1
Operator calculus

The operator calculus developed by Feynman [Fey51] makes it possible to represent functions of (noncommuting) operators as path integrals, with the integrand being the path-ordered exponential of operators, the order of which is controlled by a parameter that varies along the trajectory. This procedure is termed *Feynman disentangling*. It is also applicable to functions of matrices (say, γ-matrices which are associated with a spinor particle). When applied to the evolution operator, this procedure results in the standard path-integral representation of quantum mechanics.

In this chapter we first demonstrate the general technique using the simplest example, a free propagator in Euclidean space, and then consider the path-integral representation of quantum mechanics, as well as propagators in an external electromagnetic field.

1.1 Free propagator

Let us first consider the simplest propagator of a free scalar field which is given in the operator formalism by the vacuum expectation value of the T-product*

$$G(x-y) \quad = \quad \langle 0| \boldsymbol{T} \boldsymbol{\varphi}(x)\, \boldsymbol{\varphi}(y)\, |0\rangle \tag{1.1}$$

with $\boldsymbol{\varphi}$ being the field-operator.

The T-product (1.1) obeys the equation

$$\left(-\partial^2 - m^2\right) G(x-y) \quad = \quad \mathrm{i}\, \delta^{(d)}(x-y)\,, \tag{1.2}$$

where $d = 4$ is the dimension of space-time, however the formulas are applicable at any value of d. In the operator formalism, Eq. (1.2) is a

* The ordered products of operators were introduced by Dyson [Dys49]. This paper and other classical papers on quantum electrodynamics are collected in the book edited by Schwinger [Sch58].

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consequence of the free equations

$$\left.\begin{array}{rcl}\left(-\partial^2-m^2\right)\boldsymbol{\varphi}(x)\big|\,0\big\rangle &=& 0\,,\\ \big\langle 0\,\big|\left(-\partial^2-m^2\right)\boldsymbol{\varphi}(x) &=& 0\end{array}\right\} \tag{1.3}$$

and canonical equal-time commutators

$$\left.\begin{array}{rcl}\left[\boldsymbol{\varphi}(t,\vec{x})\,,\dot{\boldsymbol{\varphi}}(t,\vec{y})\right] &=& \mathrm{i}\,\delta^{(d-1)}(\vec{x}-\vec{y})\,,\\ \left[\boldsymbol{\varphi}(t,\vec{x})\,,\boldsymbol{\varphi}(t,\vec{y})\right] &=& 0\,.\end{array}\right\} \tag{1.4}$$

The delta-function $\delta^{(1)}(x_0-y_0)$ emerges when $(\partial/\partial x_0)^2$ is applied to the operator of the T-product in (1.1).

Problem 1.1 Derive Eq. (1.2) in the operator formalism.

Solution Let us apply the operator on the left-hand side (LHS) of Eq. (1.2) to the T-product which is defined by

$$\boldsymbol{T}\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y) \;=\; \theta(x_0-y_0)\,\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y)+\theta(y_0-x_0)\,\boldsymbol{\varphi}(y)\,\boldsymbol{\varphi}(x) \tag{1.5}$$

with

$$\theta(x_0-y_0) \;=\; \begin{cases}1 \text{ for } x_0\geq y_0\\ 0 \text{ for } x_0<y_0\,.\end{cases} \tag{1.6}$$

Equation (1.3) implies a nonvanishing result to emerge only when $(\partial/\partial x_0)^2$ is applied to the operator of the T-product. One obtains

$$\begin{aligned}\left(-\partial^2-m^2\right)\langle 0|\,\boldsymbol{T}\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y)\,|0\rangle &= -\frac{\partial}{\partial x_0}\,\langle 0|\,\boldsymbol{T}\dot{\boldsymbol{\varphi}}(x)\,\boldsymbol{\varphi}(y)\,|0\rangle\\ &= \delta^{(1)}(x_0-y_0)\,\langle 0|\,[\boldsymbol{\varphi}(y)\,,\dot{\boldsymbol{\varphi}}(x)]\,|0\rangle\\ &= \mathrm{i}\,\delta^{(d)}(x-y)\,,\end{aligned} \tag{1.7}$$

where the canonical commutation relations (1.4) are used.

The explicit solution to Eq. (1.2) for the free propagator is well-known and is most simply given by the Fourier transform:

$$G(x-y) \;=\; \int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}. \tag{1.8}$$

An extra $\mathrm{i}\varepsilon$ (with $\varepsilon\to+0$) in the denominator is due to the T-product in the definition (1.1) and unambiguously determines the integral over p_0. The propagator (1.8) is known as the Feynman propagator that respects causality.

