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ABSOLUTE SUMMABILITY FACTORS IN A SEQUENCE

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ABSTRACT. Let $\alpha \geq 0$ and $\beta > -1$. The main result gives necessary and sufficient conditions for the sequence (ε_n) in order that the sequence $(\varepsilon_n U_n)$ will be absolutely summable by the Cesàro method C^β for each sequence (U_n) which is bounded or summable by the method C^α.

Another theorem is proven when C^α and C^β are replaced by triangular methods $A = (a_{nk})$ and $B = (b_{nk})$ satisfying $\sum_{k \leq n} |\xi_{nk}| = O(\xi_{nn})$ and $\sum_{n \geq k} |\bar{\Delta} b_{nk}| = O(b_{kk})$, where $(\xi_{nk}) = (a_{nk})^{-1}$.

Let $A = (a_{nk})$ be a triangular infinite matrix of complex numbers. For a sequence[1] (U_n) of complex numbers we denote

$$U'_n = \sum_{k=0}^{n} a_{nk} U_k \tag{1}$$

and

$$u'_n = \sum_{k=0}^{n} \bar{a}_{nk} U_k, \tag{2}$$

where $\bar{a}_{nk} = a_{nk} - a_{n-1,k}$ and $a_{-1,k} = 0$. The sequence (U_n) is called *A-summable* if the limit $\lim U'_n$ exists. The sequence (U_n) is called *absolutely A-summable* or *|A|-summable* if[2] $\sum |u'_n| < \infty$. The sequence (U_n) is called *A-bounded* if $U'_n = O(1)$. If $A = C^\alpha$ is the matrix of the Cesàro method C^α of order $\alpha > -1$, then $a_{nk} = A^{\alpha-1}_{n-k}/A^\alpha_n$, where $A^\alpha_n = (n+\alpha) \cdots (1+\alpha)/n!$ for $n \geq 1$ and $A^\alpha_O = 1$. Let $B = (b_{nk})$ be a triangular infinite matrix of complex numbers. The complex numbers ε_n are called *summability factors in a sequence of the type* $(A, |B|)$ (resp. $(A_0, |B|)$ if for each sequence (U_n) which is A-summable (resp. A-bounded) the sequence $(\varepsilon_n U_n)$ is absolutely B-summable. The other types of summability factors are similarly defined. If $B = E$ is the unit matrix $E = (\delta_{nk})$ the summability factors are called *convergence factors*. To find the summability factors of a given type is to find effective (in practice, easily verifiable) necessary and sufficient conditions which assure that ε_n are the

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[1] Everywhere the free indices take on all values $0, 1, 2, \ldots$.

[2] Throughout $\sum$ denotes summation over $0, 1, 2, \ldots$, unless otherwise indicated.

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summability factors of this type. In what follows, instead of saying that ε_n are summability factors in a sequence of the types $(A, |B|)$ and $(A_O, |B|)$, we often write $\varepsilon_n \in (\mathfrak{A}, |\mathfrak{B}|)$ and $\varepsilon_n \in (\mathfrak{A}_O, |\mathfrak{B}|)$ respectively, replacing the letters A and B respectively by the Gothic letters $\mathfrak{A}$ and $\mathfrak{B}$.

The first paper concerned with finding the summability factors in a sequence was the work of Bosanquet [11] who found the summability factors in a sequence of the type (C^α, C^β) if α and β are nonnegative integers. An analogous problem for absolute summability was solved by Tyler [21]. The result of Tyler was generalized in [8] for arbitrary real $\alpha \geq 0$ and complex β with $-1 < \operatorname{Re} \beta < 2$. Recently Ahmad and Khan Mohd [3] found the summability factors in a sequence of the type $(E, |A|)$, where A is the method of Poisson-Abel.

In the present paper we find summability factors in a sequence of the types $(A, |B|)$ and $(A_O, |B|)$, where A is a normal method (i.e. $a_{kk} \neq 0$) and A satisfies the condition

$$\sum_{k=0}^{n} |\xi_{nk}| = O(1/a_{nn}), \tag{3}$$

where $(\xi_{nk}) = (a_{nk})^{-1}$ and where B satisfies the condition

$$b_k = O(b_{kk}), \tag{4}$$

where we denote

$$b_k = \sum_{n=k}^{\infty} |\bar{b}_{nk}|.$$

Since the method of Cesàro C^β does not satisfy the condition (4) at $\beta > 1$ (see [16], p. 290, or [6], p. 188), we consider separately summability factors in a sequence of the types $(C^\alpha, |C^\beta|)$ and $(C^\alpha_O, |C^\beta|)$ at $\beta > 1$.

If we denote by v'_n the B-means in the sequence to series form (2) of the sequence $(\varepsilon_n U_n)$ and applying the inverse transformation of (1), we obtain[(3)]

$$v'_n = \sum_{k=0}^{n} g_{nk} U'_k, \tag{5}$$

where

$$g_{nk} = \sum_{\nu=k}^{n} \bar{b}_{n\nu} \xi_{\nu k} \varepsilon_\nu.$$

Applying the Theorems of Peyerimhoff (see [18], Theorem 6, and [20], p. 34, or [6], Corollaries 5.2 and 5.1) to the sequence to series transformation (5) we obtain the following.

[(3)] It is more precise to denote $\bar{g}_{nk}$ instead of g_{nk} because (5) is a matrix transformation in the sequence to series form (cf. [6], p. 9).

