \in \mathscr{A}$. If $I \neq (O)$, the ideal I is said to be *non-zero*.

When $\mathscr{A}$ is a near-algebra whose elements are mappings of E into E, the set $I(E)$ is the smallest non-zero ideal of $\mathscr{A}$ whenever $I(E) \subset \mathscr{A}$. (cf. [4])

When $\mathscr{A}$ is a near-algebra whose elements are bounded mappings of E into E, then $\mathscr{C} \cap \mathscr{A}$ is an ideal of $\mathscr{A}$.

Hereafter we assume that $\mathscr{A}$ *is a near-algebra such that* $\mathscr{L} \subset \mathscr{A} \subset \mathscr{D}$.

d-sets

In order to introduce the notion of d-set we need the following definitions. We define the sets $d(f)$, $d(M)$ and $d^{-1}(N)$ as follows:

$$d(f) = \{f'(x) \mid x \in E\} \qquad \text{for } f \in \mathscr{A},$$
$$d(M) = \bigcup_{f \in M} d(f) \qquad \text{for } M \subset \mathscr{A},$$
$$d^{-1}(N) = \{f \in \mathscr{A} \mid d(f) \subset N\} \quad \text{for } N \subset \mathscr{L}.$$

We enumerate some properties of these sets.

(1) *If* $M_1 \subset M_2$, $d(M_1) \subset d(M_2)$.
If $N_1 \subset N_2$, $d^{-1}(N_1) \subset d^{-1}(N_2)$.

(2) $d(f) = (O)$ *if and only if* $f \in I(E)$.
Since every linear mapping $l \in \mathscr{L}$ is differentiable and

$$l'(x) = l \text{ for every } x \in E,$$

the following proposition is evident:

(3) *The following three conditions are mutually equivalent*: (i) $f \in \mathscr{L}$; (ii) $f \in d(f)$; (iii) $d(f) = (f)$. *Therefore*, $d(N) = N$ for $N \subset \mathscr{L}$.

(4) $M \cap \mathscr{L} \subset d(M)$ for $M \subset \mathscr{A}$.

PROOF. By (3) we have

$$M \cap \mathscr{L} = d(M \cap \mathscr{L}) \subset d(M).$$

REMARK. The equality does not always hold. Let us consider the following set M:

$$M = \{f \in \mathscr{D} \mid \sup_{x \in E} \|f(x)\| < +\infty\}.$$

Then, it is clear that $M \cap \mathscr{L} = (0)$ and $d(M) \neq (0)$.

(5) $d(d(M)) = d(M)$ *for* $M \subset \mathscr{A}$.

PROOF. This follows from (3), because $d(M) \subset \mathscr{L}$.

(6) $N \subset d^{-1}(N)$ *for* $N \subset \mathscr{L}$.

PROOF. If $l \in N$, since $(l) = d(l)$ by (3), we have $d(l) \subset N$, which means that $l \in d^{-1}(N)$.

(7) $d(d^{-1}(N)) = N$ *for* $N \subset \mathscr{L}$.

PROOF. By (3) and (6), we have $N = d(N) \subset d(d^{-1}(N))$. Now, assume that $f \in d(d^{-1}(N))$. Then, $f \in d(g)$ for some $g \in d^{-1}(N)$, or equivalently, $f \in d(g)$ for some g such that $d(g) \subset N$. Therefore, $f \in N$.

(8) $M \subset d^{-1}(d(M))$ *for* $M \subset \mathscr{A}$.

PROOF. If $f \in M$, then $d(f) \subset d(M)$, which is equivalent to $f \in d^{-1}(d(M))$.

REMARK. The converse inclusion of (8) is not always true. Let E be a separable Hilbert space and (e_n) be a complete orthonormal system. Let us consider the following mapping:

$$f(x) = \sum_{n=1}^{\infty} (x, e_n)^2 e_n.$$

Since

$$f'(x)(y) = 2 \sum_{n=1}^{\infty} (x, e_n)(y, e_n) e_n,$$

it is clear that $f \notin \mathscr{C} \cap \mathscr{D}$ and $f \in d^{-1}(d(\mathscr{C} \cap \mathscr{L}))$. Therefore, in the near-algebra $\mathscr{D}$, for $M = \mathscr{C} \cap \mathscr{D}$, we have $M \neq d^{-1}(d(M))$.