Problem 1.2 Perform the Fourier transformation of the free momentum-space propagator in the energy p_0:

$$G_\omega(t-t') \;=\; \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\,\mathrm{e}^{\mathrm{i}p_0(t-t')}\frac{\mathrm{i}}{p_0^2-\omega^2+\mathrm{i}\varepsilon},\qquad \omega \;=\; \sqrt{\vec{p}^{\,2}+m^2}\,. \tag{1.9}$$

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Solution The poles of the momentum-space propagator are at

$$p_0 \;=\; \pm\omega \mp \mathrm{i}\varepsilon\,. \tag{1.10}$$

For $t > t'$ ($t < t'$), the contour of integration can be closed in the upper (lower) half-plane which gives

$$\begin{aligned} G_\omega(t-t') \;&=\; \theta(t-t')\frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{2\omega} + \theta(t'-t)\frac{\mathrm{e}^{\mathrm{i}\omega(t-t')}}{2\omega} \\ &=\; \frac{\mathrm{e}^{-\mathrm{i}\omega|t-t'|}}{2\omega}\,. \end{aligned} \tag{1.11}$$

The Green function (1.11) obeys the equation

$$\left(-\frac{\partial^2}{\partial t^2} - \omega^2\right) G_\omega(t-t') \;=\; \mathrm{i}\,\delta^{(1)}(t-t') \tag{1.12}$$

and therefore coincides with the causal Green function for a harmonic oscillator with frequency ω.

Remark on operator notations

In mathematical language, the Green function $G(x-y)$ is termed the *resolvent* of the operator on the LHS of Eq. (1.2), and is often denoted as the matrix element of the inverse operator

$$G(x-y) \;=\; \left\langle y \left| \frac{\mathrm{i}}{-\partial^2 - m^2} \right| x \right\rangle. \tag{1.13}$$

The operators act in an infinite-dimensional Hilbert space, the elements of which in Dirac's notation [Dir58] are the *bra* and *ket* vectors $\langle g|$ and $|f\rangle$, respectively. The coordinate representation emerges when these vectors are chosen to be the eigenstates of the position operator $\boldsymbol{x}_\mu$:

$$\boldsymbol{x}_\mu |x\rangle \;=\; x_\mu |x\rangle\,. \tag{1.14}$$

These basis vectors obey the completeness condition

$$\int \mathrm{d}^d x\, |x\rangle\langle x| \;=\; 1\,, \tag{1.15}$$

while the wave functions, associated with $\langle g|$ and $|f\rangle$, are given by

$$\langle g\,|\,x\rangle \;=\; g(x)\,, \qquad \langle x\,|\,f\rangle \;=\; f(x)\,. \tag{1.16}$$

These wave functions appear in the expansions

$$|f\rangle \;=\; \int \mathrm{d}^d x\, f(x)|x\rangle\,, \qquad \langle g| \;=\; \int \mathrm{d}^d y\, g(y)\langle y|\,. \tag{1.17}$$

The action of a linear operator $\boldsymbol{O}$ on the bra and ket vectors in Hilbert space is determined by its matrix element $\langle y\,|\boldsymbol{O}|\,x\rangle$, which is also known as the *kernel* of the operator $\boldsymbol{O}$ and is denoted by

$$\langle y\,|\boldsymbol{O}|\,x\rangle \;=\; O(y,x)\,. \tag{1.18}$$

Using the expansion (1.17), one obtains

$$\langle g\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x \int \mathrm{d}^d y\, g(y)\, O(y,x)\, f(x)\,. \tag{1.19}$$

Since the kernel of the unit operator is the delta-function,

$$\langle y\,|\mathbf{1}|\,x\rangle \;=\; \langle y|x\rangle \;=\; \delta^{(d)}(x-y)\,, \tag{1.20}$$

the formula

$$\langle y\,|\boldsymbol{O}|\,x\rangle \;=\; \boldsymbol{O}\,\delta^{(d)}(x-y) \tag{1.21}$$

can also be written down as a direct consequence of Eq. (1.20), where the operator $\boldsymbol{O}$ on the right-hand side (RHS) acts on the variable x.

Therefore, when the operator acts on a function $f(x)$, the result is expressed via the kernel by the standard formula

$$\boldsymbol{O}f(y) \;\equiv\; \langle y\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x\, O(y,x)\, f(x)\,. \tag{1.22}$$

Equation (1.21) is obviously reproduced when f is substituted by a delta-function, while Eq. (1.19) takes the form

$$\langle g\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x\, g(x)\,\boldsymbol{O} f(x)\,. \tag{1.23}$$

If space-time is approximated by a discrete set of points, then the operator $\boldsymbol{O}$ is approximated by a matrix with elements $\langle y\,|\boldsymbol{O}|\,x\rangle$.