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LEMMA 1. *In order that* $\varepsilon_n \in (\mathfrak{A}, |\mathfrak{B}|)$ *and* $\varepsilon_n \in (\mathfrak{A}_0, |\mathfrak{B}|)$, *it is necessary and sufficient to satisfy the condition*[4]

$$\forall (d_n) \in m: \sum \left| \sum_{n=k}^{\infty} g_{nk} d_n \right| < \infty, \tag{6}$$

and necessary to satisfy the condition

$$\sum |g_{kk}| < \infty. \tag{7}$$

Condition (6) of Lemma 1 follows from the stronger condition

$$\sum \sum_{n=k}^{\infty} |g_{nk}| < \infty. \tag{8}$$

If the method A satisfies the condition (3), then (8) follows from the condition

$$\sum b_k \, |\varepsilon_k / a_{kk}| < \infty, \tag{9}$$

because according to condition (3)

$$\sum \sum_{n=k}^{\infty} |g_{nk}| \le \sum \sum_{\nu=k}^{\infty} b_\nu \, |\xi_{\nu k} \varepsilon_\nu| = \sum b_\nu \, |\varepsilon_\nu| \sum_{k=0}^{\nu} |\xi_{\nu k}| = O(1) \sum b_\nu \, |\varepsilon_\nu / a_{\nu\nu}| < \infty.$$

If the method B satisfies condition (4), then (9) follows from

$$\sum |b_{kk} \varepsilon_k / a_{kk}| < \infty, \tag{10}$$

i.e. from the necessary condition (7) of Lemma 1. Thus we proved

THEOREM 1. *If A satisfies condition* (3) *and B satisfies condition* (4), *then the necessary and sufficient condition for* $\varepsilon_n \in (\mathfrak{A}, |\mathfrak{B}|)$ *and* $\varepsilon_n \in (\mathfrak{A}_O, |\mathfrak{B}|)$ *is* (10).

The condition (3) is satisfied for many methods A. In particular, if A is the method of weighted means of Riesz $P = (R, p_n)$ with $p_n \neq 0$, then (3) is satisfied when $P_{n-1} = O(P_n)$ (see [15], p. 57, or [6], p. 115). If we let A be the method of Woronoi-Nörlund $Q = (WN, q_n)$ with $q_0 \neq 0$, then $\xi_{nk} = Q_k c_{n-k}$ (see [6], p. 103) and (3) is satisfied when $\sum |c_k| < \infty$, where $\sum c_k x^k = (\sum q_k x^k)^{-1}$. We remark that according to the Theorem of Kaluza (see [15], Theorem 22) the condition $\sum |c_k| < \infty$ is fulfilled for many methods Q, in particular for the Cesàro method C^α with $\alpha \ge 0$. Concerning Theorem 1 if A is the method of Euler-Knopp and $B = E$ see Espenberg [14].

[4] Conditions equivalent to (6) we obtain from the Theorems of Lorentz and Zeller (cf. [22], p. 344, or [6], p. 40).

If in Theorem 1 we let $B=C^\beta$, then according to what has been said above, we obtain summability factors in a sequence of the types $(C^\alpha, |C^\beta|)$ and $(C_O^\alpha, |C^\beta|)$ only for $-1<\beta\le 1$ or for complex β with $-1<\operatorname{Re}\beta<1$. Therefore we prove another theorem for the case $B=C^\beta$, free from the restriction $\beta\le 1$. In the proof of the neccessity we use the Theorems of Peyerimhoff [19] for the difference of products of sequences, but in the proof of sufficiency we employ calculations of the paper [5].

THEOREM 2. *If* $\alpha\ge 0$ *and* $\beta>-1$ *then necessary and sufficient conditions for* $\varepsilon_n\in(\mathfrak{C}^\alpha, |\mathfrak{C}^\beta|)$ *and* $\varepsilon_n\in(\mathfrak{C}_O^\alpha, |\mathfrak{C}^\beta|)$ *are*

$$\sum (k+1)^{\alpha-\beta}|\varepsilon_k|<\infty \tag{11}$$

and

$$\sum (k+1)^{\alpha-1}|\Delta^\alpha\varepsilon_k|<\infty \tag{12}$$

if $\beta\le\alpha+1$, *but are*

$$\sum (k+1)^{-1}|\varepsilon_k|<\infty. \tag{13}$$

and (12) *if* $\beta>\alpha+1$.

Proof. We need the following formulas for Cesàro numbers. The formula of the sum of products (see [23], p. 77, or [6], p. 77)

$$\sum_{\kappa=\nu}^{m} A_{m-\kappa}^{\sigma}A_{\kappa-\nu}^{\tau}=A_{m-\nu}^{\sigma+\tau+1}, \tag{14}$$

is valid for arbitrary complex σ and τ; the formula of Chow and Peyerimhoff (see [13], p. 461, and [17], p. 418, and [6], p. 80)

$$\sum_{n=k}^{\infty} A_{n-k}^{\sigma}/(nA_n^{\tau})=1/(kA_k^{\tau-\sigma-1}), \tag{15}$$

is valid for all complex σ with τ with $\operatorname{Re}\tau>-1$ and $\operatorname{Re}(\tau-\sigma)>0$, where $k=1,2,\ldots$; the formula (see [6], p. 81)

$$\sum_{n=k}^{\infty} A_{n-k}^{\sigma}/A_n^{\tau}=\frac{\tau}{\tau-\sigma-1}\Big/A_k^{\tau-\sigma-1} \tag{16}$$

is valid for all complex σ and τ with $\operatorname{Re}\tau\ge 0$ and $\operatorname{Re}(\tau-\sigma)>1$; the formula of Bosanquet (see [10], p. 487–488; or [6], p. 82)

$$\sum_{\kappa=\nu}^{m} A_{n-\kappa}^{\sigma-1}A_{\kappa-\nu}^{-\tau-1}=O(1)A_{n-\nu}^{\sigma-1}A_{m-\nu}^{-\tau} \tag{17}$$

is valid for $0<\sigma<1$ and $0\le\tau<1$, where $0<\nu\le m\le n$. We often will make use of the following asymptotic formula without stating it explicitly:

$$A_n^\sigma\sim\Gamma^{-1}(\sigma+1)\cdot(n+1)^\sigma$$

for arbitrary complex $\sigma\ne -1,-2,\ldots$; moreover $A_n^\sigma>0$ for $\sigma>-1$.

