PERIODIC SOLUTION AND STABILITY PROBLEM

Diliberto's mathematical theory (1) on the periodic surfaces originates from a conjecture that the existence of a family of periodic surfaces, which, if true, would establish the stability of the orbits and the validity of the algorithm for finding approximate solutions. The union of the trajectories through an invariant curve is called a periodic surface.

The equations of motion for a dynamical system of two degrees of freedom, including the motion of an artificial satellite around the oblate Earth's gravitational field, are transformed to

$$\frac{d\theta_i}{d\varphi} = 1 + \lambda\, \Theta_i\,(\theta_1,\, \theta_2,\, r_1,\, r_2), \qquad (i = 1, 2)$$
$$\frac{dr_i}{d\varphi} = \lambda R_i\,(\theta_1,\, \theta_2,\, r_1,\, r_2),$$

where $\lambda = 0$ corresponds to the undisturbed motion. A periodic surface of this system of equations is the graph of a pair of analytic functions

$$r_i = S_i\,(\theta_1,\, \theta_2,\, \lambda)$$

defined for all (θ_1, θ_2), and for some neighbourhood of $\lambda = 0$, with period 2π in θ_i and such that, if $p_i = S_i\,(r_1, r_2, \lambda)$, then the solution $r_i\,(\varphi)$, $\theta_i(\varphi)$ taking on the initial values $r_i\,(\varphi_0) = p_i$, $\theta_i(\varphi_0) = r_i$ satisfy

$$r_i(\varphi) = S_i[\theta_1(\varphi),\, \theta_2(\varphi),\, \lambda]$$

for $-\infty < \varphi < +\infty$. The torus which is an example of a periodic two-surface depends on two parameters p_1 and p_2, the radii of its normal cross sections. The central mathematical problem is whether there exists a nested complete family of periodic two-surfaces when the perturbation parameter λ is not zero.

Diliberto *et al.* (2) described their expansion technique for finding the periodic surfaces, and a simple requisite condition which a co-ordinate system must satisfy so as to allow the possibility of such expansions and the determination of at least one co-ordinate system which satisfies the prerequisite condition. It was shown that the energy integral has a simple form in terms of the co-ordinates used. From this it follows that, if one component of the surface is known to a given order, then the other component can be found to the same order directly from the energy integral. Diliberto *et al.* initiated numerical studies for the application of the theory to the study of the orbits of artificial satellites and gave the formulae for the approximate numerical solution.

Thus they announced, for the case where the only term in the perturbation potential is the second zonal harmonic and is sufficiently small, the existence of the family of periodic orbits of arbitrary inclination, the existence of a quadruple family of periodic orbits of arbitrary eccentricity, and the property that these two families plus that at zero inclination include all periodic orbits with common axial angular momentum with periods continuously approaching that of the corresponding Keplerian orbits as the perturbation tends to zero.

For the motion of a satellite around an axially symmetric planet Haseltine (3) proved that there are three one-parameter families, in the sense that, if a value of the parameter is chosen, then the corresponding periodic orbit exists when the perturbation potential is small enough.

Kyner (4) discussed the mathematical problem of the orbits about an oblate planet. He referred to the method of averaging for constructing a new set of approximating formulae, which cannot be a solution but have the novel feature of being accompanied by error estimates. This method of averaging is developed by Struble (5) independently of the more general theory developed by Bogoliubov and Mitropolski. The first-order formulae are free from the difficulties common to most general perturbation method as in the solutions at the critical inclination and

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for small eccentricities. Thus Kyner proved the existence of periodic solutions after the fashion of the recent works of Cesari and Hale but by a method different from Diliberto's. It was shown that the only possible generating orbits are circular orbits of arbitrary inclination, orbits in the equatorial plane, and orbits at the critical inclination. Macmillan's classical paper is criticized.

According to Diliberto (**6**), Krein studied linear differential equations with periodic coefficients and proved that there are certain Hamiltonian systems such that arbitrary periodic Hamiltonian perturbations of them have only stable motions. By Floquet's theorem this implies that in terms of a perturbation parameter the given Hamiltonian system is diagonalizable to a constant matrix whose roots are all purely imaginary. Diliberto proves that for a linear perturbation these characteristic roots are analytic in the parameter. This establishes the stability of the motion.

Some time ago Siegel and Moser (**7**) discussed the stability of a motion of two degrees of freedom and obtained several elegant theorems. Kolmogorov and Arnold proved the existence of almost-periodic solutions of Hamiltonian systems. Moser (**8**) invented a new technique, which he called a smoothing operator, for constructing the solution of a non-linear differential equation by extending the usual iteration process of Picard. He then proceeded (**9**) to prove a criterion of stability of periodic solutions, and studied the perturbation theory for almost-periodic solutions for undamped non-linear differential equations. Recently Moser (**10**) referred to the invariant-point theorem by Poincaré and Birkhoff for the existence of periodic solutions, that is, by studying the area-preserving mappings of a ring-domain into itself. Moser (**11**) proved a theorem which guarantees the existence of closed invariant curves of such a mapping. Closed invariant curves correspond to almost-periodic solutions of the differential equation which generates the mapping, and are important for the study of stability of periodic solutions.

Cronin (**12**), on the other hand, introduced the idea of topological degree for a criterion of stability of periodic solutions in the perturbation problems, which is identical with the criterion of Andronov and Witt on the van der Pol equations.

Conley (**13**) proved the existence of some new long-period periodic solution in the plane restricted three-body problem. Auslander (**14**) and Seibert (**15**) considered the prolongation of orbits by his method of continuing the orbits beyond their omega limit sets in the sense of Birkhoff and generalized the stability in the sense of Liapounov.

Choudhry (**16**) has discussed the existence of direct and retrograde symmetric periodic orbits, as referred to the rotating axes, in the restricted three-body problem in three dimensions. The periodic orbits obtained by analytic continuation from the generating periodic orbits touch the generating orbits, so that they correspond to the periodic solutions of Schwarzschilds's type.

Huang (**17**), on the other hand, obtained a stability criterion of the periodic orbits in the restricted three-body problem based on the eigenvalues of a fourth order matrix.

New classes of periodic orbits in the restricted three-body problem have been found by Aksenov (**18**) enclosing both of the finite mass bodies, and by Demin (**19**) in the vicinity of trajectories for the problem of two fixed centres with Thiele's variables, both according to Poincaré's method. Krasinski (**20**) has developed a theory of double collision trajectories in the restricted three-body problem, symmetric and asymmetric, with Levi-Civita's regularization, similar to Poincaré's theory of periodic orbits. Petrovskaya (**21**) has found the values of disturbing masses and other characteristics which give the convergence of the series representing the periodic solutions of the planar restricted three-body problem. Volkov (**22**) has obtained symmetric periodic solutions for the three-body problem with finite dimensions as a continuation of the works of Kondurar. Merman (**23**) proved the existence of almost-periodic solutions in the planar restricted three-body problem.

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