1.2 Euclidean formulation

Equation (1.8) can be obtained alternatively by inverting the operator on the LHS of Eq. (1.2). Before doing that, it is convenient to make an analytic continuation in the time-variable t, and to pass to the Euclidean formulation of quantum field theory (QFT) where one substitutes

$$t \;=\; -\mathrm{i}\, x_4\,. \tag{1.24}$$

The four-momentum operator in Minkowski space reads as

$$\boldsymbol{p}^{\mu}_{\mathrm{M}} \;=\; \mathrm{i}\,\partial^{\mu}_{\mathrm{M}} \;\equiv\; \left(\mathrm{i}\frac{\partial}{\partial t}, -\mathrm{i}\frac{\partial}{\partial \vec{x}}\right) \qquad \boxed{\text{Minkowski space}}\,, \tag{1.25}$$

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while its Euclidean counterpart is given by

$$\boldsymbol{p}_{\mathrm{E}}^{\mu} \;=\; -\mathrm{i}\,\partial_{\mathrm{E}}^{\mu} \;\equiv\; \left(-\mathrm{i}\frac{\partial}{\partial\vec{x}}, -\mathrm{i}\frac{\partial}{\partial x_4}\right) \qquad \boxed{\text{Euclidean space}}\,. \tag{1.26}$$

These two formulas together with Eq. (1.24) yield

$$E \;\equiv\; p_0 \;=\; -\mathrm{i}\,p_4 \tag{1.27}$$

for the relation between energy and the fourth component of the Euclidean four-momentum.

The passage to Euclidean space results in changing the Minkowski signature of the metric $g_{\mu\nu}$ to the Euclidean one:*

$$\begin{array}{ccc} (+---) & \longrightarrow & (++++) \\ \boxed{\text{Minkowski signature}} & \longrightarrow & \boxed{\text{Euclidean signature}}\,. \end{array} \tag{1.28}$$

As such, one finds

$$p_{\mathrm{M}}^2 = p_0^2 - \vec{p}^{\,2} \quad\longrightarrow\quad -p_{\mathrm{E}}^2 = -\vec{p}^{\,2} - p_4^2\,. \tag{1.29}$$

The exponent in the Fourier transformation changes analogously:

$$-p_\mu x^\mu = -Et + \vec{p}\vec{x} \quad\longrightarrow\quad p_{\mathrm{E}}^{\mu} x_{\mathrm{E}}^{\mu} = \vec{p}\vec{x} + p_4 x_4\,. \tag{1.30}$$

This reproduces the standard Fourier transformation in Euclidean space

$$\left.\begin{array}{rcl} f(p) & = & \displaystyle\int \mathrm{d}^d x\, \mathrm{e}^{-\mathrm{i}px} f(x)\,, \\ f(x) & = & \displaystyle\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}px} f(p)\,. \end{array}\right\} \tag{1.31}$$

We shall use the same notation v^μ for a four-vector in Minkowski and Euclidean spaces:

$$\left.\begin{array}{rcll} v_{\mathrm{M}}^{\mu} & = & (v_0, \vec{v}) & \boxed{\text{Minkowski space}}\,, \\ v_{\mathrm{E}}^{\mu} & = & (\vec{v}, v_4) & \boxed{\text{Euclidean space}}\,, \end{array}\right\} \tag{1.32}$$

* An older generation will be familiar with the Euclidean notation which is used throughout the book by Akhiezer and Berestetskii [AB69]. In contrast, the two canonical books on quantum field theory by Bogoliubov and Shirkov [BS76] and by Bjorken and Drell [BD65] use the Minkowskian notation instigated by Feynman. The modern generation of textbooks on quantum field theory includes those by Brown [Bro92] and Weinberg [Wei98].

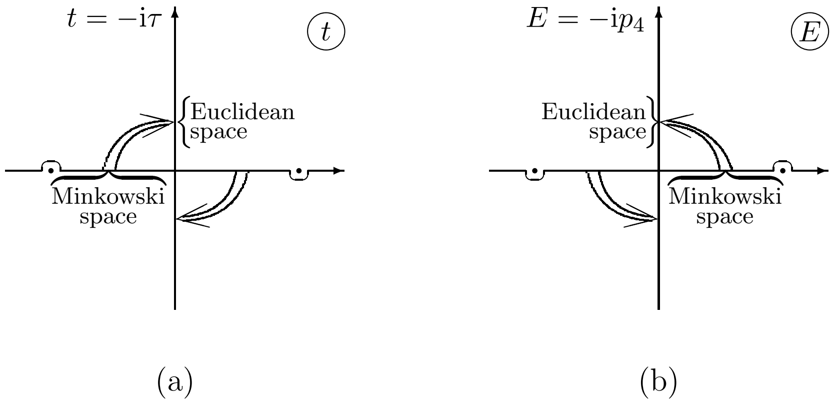

Fig. 1.1. Direction of Wick's rotation from Minkowski to Euclidean space (indicated by the arrows) for (a) time and (b) energy. The dots represent singularities of a free propagator in (a) coordinate and (b) momentum spaces. The contours of integration in Minkowski space are associated with causal Green functions. They can obviously be deformed in the directions of the arrows.

with

$$v_0 = -\mathrm{i}v_4 \,. \tag{1.33}$$

The only difference resides in the metric. We do not distinguish between upper and lower indices in Euclidean space.