We recall that the difference $\Delta^\sigma \mu_n$ of arbitrary order σ of the sequence (μ_n) defined by

$$\Delta^\sigma \mu_k = \sum_{\nu=k}^{\infty} A_{\nu-k}^{-\sigma-1} \mu_\nu,$$

whenever the series converges, and denote $\Delta \mu_k = \Delta^1 \mu_k = \mu_k - \mu_{k+1}$.

We begin from the conclusion of effective necessary conditions for $\varepsilon_n \in (\mathfrak{C}^\alpha, |\mathfrak{C}^\beta|)$. If $A = C^\alpha$ and $B = C^\beta$ we have (see formulas (15.25), (8.9) and (15.21) in [6])

(18) $$\xi_{nk} = A_k^\alpha A_{n-k}^{-\alpha-1}$$

and

(19) $$n A_n^\beta \bar{b}_{nk} = (n+\beta) A_{n-k}^{\beta-2} - \beta A_{n-k}^{\beta-1}$$

because $n A_n^\beta \bar{b}_{nk} = \Delta(k A_{n-k}^{\beta-1})$ and $k A_{n-k}^{\beta-1} = (n+\beta) A_{n-k}^{\beta-1} - \beta A_{n-k}^{\beta}$ (see [6], p. 204).

In view of (18) and (19) we have $g_{kk} = A_k^\alpha \varepsilon_k / A_k^\beta$ and the necessary condition (7) of Lemma 1 reduces to (11).

In order to discover other effective necessary conditions, we choose $d_n = A_n^\beta / A_n^{\beta+i\varphi}$ for real $\varphi \neq 0$; then from (19), using formulas (16) and (15), we obtain

(20) $$\sum_{n=k}^{\infty} \bar{b}_{nk} d_n = h / A_k^{1+i\varphi},$$

where $h = i\varphi(1+i\varphi)^{-1}$. Since $C^0 \subset C^\alpha$ when $\alpha > 0$, then assuming $\alpha = 0$ in (18) we deduce

$$\sum \left| \sum_{n=k}^{\infty} g_{nk} d_n \right| = \sum \left| \varepsilon_k \sum_{n=k}^{\infty} \bar{b}_{nk} d_n \right| = |h| \sum |\varepsilon_k / A_k^{1+i\varphi}| < \infty$$

from the condition (6) of Lemma 1. This means that for $\varepsilon_n \in (\mathfrak{C}^\alpha, |\mathfrak{C}^\beta|)$ it is necessary to fulfill the condition (13).

If $\alpha > 0$, then from (18) and (20) we conclude that

$$\sum_{n=k}^{\infty} g_{nk} d_n = \sum_{\nu=k}^{\infty} \xi_{\nu k} \varepsilon_\nu \sum_{n=\nu}^{\infty} \bar{b}_{n\nu} d_n = h A_k^\alpha \Delta^\alpha (\varepsilon_k / A_k^{1+i\varphi}),$$

since in view of (13) the repeated series is absolutely convergent. This together with condition (6) of Lemma 1 yields the necessity of

(21) $$\sum A_k^\alpha |\Delta^\alpha (\varepsilon_k / A_k^{1+i\varphi})| < \infty.$$

We shall prove that (11), (13) and (21) imply the necessity of the condition (12).

If α is a natural number, then by the known formula for difference of products of sequences (cf. [15], p. 129, or [6], p. 180)

(22) $$\delta_k\Delta^\alpha\varepsilon_k=\Delta^\alpha(\varepsilon_k\delta_k)-\sum_{\lambda=1}^{\alpha}\binom{\alpha}{\lambda}\Delta^\lambda\delta_k\cdot\Delta^{\alpha-\lambda}\varepsilon_{k+\lambda},$$

where $\delta_k=1/A_k^{1+i\varphi}$. Since (cf. [16], p. 288) by (16) we obtain

(23) $$\Delta^\lambda\delta_k=h_\lambda/A_k^{1+\lambda+i\varphi},$$

where $h_\lambda=1-\lambda(1+\lambda+i\varphi)^{-1}$, then with the help of (22) from (21) for natural number $\alpha=a$ we lead out

(24) $$\sum(k+1)^{\alpha-1}|\Delta^\alpha\varepsilon_k|=O(1)+O(1)\sum_{\lambda=1}^{a}\sum_{k=\lambda}^{\infty}(k+1)^{\alpha-\lambda-1}|\Delta^{\alpha-\lambda}\varepsilon_k|.$$

If $\alpha=1$, then (24) implies the necessity of condition (12), that is the

(25) $$\sum|\Delta\varepsilon_k|<\infty,$$

because (13) is necessary. The condition (25) is necessary also for arbitrary $\alpha>1$ in view of the inclusion $C^1\subset C^\alpha$. If $\alpha=2$, then from (24) the necessity of (12) follows, since the series on the right hand side in (24) are convergent by (25) and (13). Therefore the condition $\sum(k+1)|\Delta^2\varepsilon_k|<\infty$ is necessary also for $\alpha>2$ in view of the inclusion $C^2\subset C^\alpha$. Continuing in the same way, we arrive at the necessity of (12) for all integers $\alpha\ge 0$.