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DEFINITION. *A subset M of $\mathscr{A}$ is said to be a d-set if* $d^{-1}(d(M)) = M$.
The followings are important properties of d-sets.

(9) *If M is a d-set,* $d(M) = M \cap \mathscr{L}$.

PROOF. Since $M \cap \mathscr{L} \subset d(M)$ by (4), we have only to prove that $d(M) \subset M$. Now, since $d(M) \subset \mathscr{L}$, it follows from (6) that

$$d(M) \subset d^{-1}(d(M)) = M.$$

(10) *The following three conditions are mutually equivalent*: (i) *M is a d-set*; (ii) *$f \in M$ if and only if $d(f) \subset M$*; (iii) *$M = d^{-1}(N)$ for some $N \subset \mathscr{L}$.*

PROOF. (i) $\to$ (ii): If $f \in M$, since $d(M) \subset M$ by (9), we have $d(f) \subset d(M) \subset M$. Conversely, if $d(f) \subset M$, we have $d(f) \subset d(M)$ by (5), hence it follows that $f \in d^{-1}(d(M)) = M$.

(ii) $\to$ (iii): For $N = d(M)$, we have $d^{-1}(N) = d^{-1}(d(M)) \supset M$ by (8). Conversely, if $f \in d^{-1}(N)$, then, since $d(g) \subset M$ whenever $g \in M$, we have

$$d(f) \subset N = d(M) = \bigcup_{g \in M} d(g) \subset M.$$

Therefore, $f \in M$.

(iii) $\to$ (i): It follows from (7) that

$$d^{-1}(d(M)) = d^{-1}(d(d^{-1}(N))) = d^{-1}(N) = M.$$

(11) *If M_1 and M_2 are d-sets and $d(M_1) = d(M_2)$, then $M_1 = M_2$.*

PROOF. $M_1 = d^{-1}(d(M_1)) = d^{-1}(d(M_2)) = M_2$.

d-ideals

If I is an ideal of the near-algebra $\mathscr{A}$, then $I \cap \mathscr{L}$ is an ideal of the Banach algebra $\mathscr{L}$. Conversely, we have the following proposition.

(12) *If J is an ideal of the Banach algebra $\mathscr{L}$, then $d^{-1}(J)$ is an ideal of the near-algebra $\mathscr{A}$.*

PROOF. To prove that $d^{-1}(J)$ is linear, we assume that $f \in d^{-1}(J)$ and $g \in d^{-1}(J)$. Then, since J is linear, we have

$$d(\alpha f + \beta g) \subset \alpha d(f) + \beta d(g) \subset J + J = J,$$

which implies that $\alpha f + \beta g \in d^{-1}(J)$. Next, assume that $f \in d^{-1}(J)$ and $g \in \mathscr{A}$. Then, since J is an ideal, we have

$$d(fg) \subset d(f)d(g) \subset Jd(g) \subset J$$

and

$$d(gf) \subset d(g)d(f) \subset d(g)J \subset J.$$

Therefore, fg and gf belong to $d^{-1}(J)$.

DEFINITION. An ideal of $\mathscr{A}$ is said to be a d-ideal if it is a d-set.

Therefore, for any ideal J of $\mathscr{L}$, $d^{-1}(J)$ is a d-ideal of $\mathscr{A}$.

REMARK 1. Since $I(E) = d^{-1}((O)$, $I(E)$ is a d-ideal of every $\mathscr{A}$ such that $I(E) \subset \mathscr{A}$.

REMARK 2. As we have shown in the remark after (8) of the preceeding section, the ideal $\mathscr{C} \cap \mathscr{D}$ of $\mathscr{D}$ is not a d-ideal. However, it has been proved in [3] that $\mathscr{C} \cap \mathscr{A}$ is a d-ideal of some near-algebra $\mathscr{A}$ such that $\mathscr{L} \subset \mathscr{A} \subset \mathscr{D}$.

($\mathscr{L}$)-closed d-ideals

DEFINITION. A subset M of $\mathscr{A}$ is said to be ($\mathscr{L}$)-closed if $M \cap \mathscr{L}$ is closed under the norm topology of $\mathscr{L}$.