Using Eqs. (1.24) and (1.26), we see that in Euclidean space Eq. (1.2) takes the form

$$\left(-\partial^2 + m^2\right) G(x-y) = \delta^{(d)}(x-y) \tag{1.34}$$

with a positive sign in front of m^2.

The passage to the Euclidean formulation is justified in perturbation theory where it is associated with the Wick rotation. The direction in which the rotation is performed is unambiguously prescribed by the $+\mathrm{i}\varepsilon$ term in Eq. (1.8), and is depicted in Fig. 1.1. The variable $t = x_0$ rotates through $-\pi/2$, while $E = p_0$ rotates through $\pi/2$.

Figure 1.1a explains the sign in Eq. (1.24). Figure 1.1b and Eq. (1.27) implies that the integration over p_4 goes in the opposite direction, so that

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}p_0}{2\pi} \cdots = \mathrm{i} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_4}{2\pi} \cdots \,. \tag{1.35}$$

Thus when passing into Euclidean variables, Eq. (1.8) becomes

$$G(x-y) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}p(y-x)} \frac{1}{p^2+m^2}. \tag{1.36}$$

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Note that the RHS of Eq. (1.36) is nothing but the Fourier transform of the free momentum-space Euclidean propagator, and there is no need to retain an iε in the denominator since the integration prescription is now unambiguous.

It is now clear why we keep the same notation for the coordinate-space Green functions: the Feynman propagator in Minkowski space and the Euclidean propagator. They are the same analytic function of the time-variable.

Problem 1.3 Repeat the calculation of Problem 1.2 in Euclidean space.

Solution According to Eq. (1.36) we need to calculate

$$G_\omega(\tau-\tau') \;=\; \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_4}{2\pi}\, \mathrm{e}^{\mathrm{i}p_4(\tau'-\tau)} \frac{1}{p_4^2+\omega^2}\,. \tag{1.37}$$

The integral on the RHS can be calculated for $\tau > \tau'$ ($\tau < \tau'$) by closing the contour in the lower (upper) half-plane, and taking the residues at $p_4 = -\mathrm{i}\omega$ ($p_4 = \mathrm{i}\omega$), respectively. This yields

$$\begin{aligned} G_\omega(\tau-\tau') \;&=\; \theta(\tau-\tau')\frac{\mathrm{e}^{\omega(\tau'-\tau)}}{2\omega} + \theta(\tau'-\tau)\frac{\mathrm{e}^{\omega(\tau-\tau')}}{2\omega} \\ &=\; \frac{\mathrm{e}^{-\omega|\tau-\tau'|}}{2\omega}\,. \end{aligned} \tag{1.38}$$

The Euclidean Green function (1.38) can obviously be obtained from the Minkowskian one, Eq. (1.11), by the substitution

$$\tau \;=\; \mathrm{i}t\,, \qquad \tau' \;=\; \mathrm{i}t' \tag{1.39}$$

and vice versa. $G_\omega(\tau-\tau')$ obeys the equation

$$\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) G_\omega(\tau-\tau') \;=\; \delta^{(1)}(\tau-\tau') \tag{1.40}$$

and, therefore, is the Green function for a Euclidean harmonic oscillator with frequency ω.

As we shall see in a moment, the Euclidean formulation makes path integrals well-defined, and allows nonperturbative investigations analogous to statistical mechanics to be carried out. There are no reasons, however, why Minkowski and Euclidean formulations should always be equivalent nonperturbatively.

Remark on Euclidean γ-matrices

The γ-matrices in Minkowski space satisfy

$$\{\gamma_{\mathrm{M}}^\mu, \gamma_{\mathrm{M}}^\nu\} \;=\; 2\,g^{\mu\nu}\,\mathbb{I}\,, \tag{1.41}$$

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where $\mathbb{I}$ denotes the unit matrix. Therefore, γ_0 is Hermitian while the Minkowskian spatial γ-matrices are anti-Hermitian.

Analogously, the Euclidean γ-matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}\,\mathbb{I}\,, \tag{1.42}$$

so that all of them are Hermitian. We compose them from 2×2 matrices as

$$\gamma_4 = \gamma_0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{1.43}$$

and

$$\vec{\gamma} = \begin{pmatrix} 0 & -\mathrm{i}\vec{\sigma} \\ \mathrm{i}\vec{\sigma} & 0 \end{pmatrix}, \tag{1.44}$$

where $\vec{\sigma}$ are the usual Pauli matrices. Note that the Euclidean spatial γ-matrices differ from the Minkowskian ones by a factor of i.

The free Dirac equation in Euclidean space reads as

$$\left(\hat{\partial} + m\right)\psi = 0\,, \qquad \hat{\partial} = \gamma_\mu \partial_\mu \tag{1.45}$$

or

$$(\mathrm{i}\widehat{\boldsymbol{p}} + m)\,\psi = 0 \tag{1.46}$$

with $\boldsymbol{p}$ given by Eq. (1.26).