If α is not an integer, then we take advantage of the following fine results of Peyerimhoff (see [19], p. 9 and p. 12) which are generalizations of the formula (22) for differences of any real order $\alpha>0$.

LEMMA 2. *Let γ be a real number and let l be an integer such that $0\le l\le\gamma\le l+1$ and $l\le a$. If $\delta_k=O(1)$ and ε_k satisfies the condition*

(26) $$\varepsilon_k=O(1),$$

then

$$\Delta^\alpha(\varepsilon_k\delta_k)=\sum_{\lambda=0}^{l}\binom{\alpha}{\lambda}\Delta^\lambda\delta_k\cdot\Delta^{\alpha-\lambda}\varepsilon_{k+\lambda}+R_k,$$

where (if the following series are convergent)

$$R_k=O(1)\sup_{\nu\ge k}|\Delta^\gamma\delta_\nu|\cdot\sum_{\kappa=k}^{\infty}(\kappa+1-k)^{\gamma-\alpha-1}|\varepsilon_\kappa|.$$

LEMMA 3. *Let γ and c be real numbers and let l be an integer such that $0\le l\le\gamma\le l+1$ and $0<\alpha-\gamma<c<1$. If $\delta_k=O(1)$ and ε_k satisfies (26), then*

$$\Delta^\alpha(\varepsilon_k\delta_k)=\sum_{\lambda=0}^{l}\binom{\alpha}{\lambda}\Delta^\lambda\delta_k\cdot\Delta^{\alpha-\lambda}\varepsilon_{k+\lambda}+\varepsilon_k\Delta^\alpha\delta_k+R'_k,$$

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where

$$R'_k = O(1) \sup_{\nu \geq k} |\Delta^\gamma \delta_\nu| \cdot \{k^{\gamma-\alpha} |\varepsilon_k| + \sum_{\kappa=2k+1}^{\infty} \kappa^{\gamma-\alpha-1} |\varepsilon_\kappa|$$

$$+ \sum_{\kappa=k}^{2k} (\kappa+1-k)^{\gamma+c-\alpha-1} |\Delta^c \varepsilon_\kappa| + k^{\gamma-\alpha+1} \sum_{\kappa=2k+1}^{\infty} \kappa^{c-2} |\Delta^c \varepsilon_\kappa|\}.$$

Later on the following Lemma of Andersen (see [4], p. 31, or [9], p. 168, or [6], p. 179) will be needed.

LEMMA 4. *From the conditions* (26) *and* (12) *for any* τ *with* $0 < \tau \leq \alpha$ *it follows that* $\sum (k+1)^{\tau-1} |\Delta^\tau \varepsilon_k| < \infty$.

At first let $0 < \alpha < 1$. In Lemma 2 we choose $l = 0$ and put $0 \leq \gamma < \alpha$ at $\beta < \alpha + 1$, but $\alpha < \gamma < 1$ at $\beta \geq \alpha + 1$. With this choice of l and γ we denote

$$e_k = (k+1)^{-1} \varepsilon_k$$

and apply Lemma 2 to e_k having chosen $\delta_k = (k+1)/A_k^{1+i\varphi} = (1+i\varphi)/A_{k+1}^{i\varphi}$, which is admissible, because from (13) it follows that $e_k = O(1)$. Thus

$$\delta_k \Delta^\alpha e_k = \Delta^\alpha (\varepsilon_k / A_k^{1+i\varphi}) - R_k, \tag{27}$$

where

$$R_k = O(1)(k+1)^{-\gamma} \sum_{\kappa=k}^{\infty} (\kappa+1-k)^{\gamma-\alpha-1} |\varepsilon_\kappa|,$$

since similarly to (23) in our present case $\Delta^\gamma \delta_\nu = i\varphi h_{\gamma-1}/A_{\nu+1}^{\gamma+i\varphi}$. Consequently

$$\sum (k+1)^\alpha |R_k| = O(1) \sum |e_\kappa| \sum_{k=0}^{\kappa} (k+1)^{\alpha-\gamma} (\kappa+1-k)^{\gamma-\alpha-1} \tag{28}$$

and so $\sum (k+1)^\alpha |R_k| = O(1) \sum (\kappa+1)^{\alpha-\gamma-1} |\varepsilon_\kappa| < \infty$, by condition (11) for $\beta < \alpha + 1$, since in this case we can take $\gamma \geq \beta - 1$ and then $\alpha - \gamma - 1 \leq \alpha - \beta$. If however $\beta \geq \alpha + 1$, then by our choice of γ we have $\alpha - \gamma > \alpha - 1 > -1$ and $\gamma - \alpha - 1 > -1$, after which using the formula (14) to (28) we can write $\sum (k+1)^\alpha |R_k| = O(1) \sum |e_k| < \infty$ by condition (13). Now with the aid of (27) the necessity of the condition

$$\sum (k+1)^\alpha |\Delta^\alpha e_k| < \infty \tag{29}$$

follows from (21), (13) and (11) at $0 < \alpha < 1$. Now by the formula (cf. [12], p. 77, or [6], p. 197)

$$\Delta^\alpha \varepsilon_k = (k+1+\alpha) \Delta^\alpha e_k - \alpha \Delta^{\alpha-1} e_k \tag{30}$$

and the definition of $\Delta^{\alpha-1} e_k$ we obtain

$$\sum (k+1)^{\alpha-1} |\Delta^\alpha \varepsilon_k| = O(1) \sum (k+1)^\alpha |\Delta^\alpha e_k| + O(1) \sum |e_k| < \infty \tag{31}$$

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from conditions (29) and (13). By the same token the necessity of (12) for $0<\alpha<1$ is proved.