The collection of all ($\mathscr{L}$)-closed subsets of $\mathscr{A}$ defines a topology on $\mathscr{A}$, which is the strongest among the topologies under which the mapping $l \to l$ of $\mathscr{L}$ into $\mathscr{A}$ becomes continuous.

(13) *d-ideals $I(E)$ and $d^{-1}(\mathscr{C} \cap \mathscr{L})$ are ($\mathscr{L}$)-closed.*

PROOF. $I(E)$ is ($\mathscr{L}$)-closed, because $I(E) \cap \mathscr{L} = (O)$. Since $d^{-1}(\mathscr{C} \cap \mathscr{L})$ is a d-set, we have by (9) that

$$d^{-1}(\mathscr{C} \cap \mathscr{L}) \cap \mathscr{L} = d(d^{-1}(\mathscr{C} \cap \mathscr{L}) = \mathscr{C} \cap \mathscr{L}.$$

Since $\mathscr{C} \cap \mathscr{L}$ is closed in $\mathscr{L}$, $d^{-1}(\mathscr{C} \cap \mathscr{L})$ is ($\mathscr{L}$)-closed.

As the converse, we prove the following theorem which is the main result of this paper.

THEOREM 1. *Let I be an arbitrary ($\mathscr{L}$)-closed d-ideal of $\mathscr{A}$. Then, we have either $I = I(E)$ or $I \supset d^{-1}(\mathscr{C} \cap L)$.*

2. *When E is a separable Hilbert space and I is an ($\mathscr{L}$)-closed d-ideal, we have either $I = I(E)$ or $I = d^{-1}(\mathscr{C} \cap \mathscr{L})$.*

3. *When E is a separable Hilbert space and $f(O) = O$ for every $f \in \mathscr{A}$, then $d^{-1}(\mathscr{C} \cap \mathscr{L})$ is the only ($\mathscr{L}$)-closed d-ideal of $\mathscr{A}$.*

PROOF. 1. Since I is ($\mathscr{L}$)-closed, $I \cap \mathscr{L}$ is a closed subset of $\mathscr{L}$. Moreover, $I \cap \mathscr{L}$ is evidently an ideal of $\mathscr{L}$. Therefore, since $I \cap \mathscr{L}$ is a closed ideal of the Banach algebra $\mathscr{L}$, we have either $I \cap \mathscr{L} = (O)$ of $I \cap \mathscr{L} \supset \mathscr{C} \cap \mathscr{L}$. From the definition of d-sets and (9) it follows that

$$I = d^{-1}(d(I)) = d^{-1}(I \cap \mathscr{L}) = d^{-1}((O)) = I(E)$$

or

$$I = d^{-1}(I \cap \mathscr{L}) \supset d^{-1}(\mathscr{C} \cap \mathscr{L}).$$

2. When E is a separable Hilbert space, by the Calkin's theorem [1], $\mathscr{C} \cap \mathscr{L}$ is the only non-zero closed ideal of $\mathscr{L}$. Therefore, we have either

$I \cap \mathscr{L} = (O)$ or $I \cap \mathscr{L} = \mathscr{C} \cap \mathscr{L}$, hence it follows that we have either $I = I(E)$ or $I = d^{-1}(\mathscr{C} \cap \mathscr{L})$.

3. In this case, we have $I(E) \cap \mathscr{A} = (O)$. Therefore, the case when $I = I(E)$ does not occur.

References

[1] J. W. Calkin, 'Two-sided ideals and congruences in the ring of bounded operators in Hilbert space', *Ann. of Math.* 42 (1941), 839–873.

[2] S. Yamamuro, 'On the spaces of mappings on Banach spaces', *J. Aust. Math. Soc.* 7 (1967), 160–164.

[3] S. Yamamuro, 'On near-algebras of mappings on Banach spaces', *Proc. Japan Acad.* 41 (1965), 889–892.

[4] S. Yamamuro, 'Ideals and homomorphisms in some near-algebras', *Proc. Japan Acad.* 42 (1966), 427–432.

[5] H. Zassenhaus, *Lehrbuch der Gruppentheorie* (Berlin, 1937).

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