1.3 Path-ordering of operators

There are no problems in defining a function of an operator A, say via the Taylor series. For instance,

$$\mathrm{e}^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{1.47}$$

However, it is more complicated to define a function of several noncommuting operators (or matrices), e.g. A and B having

$$[A, B] \neq 0\,, \tag{1.48}$$

since the order of operators is now essential. In particular, one has

$$\mathrm{e}^{A+B} \neq \mathrm{e}^A\,\mathrm{e}^B, \tag{1.49}$$

so that the law of addition of exponents fails. Certainly, the exponential on the LHS is a well-defined function of $A + B$, but since A and B

are intermixed in the Taylor expansion, this expansion is of little use in practice. We would like to have an expression where all Bs are written, say, to the right of all As. Generically, this is a problem of representing a symmetric ordering of operators via a normal ordering.

This can be achieved by the following formal trick [Fey51].

Let us write

$$\begin{aligned} e^{A+B} &= \lim_{M\to\infty}\left[1+\frac{1}{M}(A+B)\right]^M \\ &= \lim_{M\to\infty}\underbrace{\left[1+\frac{1}{M}(A+B)\right]\cdots\left[1+\frac{1}{M}(A+B)\right]}_{M \text{ times}}. \end{aligned} \tag{1.50}$$

The structure of the product on the RHS prompts us to introduce an index i running from 1 to M and replace $(A+B)$ in each multiplier by (A_i+B_i). Therefore, one writes

$$\begin{aligned} e^{A+B} &= \lim_{M\to\infty}\prod_{i=1}^{M}\left[1+\frac{1}{M}(A_i+B_i)\right] \\ &= \lim_{M\to\infty}\left[1+\frac{1}{M}(A_M+B_M)\right]\cdots\left[1+\frac{1}{M}(A_1+B_1)\right], \end{aligned} \tag{1.51}$$

where the index i controls the order of the operators which are all treated *differently*. The ordering is such that the larger i is, the later the operator with the index i acts. This order of operators is prescribed by quantum mechanics, where initial and final states are represented by ket and bra vectors, respectively.

Equation (1.51) can be rewritten as

$$e^{A+B} = \boldsymbol{P}\lim_{M\to\infty}\exp\left[\frac{1}{M}\sum_{i=1}^{M}(A_i+B_i)\right], \tag{1.52}$$

where the symbol $\boldsymbol{P}$ denotes the ordering operation. There is no ambiguity on the RHS of Eq. (1.52) concerning ordering A_i and B_i with the same index i, since such terms are $\mathcal{O}\big(M^{-2}\big)$ and are negligible as $M\to\infty$.

To describe the continuum limit as $M\to\infty$, one introduces the continuum variable $\sigma = i/M$ which belongs to the interval $[0,1]$. The continuum limit of Eq. (1.52) reads as

$$e^{A+B} = \boldsymbol{P}\exp\left\{\int_0^1 d\sigma\,[A(\sigma)+B(\sigma)]\right\}, \tag{1.53}$$

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where $A(i/M) = A_i$ and $B(i/M) = B_i$, while the operator $A(\sigma) + B(\sigma)$ acts at order σ.

Equation (1.53) is, in fact, obvious since it only involves the operator $A + B$, which commutes with itself. For commuting operators there is no need for ordering so that $A(\sigma) + B(\sigma)$ does not depend on σ in this case. The integral in the exponent on the RHS of Eq. (1.53) can then be performed, and reproduces the LHS.

Equation (1.53) can however be manipulated as though $A(\sigma)$ and $B(\sigma)$ were just functions rather than operators since the order would be specified automatically by the path-ordering operation. This is analogous to the well-known fact that operators can be written in an arbitrary order under the T-product. Therefore, we can rewrite Eq. (1.53) as

$$\mathrm{e}^{A+B} \quad = \quad \boldsymbol{P}\, \mathrm{e}^{\int_0^1 \mathrm{d}\sigma' A(\sigma')}\, \mathrm{e}^{\int_0^1 \mathrm{d}\sigma B(\sigma)}. \tag{1.54}$$

This is the operator analog of the law of addition of exponents.

Problem 1.4 Calculate explicitly the first term of the expansion of $\exp(A + B)$ in B.

Solution Expanding the RHS of Eq. (1.54) in B, one finds

$$\mathrm{e}^{A+B} \quad = \quad \mathrm{e}^{A} + \int_0^1 \mathrm{d}\sigma\, \mathrm{e}^{\int_\sigma^1 \mathrm{d}\sigma' A(\sigma')} B(\sigma)\, \mathrm{e}^{\int_0^\sigma \mathrm{d}\sigma' A(\sigma')} + \cdots . \tag{1.55}$$

There is no need for a path-ordering sign in this formula, since the order of the operators A and B is written explicitly. There is also no ambiguity in defining the exponentials of the operator A as already explained.