If $\alpha \geq 1$, then, as proved above, condition (25) is necessary from which the necessity of (26) follows. Therefore we can adapt Lemma 2 immediately to ε_k, assuming $\delta_k = 1/A_k^{1+i\varphi}$. To this end we denote $a=[\alpha]$ and choose, in the Lemma 2, the parameters $l=a$ and $\gamma=\alpha-\sigma$, where $0<\sigma<\alpha-a$. Then by Lemma 2 instead of (22) we have

$$\delta_k \Delta^\alpha \varepsilon_k = \Delta^\alpha(\varepsilon_k\delta_k) - \sum_{\lambda=1}^{a} \binom{\alpha}{\lambda} \Delta^\lambda \delta_k \cdot \Delta^{\alpha-\lambda}\varepsilon_{k+\lambda} - R_k, \tag{32}$$

where in view of (23) we have

$$R_k = O(1)(k+1)^{-1-\gamma} \sum_{\kappa=k}^{\infty} (\kappa+1-k)^{\gamma-\alpha-1} |\varepsilon_\kappa|.$$

From here, denoting $K=[\kappa/2]$ and taking into account that $\gamma-\alpha-1= -\sigma-1<-1$ and $\alpha-\gamma>0$, we obtain

$$\begin{aligned}\sum (k+1)^\alpha |R_k| &= O(1) \sum |\varepsilon_\kappa| \left(\sum_{k=0}^{K} + \sum_{k=K}^{\kappa} \right) (\kappa+1-k)^{\gamma-\alpha-1}(k+1)^{\alpha-\gamma-1} \\ &= O(1) \sum |\varepsilon_\kappa| \{(K+1)^{\gamma-\alpha-1}(K+1)^{\alpha-\gamma} + (\kappa+1)^{\alpha-\gamma-1}\} \\ &= O(1) \sum |e_\kappa| + O(1) \sum (\kappa+1)^{\alpha-\gamma-1} |\varepsilon_\kappa| < \infty\end{aligned}$$

by conditions (13) and (11) if $\beta<\alpha+1$. Therefore from (32), in view of (21) and (23), the estimate (24) follows. If however $\beta \geq \alpha+1$, then we can apply to ε_k Lemma 3. In this connection we choose, as above, the parameters $l=a$ and $\gamma=\alpha-\sigma$, but we choose σ such that $0<\sigma<\frac{1}{2}$, after which we can choose the parameter $c=2\sigma$. Then by Lemma 3 instead of (32) we have

$$\delta_k \Delta^\alpha \varepsilon_k = \Delta^\alpha(\varepsilon_k\delta_k) - \varepsilon_k \Delta^\alpha \delta_k - \sum_{\lambda=1}^{a} \binom{\alpha}{\lambda} \Delta^\lambda \delta_k \cdot \Delta^{\alpha-\lambda}\varepsilon_{k+\lambda} - R_k', \tag{33}$$

where according to (23) we find $\sum (k+1)^\alpha |\varepsilon_k \Delta^\alpha \delta_k| = O(1) \sum |e_k| < \infty$ by condition (13). Applying Lemma 4 with $\tau=2\sigma$ and the formula (14), we obtain

$$\begin{aligned}\sum (k+1)^\alpha |R_k'| = O(1) \Big\{ &\sum |e_k| + \sum (k+1)^{\sigma-1} \sum_{\kappa=2k+1}^{\infty} \kappa^{-\sigma-1} |\varepsilon_\kappa| \\ &+ \sum (k+1)^{\sigma-1} \sum_{\kappa=k}^{\infty} (\kappa+1-k)^{\sigma-1} |\Delta^{2\sigma}\varepsilon_\kappa| \\ &+ \sum \sum_{\kappa=2k+1}^{\infty} \kappa^{2\sigma-2} |\Delta^{2\sigma}\varepsilon_\kappa| \Big\} \\ = O(1) + O(1) &\sum |e_\kappa| + O(1) \sum (\kappa+1)^{2\sigma-1} |\Delta^{2\sigma}\varepsilon_\kappa| < \infty\end{aligned}$$

by the conditions (13) and (25); since (26) follows from (25) and $0<2\sigma<1$, so

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that Lemma 4 is applicable. As we remarked above, condition (25) is necessary if $\alpha \geq 1$. Thus for $\beta \geq \alpha+1$, in view of (21) and (23), the evaluation (24) also follows.

Having proved the inequality (24), we can easily prove the necessity of condition (12) also for non-integer $\alpha > 1$. In fact, if $a=1$, then (12) follows from (24), since $0 < \alpha - a < 1$ and the series on the right hand side in (24) converges by the necessity of (12) for $0 < \alpha < 1$ and the inclusion $C^{\alpha-a} \subset C^{\alpha}$. If $a=2$, then $0 < \alpha - 2 < 1$ and $1 < \alpha - 1 < 2$, and applying what was proved above in the cases $0 < \alpha < 1$ and $1 < \alpha < 2$ the convergence of both series on the right hand side in (24) is necessary and consequently condition (12) is necessary. Continuing in the same way we also arrive at the necessity of (12) for all non-integers $\alpha > 1$. Consequently (12) is necessary for $\varepsilon_n \in (\mathfrak{C}^{\alpha}, |\mathfrak{C}^{\beta}|)$ for all real $\alpha \geq 0$.