Since the order is explicit, one drops the formal dependence of A and B on the ordering parameter which gives

$$\mathrm{e}^{A+B} \quad = \quad \mathrm{e}^{A} + \int_0^1 \mathrm{d}\sigma\, \mathrm{e}^{(1-\sigma)A} B\, \mathrm{e}^{\sigma A} + \cdots . \tag{1.56}$$

Formulas (1.55) and (1.56) are known from time-dependent perturbation theory in quantum mechanics.

Problem 1.5 Using Eq. (1.56), derive

$$\frac{1}{A+B} \quad = \quad \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \cdots \tag{1.57}$$

for small B.

Solution Exponentiating and using Eq. (1.56), we obtain

$$\frac{1}{A+B} = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau(A+B)}$$
$$= \int_0^\infty \mathrm{d}\tau \Big[\mathrm{e}^{-\tau A} - \tau \int_0^1 \mathrm{d}\sigma\, \mathrm{e}^{\tau(\sigma-1)A} B\, \mathrm{e}^{-\tau\sigma A} \Big] + \cdots . \tag{1.58}$$

Introducing the new variables

$$\tau_1 = \tau\,(1-\sigma)\,, \qquad \tau_2 = \tau\sigma\,, \tag{1.59}$$

we rewrite the RHS of Eq. (1.58) as

$$\frac{1}{A} - \int_0^\infty \mathrm{d}\tau_1\, \mathrm{e}^{-\tau_1 A} B \int_0^\infty \mathrm{d}\tau_2\, \mathrm{e}^{-\tau_2 A} + \cdots = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \cdots \tag{1.60}$$

which proves Eq. (1.57).

1.4 Feynman disentangling

The operator on the LHS of Eq. (1.34) can be inverted as follows:

$$G(x-y) = \frac{1}{-\partial^2 + m^2} \delta^{(d)}(x-y)$$
$$= \frac{1}{2} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{\frac{1}{2}\tau\left(\partial^2 - m^2\right)} \delta^{(d)}(x-y)$$
$$= \frac{1}{2} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{1}{2}m^2\tau}\, \boldsymbol{P}\, \mathrm{e}^{\frac{1}{2}\int_0^\tau \mathrm{d}t\, \partial^2(t)}\, \delta^{(d)}(x-y)\,, \tag{1.61}$$

where we have formally labeled the derivatives using an ordering parameter $t \in [0, \tau]$, which is an analog of σ from the previous section. This is the general procedure upon which the Feynman disentangling is built.

Since the operators ∂_μ and ∂_ν commute in the free case, we could manage without introducing the t-dependence, however the operators do not commute in general. The simple example of the nonrelativistic Hamiltonian and the propagator in an external electromagnetic field are considered later in this chapter. Other cases where the disentangling is needed are related to inverting an operator which is also a matrix in some symmetry space.

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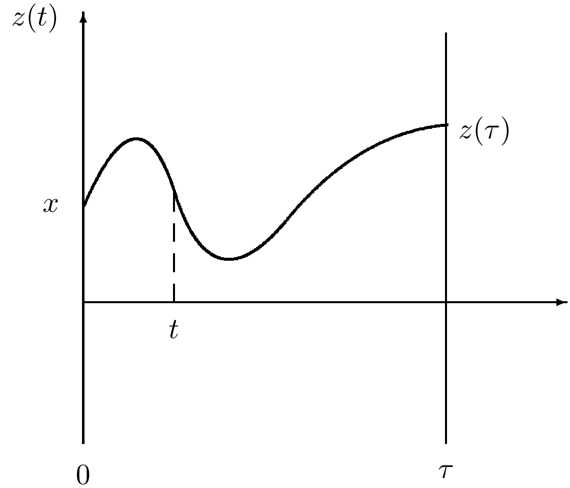

Fig. 1.2. Trajectory $z_\mu(t)$. The operator $\partial_\mu(t)$ acts at the order t.

Continuing with the disentangling, the RHS of Eq. (1.61) can be rewritten as

$$G(x-y) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{1}{2}m^2\tau} \int_{z_\mu(0)=x_\mu} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\, \dot{z}_\mu^2(t)} \times \boldsymbol{P}\, \mathrm{e}^{\int_0^\tau \mathrm{d}t\, \dot{z}_\mu(t)\partial_\mu(t)}\, \delta^{(d)}(x-y)\,, \tag{1.62}$$

where the integration runs over all trajectories $z_\mu(t)$ which begin at the point x, as depicted in Fig. 1.2.

Since the operator $\partial_\mu(t)$ acts at the order t, these operators are ordered along the trajectory $z_\mu(t)$ with $\boldsymbol{P}$, in Eq. (1.62), denoting the path-ordering operator. Note, that $\dot{z}_\nu(t)$ and $\partial_\mu(t)$ commute since

$$\partial_\mu(t)\dot{z}_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\mu\nu} = 0 \tag{1.63}$$

so that their order is not essential in Eq. (1.62). With these rules of manipulation, Eq. (1.62) can be proven by the "translation"

$$z_\mu(t) \to z'_\mu(t) = z_\mu(t) + \int_0^t \mathrm{d}t'\, \partial_\mu(t') \tag{1.64}$$

of the integration variable $z_\mu(t)$ in the Gaussian integral.

The integral over the functions $z_\mu(t)$ in Eq. (1.62) is called a *path integral* or a *functional integral*.

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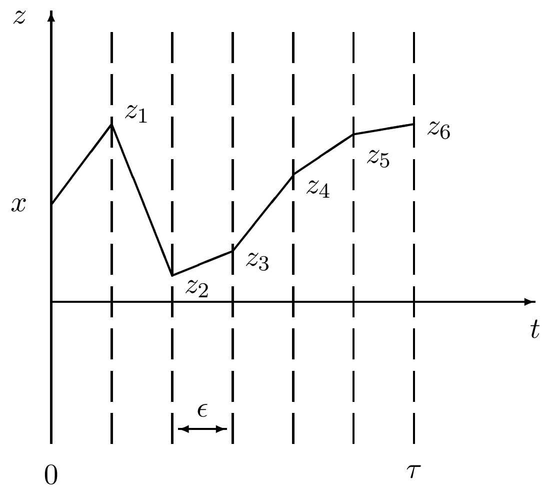

Fig. 1.3. Discretization of trajectory $z_\mu(t)$ (depicted for $M = 6$).

The continual path integral can be approximated by a finite one. To this end, let us choose M points $t_i = i\epsilon$, where ϵ is the discretization step, and $M = \tau/\epsilon$. We then connect the points

$$z_0 \;=\; x\,, \quad z_i \;=\; z(i\epsilon) \qquad i = 1, 2, \ldots, M \tag{1.65}$$

by straight lines. Such a discretization of the trajectory $z_\mu(t)$ is depicted in Fig. 1.3. The measure in Eq. (1.62) can then be discretized by

$$\int \mathcal{D} z_\mu(t) \cdots \;=\; \prod_{i=1}^{M} \int \frac{\mathrm{d}^d z_i}{(2\pi\epsilon)^{d/2}} \cdots\,. \tag{1.66}$$

The explicit form of the operator ∂_μ in Eq. (1.34) was not essential in deriving Eq. (1.62). If ∂_μ in Eq. (1.34) is replaced by an arbitrary operator D_μ with noncommuting components, then Eq. (1.62) holds with $\partial_\mu(t)$ substituted by $D_\mu(t)$. The discretized path-ordered exponential of a general operator $D_\mu(t)$ is given by

$$\boldsymbol{P}\, \mathrm{e}^{\int_0^\tau \mathrm{d}t\, \dot z^\mu(t) D_\mu(t)} \;=\; \lim_{\epsilon \to 0} \prod_{i=1}^{M} \left[1 + (z_i - z_{i-1})^\mu\, D_\mu(i\epsilon)\right]. \tag{1.67}$$

The order of multiplication here is the same as in Eq. (1.51).

The explicit form of the operator ∂_μ is essential when we calculate how it acts on the delta-function as prescribed by the RHS of Eq. (1.62). For the free case, when the t-dependence of $\partial_\mu(t)$ is not essential, one simply

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finds

$$\boldsymbol{P}\,\mathrm{e}^{\int_0^\tau \mathrm{d}t\,\dot{z}^\mu(t)\partial_\mu(t)} = \exp\left\{[z^\mu(\tau)-x^\mu]\,\frac{\partial}{\partial x^\mu}\right\}, \tag{1.68}$$

which is nothing but the shift operator. Applying the operator on the RHS of Eq. (1.68) to the delta-function, one obtains

$$\boldsymbol{P}\,\mathrm{e}^{\int_0^\tau \mathrm{d}t\,\dot{z}^\mu(t)\partial_\mu(t)}\,\delta^{(d)}(x-y) = \delta^{(d)}(z\,(\tau)-y)\,. \tag{1.69}$$

Therefore, $z_\mu(\tau)$ has to coincide with y_μ owing to the delta-function, which disappears after the integration over $z_\mu(\tau)$ has been performed. Thus the final answer is

$$G(x-y) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\frac{1}{2}\tau m^2} \int\limits_{\substack{z_\mu(0)=x_\mu\\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\,\dot{z}_\mu^2(t)}. \tag{1.70}$$

This path integral goes over all trajectories $z_\mu(t)$ that connect the initial point x_μ and the final point y_μ.

Problem 1.6 Derive Eqs. (1.62) and (1.70) by introducing a path integral over velocity $v_\mu(t) = \dot{z}_\mu(t)$.

Solution The operator on the RHS of Eq. (1.61) can be disentangled using the following Gaussian path integral:

$$\boldsymbol{P}\,\mathrm{e}^{\frac{1}{2}\int_0^\tau \mathrm{d}t D_\mu^2(t)} = \int \mathcal{D}v_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\,v_\mu^2(t)}\,\boldsymbol{P}\,\mathrm{e}^{\int_0^\tau \mathrm{d}t\,v^\mu(t)D_\mu(t)}. \tag{1.71}$$

This formula holds for an arbitrary operator D_μ and can be proven formally by calculating the Gaussian integral after shifting $v_\mu(t)$.