We will now prove that the necessary conditions (11), (12) and (13) are sufficient in order that $\varepsilon_n \in (\mathfrak{C}^{\alpha}_{O}, |\mathfrak{C}^{\beta}|)$ and $\varepsilon_n \in (\mathfrak{C}^{\alpha}, |\mathfrak{C}^{\beta}|)$. To this end we will first prove, for the case $\beta \leq \alpha+1$, that (8) follows from (11) and (12), which in view of (18) and (19) means

$$\sum A_k^{\alpha} \sum_{n=k}^{\infty} (nA_n^{\beta})^{-1} \left|(n+\beta)\Delta^{\alpha}(A_{n-k}^{\beta-2}\varepsilon_k) - \beta\Delta^{\alpha}(A_{n-k}^{\beta-1}\varepsilon_k)\right| < \infty. \tag{34}$$

Now together with $a=[\alpha]$ we denote $b=[\beta]$. By successively applying partial summation a times and using formula (22) we, for example, obtain

$$\Delta^{\alpha}(A_{n-k}^{\beta-1}\varepsilon_k) = \sum_{j=0}^{a} \binom{a}{j} \sum_{\nu=k}^{n-j} A_{\nu-k}^{a-\alpha-1} A_{n-j-\nu}^{j+\beta-a-2} \Delta^j \varepsilon_{\nu}.$$

Therefore in place of (34) it is sufficient to prove that for all $j=0,\ldots,a$, conditions (11) and (12) imply

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |B_j| < \infty \tag{35}$$

and

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta} |B_j'| < \infty, \tag{36}$$

where

$$B_j = \sum_{\nu=k}^{n} A_{\nu-k}^{a-\alpha-1} A_{n-\nu}^{j+\beta-a-1} \Delta^j \varepsilon_{\nu}, \qquad B_j' = \sum_{\nu=k}^{n} A_{\nu-k}^{a-\alpha-1} A_{n-\nu}^{j+\beta-a-2} \Delta^j \varepsilon_{\nu}.$$

As in [5], the proof that (11) and (12) imply (35) and (36) depends on the behavior of the sequences $(A_n^{j+\beta-a-1})$ and $(A_n^{j+\beta-a-2})$ in conditions (35) and (36). Three separate cases must be considered, but taking into account Theorem 1 it remains to consider only the case $b \geq 1$.

1. The proof is easiest if for j in formulas (35) and (36)

$$\sum |A_n^{j+\beta-a-1}|<\infty, \qquad \sum |A_n^{j+\beta-a-2}|<\infty. \tag{37}$$

Then (11) implies

$$\sum (k+1)^\alpha \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |B_j| = O(1) \sum (k+1)^\alpha \sum_{\nu=k}^{\infty} (\nu+1)^{-\beta-1} |A_{\nu-k}^{a-\alpha-1} \Delta^j \varepsilon_\nu|$$
$$= O(1) \sum (\nu+1)^{\alpha-\beta-1} |\Delta^j \varepsilon_\nu| < \infty$$

for all[5] $j=0,\ldots,a-b-1$ if $\beta>b$ (i.e., if β is not an integer) and for all $j=0,\ldots,a-\beta$ if $\beta=b$ (i.e., if β is an integer) and similarly

$$\sum (k+1)^\alpha \sum_{n=k}^{\infty} (n+1)^{-\beta} |B_j'| = O(1) \sum (\nu+1)^{\alpha-\beta} |\Delta^j \varepsilon_\nu| < \infty$$

for all[5] $j=0,\ldots,a-b$ if $\beta>b$ and for all $j=0,\ldots,a-\beta+1$ if $\beta=b$.

2. For those j in (35) and (36), for which (37) is not satisfied, we need the following Lemma of Andersen.

LEMMA 5. *If the conditions* $\sigma \ge 0, \tau > -1$ *and* $\sigma+\tau \ge 0$ *are satisfied, then condition* (26) *implies the equality* $\Delta^\tau(\Delta^\sigma \varepsilon_k) = \Delta^{\sigma+\tau} \varepsilon_k$.

The proof is due to Andersen (see [4], p. 20, or [6], p. 177). Another proof is due to Bosanquet (see [9], p. 167).

We remark, that for $\alpha \ge 1$ condition (26) follows from conditions (13) and (12). This proof may be found in [5], p. 61, Lemma 7, and in [6], p. 198–201. Consequently, if $j=0$, then from the definition of difference and for $j \ge 1$ by Lemma 5 from condition (26), we have (cf. [5], p. 52)

$$B_j = C_j - D_j, \qquad B_j' = C_j' - D_j',$$

where

$$C_j = A_{n-k}^{j+\beta-a-1} \Delta^{j+\alpha-a} \varepsilon_k, \qquad C_j' = A_{n-k}^{j+\beta-a-2} \Delta^{j+\alpha-a} \varepsilon_k,$$

$D_j = D_j' = 0$ for $\alpha = a$ (i.e., α is an integer), but for $\alpha > a$

$$D_j = \sum_{\kappa=0}^{n-k} A_\kappa^{j+\beta-a-2} d_{n\kappa k}, \qquad D_j' = \sum_{\kappa=0}^{n-k} A_\kappa^{j+\beta-a-3} d_{n\kappa k},$$

$$d_{n\kappa k} = \sum_{\nu=n-\kappa+1}^{\infty} A_{\nu-k}^{a-\alpha-1} \Delta^j \varepsilon_\nu.$$

Applying formula (16), we see that from condition (12) for $a=0$ and by Lemma 4 from conditions (26) and (12) for $a \ge 1$ for all $j = a-b, \ldots, a$ (i.e.,

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(5) If those $j \ge 0$ exist.

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those j for which $A_n^{j+\beta-a-1}\geq 0$) we obtain

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |C_j| = O(1) \sum (k+1)^{j+\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_k| < \infty,$$

but for all $j=a-b+1,\ldots,a$ we obtain

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta} |C_j'| = O(1) \sum (k+1)^{j+\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_k| < \infty.$$

If $\alpha > a$, then applying formula (17), as is shown on page 56 of [5], by Lemma 5 and 4 from conditions (26) and (12) for $a\geq 1$ the estimate

$$d_{n\kappa k} = O(1) A_{n-k-\kappa}^{a-\alpha} \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_s| \tag{38}$$

follows. Consequently, for all $j=a-b+1,\ldots,a$ from the above mentioned conditions and Lemmas, applying formulas (14) and (16)

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |D_j| = O(1) \sum (k+1)^{j-1} \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_s|$$
$$= O(1) \sum (s+1)^{j+\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_s| < \infty$$

follows, if $j\neq 0$. In the same way, applying (38), for all $j=a-b+2,\ldots,a$ conditions (26) and (12) imply

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta} |D_j'| = O(1) \sum (s+1)^{j+\alpha-a-1} |\Delta^{j+\alpha-a}\varepsilon_s| < \infty.$$

If $b=a+1$, then $j=a-b+1=0$ and $(k+1)^{j-1}\neq O(1)A_k^{j-1}$. In this case instead of D_0 it is more convenient to consider B_0. By formula (30) we have $B_0=(a-\alpha)T+(k+1+\alpha-a)V$, where (cf. [5], p. 66) $T=\Delta^{\alpha-a-1}(A_{n-k}^{\beta-b}e_k)$ and $V=\Delta^{\alpha-a}(A_{n-k}^{\beta-b}e_k)$. By means of formulas (16) and (14), from condition (13) we deduce

$$\sum (k+1)^{\alpha} \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |T| = O(1) \sum |e_\nu| < \infty.$$

Since $e_k=O(1)$, we can represent V in the form of a difference of expressions C_0 and D_0, in which ε_k is substituted by e_k. Therefore in exactly the same way as for C_j and D_j we obtain the result

$$\sum (k+1)^{\alpha+1} \sum_{n=k}^{\infty} (n+1)^{-\beta-1} |V| = O(1) \sum (s+1)^{\alpha-a} |\Delta^{\alpha-a}e_s|,$$

but from the proof of (31) we see, that from (30) it follows that

$$\sum (k+1)^{\alpha-a} |\Delta^{\alpha-a}e_k| = O(1) \sum |e_k| + O(1) \sum (k+1)^{\alpha-a-1} |\Delta^{\alpha-a}\varepsilon_k| < \infty.$$

from the conditions (13), (26) and (12), using Lemma 4.

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3. We have not yet proved the corresponding estimates for D_{a-b} and D'_{a-b+1} for $\alpha>a$ with $\beta>b$. Instead of this, it is more convenient to consider B_{a-b} and B'_{a-b+1}.

We have (by the definition of difference) $B_{a-b}=\Delta^{\alpha-a}(A^{\beta-b-1}_{n-k}\Delta^{a-b}\varepsilon_k)$. Using Lemma 2 with $\delta_k=A^{\beta-b-1}_{n-k}$ and $\alpha-a$ instead of α, putting $0<\gamma<\alpha-a$ and $l=0$, we obtain $B_{a-b}=A^{\beta-b-1}_{n-k}\Delta^{\alpha-b}\varepsilon_k+R_k$, since from $b\geq 1$ we have $a\geq 1$ and so condition (26) is satisfied and by Lemma 5 the equation $\Delta^{\alpha-a}(\Delta^{a-b}\varepsilon_k)=\Delta^{\alpha-b}\varepsilon_k$ holds. From here by formula (16) we deduce

$$\sum (k+1)^\alpha \sum_{n=k}^{\infty} (n+1)^{-\beta-1}A^{\beta-b-1}_{n-k}\,|\Delta^{\alpha-b}\varepsilon_k|=O(1)\sum (k+1)^{\alpha-b-1}\,|\Delta^{\alpha-b}\varepsilon_k|<\infty$$

from conditions (26) and (12) by Lemma 4. Hence $\Delta^\gamma\delta_\nu=O(1)$, then

$$\begin{aligned}\sum (k+1)^\alpha \sum_{n=k}^{\infty} (n+1)^{-\beta-1}\,|R_k| &= O(1)\sum (k+1)^{\alpha-\beta}\sum_{\kappa=k}^{\infty}(\kappa+1-k)^{\gamma-(\alpha-a)-1}\,|\Delta^{a-b}\varepsilon_\kappa|\\ &= O(1)\sum (\kappa+1)^{\alpha-\beta}\,|\Delta^{a-b}\varepsilon_\kappa|<\infty\end{aligned}$$

from condition (11), because denoting $K=[\kappa/2]$, since $\gamma<\alpha-a$ we obtain

$$\begin{aligned}\sum_{k=0}^{\kappa}(\kappa+1-k)^{\gamma-\alpha+a-1}(k+1)^{\alpha-\beta}&=O(1)(K+1)^{\gamma-\alpha+a-1}L_\kappa\\ &\quad+O(1)(\kappa+1)^{\alpha-\beta}=O(1)(\kappa+1)^{\alpha-\beta},\end{aligned}$$

where $L_\kappa=(K+1)^{\alpha-\beta+1}$ for $\beta<\alpha+1$, but $L_\kappa=\ln(K+2)$ for $\beta=\alpha+1$.

We estimate B'_{a-b+1} using the calculations of [5]. Applying partial summation and formula (14) we obtain

$$B'_{a-b+1}=E'+F',$$

where

$$E'=A^{a-\alpha+\beta-b-1}_{n-k}\Delta^{a-b+1}\varepsilon_{n+1},$$

$$F'=\sum_{\nu=k}^{n} f_{n\nu k}\Delta^{a+2-b}\varepsilon_\nu,\qquad f_{n\nu k}=\sum_{\kappa=k}^{\nu}A^{a-\alpha-1}_{\kappa-k}A^{\beta-b-1}_{n-\kappa}.$$

If $a-\alpha+\beta-b\leq 0$, then from condition (11) it follows that

$$\sum (k+1)^\alpha \sum_{n=k}^{\infty}(n+1)^{-\beta}\,|E'|=O(1)\sum (n+1)^{\alpha-\beta}\,|\Delta^{a-b+1}\varepsilon_{n+1}|<\infty. \tag{39}$$

If $a-\alpha+\beta-b>0$, then $a\geq 1$ (since $\beta>\alpha-a+1$) and by Lemma 4 and formula (14), conditions (26) and (12) yield that (39) holds with $\alpha-\beta$ replaced

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by $a-b$. In order to estimate F' we observe that condition (13) implies

$$\Delta^{a+2-b}\varepsilon_\nu=\sum_{s=\nu}^{\infty}A_{s-\nu}^{\alpha-a-2}\Delta^{\alpha+1-b}\varepsilon_s \tag{40}$$

since $\max\{\alpha-a-2, b-\alpha-2\}<-1$ and therefore

$$\sum_{s=\nu}^{t}|A_{s-\nu}^{\alpha-a-2}A_{t-s}^{b-\alpha-2}|=O(1)(t-\nu+1)^{-1}.$$

Substituting the equation (40) into F', we see that it can be expressed as the sum of two parts, $F'=G'+H'$, where

$$G'=\sum_{s=k}^{n}\Delta^{\alpha+1-b}\varepsilon_s\cdot\sum_{\nu=k}^{s}A_{s-\nu}^{\alpha-a-2}f_{n\nu k},\qquad H'=\sum_{s=n+1}^{\infty}\Delta^{\alpha+1-b}\varepsilon_s\cdot\sum_{\nu=k}^{n}A_{s-\nu}^{\alpha-a-2}f_{n\nu k}.$$

Inverting the order of summation twice, using formula (14), afterwards partial summation and once again inverting the order of summation, we obtain $G'=C'_{a-b+1}+J'$, and, consequently,

$$F'=C'_{a-b+1}+J'+H',$$

where

$$J'=\sum_{\kappa=k}^{n}A_{n-\kappa}^{\beta-b-2}\sum_{s=\kappa+1}^{n}\Delta^{\alpha+1-b}\varepsilon_s\cdot\sum_{p=k}^{\kappa}A_{p-k}^{a-\alpha-1}A_{s-p}^{\alpha-a-1}.$$

The expression C'_{a-b+1} was estimated above (case 2). Using formula (17) and taking into account that $\beta>1$, similarly to estimate (12) of [5], we obtain

$$\sum_{n=k}^{\infty}(n+1)^{-\beta}|J'|=O(1)\sum_{s=k}^{\infty}(s+1)^{-\beta}A_{s-k}^{\beta-b-1}|\Delta^{\alpha+1-b}\varepsilon_s|.$$

From here by inversion of the order of summation and using formula (14) we find

$$\sum(k+1)^{\alpha}\sum_{n=k}^{\infty}(n+1)^{-\beta}|J'|=O(1)\sum(s+1)^{\alpha-b}|\Delta^{\alpha+1-b}\varepsilon_s|<\infty \tag{41}$$

by Lemma 4 from conditions (26) and (12) for $a\geq 1$ and from condition (12) for $a=0$. Further, choosing a number η such that $0<\eta<\min\{\alpha-a, \beta-b\}$ and applying (17) we find

$$H'=O(1)A_{n-k}^{\beta-b-\eta-1}\sum_{s=n}^{\infty}A_{s-n}^{\eta-1}|\Delta^{\alpha+1-b}\varepsilon_s|$$

(see [5], p. 65) after which in a similar manner to (41) we obtain

$$\sum(k+1)^{\alpha}\sum_{n=k}^{\infty}(n+1)^{-\beta}|H'|=O(1)\sum(s+1)^{\alpha-b}|\Delta^{\alpha+1-b}\varepsilon_s|<\infty.$$

Now let $\beta > \alpha + 1$. Taking into account that which has been proved, for $\varepsilon_n \in (\mathfrak{C}^\alpha, |\mathfrak{C}^{\alpha+1}|)$, conditions (11) and (12) are sufficient, and by virtue of the conclusion $|C^{\alpha+1}| \subset |C^\beta|$ also sufficient for $\varepsilon_n \in (\mathfrak{C}^\alpha, |\mathfrak{C}^\beta|)$. The necessity of these conditions was proved above.

Thus, in view of Theorem 1 with $A = C^\alpha$ and $B = C^\beta$, we have proved Theorem 2.

For integral α and $-1 < \operatorname{Re} \beta < 2$ Theorem 2 is proved by the author (see [7], Corollary 2.4), for integral α and $\beta = 0$ by Abel and Tjurnpu (see [2], Theorem 12) and for complex α and $\beta = 0$ by Abel (see [1], Theorem 11).

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