Substituting $D_\mu(t) = \partial_\mu(t)$ and calculating the action of the path-ordered exponential on $\delta^{(d)}(x-y)$, we obtain

$$G(x-y) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\frac{1}{2}\tau m^2}\int \mathcal{D}v_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\,v_\mu^2(t)}\,\delta^{(d)}\left(x+\int_0^\tau \mathrm{d}t\,v(t)-y\right). \tag{1.72}$$

The integration over $\mathcal{D}v_\mu(t)$ in this formula has no restrictions.

To derive Eq. (1.70) from Eq. (1.72), let us note that the discretized velocities read as

$$v_i^\mu = \frac{z_i^\mu - z_{i-1}^\mu}{\epsilon}\,. \tag{1.73}$$

Since

$$\int_0^\tau \mathrm{d}t\,v^2(t) \quad\rightarrow\quad \epsilon\sum_{i=1}^M v_i^2\,, \tag{1.74}$$

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the measure

$$\int \mathcal{D}v_\mu(t)\cdots \;=\; \prod_{i=1}^{M}\int \frac{\mathrm{d}^d v_i}{(2\pi/\epsilon)^{d/2}}\cdots \tag{1.75}$$

obviously recovers Eq. (1.66) after calculating the Jacobian from the variables v_i to the variables z_i. Therefore, Eq. (1.72) reproduces Eq. (1.70) provided

$$z^\mu(t) \;=\; x^\mu + \int_0^t \mathrm{d}t' v^\mu(t')\,. \tag{1.76}$$

Remark on definition of the measure

The discretized trajectory in Fig. 1.3 can be written analytically as the expansion

$$z^\mu(t) \;=\; \sum_{i=1}^{M} z_i^\mu f_i(t) + x^\mu(1-t/\epsilon)\,\theta(\epsilon-t)\,, \tag{1.77}$$

where the basis functions

$$f_i(t) = \begin{cases} 1+(t/\epsilon-i) & \text{for} \quad t\in[(i-1)\epsilon, i\epsilon]\,, \\ 1-(t/\epsilon-i) & \text{for} \quad t\in[i\epsilon,(i+1)\epsilon]\,, \\ 0 & \text{otherwise} \end{cases} \tag{1.78}$$

are nonvanishing only for the ith and $(i+1)$th intervals. The measure (1.66) is defined, therefore, via the coefficients z_i as a multiple product of $\mathrm{d}z_i$.

While the basis functions $f_i(t)$ are not orthogonal:

$$\frac{1}{\epsilon}\int_0^\tau \mathrm{d}t\, f_i(t) f_j(t) \;=\; \frac{2}{3}\delta_{ij} + \frac{1}{6}\delta_{i(j+1)} + \frac{1}{6}\delta_{i(j-1)}\,, \tag{1.79}$$

the orthogonal set appears in the expansion of the velocity

$$\dot z^\mu(t) \;=\; \sum_{i=1}^{M}(z_i^\mu - z_{i-1}^\mu)\phi_i(t)\,, \tag{1.80}$$

where

$$\phi_i(t) = \begin{cases} 1/\epsilon & \text{for} \quad t\in[(i-1)\epsilon, i\epsilon]\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.81}$$

This shows why the discretized velocities from Problem 1.6 are natural variables.

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One can choose, instead, another set of (orthogonal) basis functions and expand

$$z^{\mu}(t) \;=\; \sum_{n=1}^{M} c_n^{\mu}\phi_n(t) \tag{1.82}$$

with some coefficients c_n^{μ}. Then the measure (1.66) takes the form

$$\mathcal{D}z_{\mu}(t)\cdots \;\propto\; \prod_{n=1}^{M} \mathrm{d}^d c_n \cdots \tag{1.83}$$

modulo a c-independent Jacobian. Mathematically, this implies that one approximates the functional space by M-dimensional spaces.

1.5 Calculation of the Gaussian path integral

The Gaussian path integral (1.70) can be calculated easily using the following trick.* Let us substitute the variable $z_{\mu}(t)$ by a new variable $\xi_{\mu}(t)$, which are related by the formula

$$z_{\mu}(t) \;=\; \frac{y_{\mu}-x_{\mu}}{\tau}\,t+\xi_{\mu}(t)+x_{\mu}\,. \tag{1.84}$$

The boundary conditions for the variable $\xi(t)$ are determined by Eq. (1.84) to be

$$\xi_{\mu}(0) \;=\; \xi_{\mu}(\tau) \;=\; 0\,. \tag{1.85}$$

On substituting Eq. (1.84) into the exponent in Eq. (1.70), one finds

$$\int_0^{\tau} \mathrm{d}t\, \dot{z}^2(t) \;=\; \frac{(y-x)^2}{\tau}+2\frac{(y-x)}{\tau}\left[\xi(\tau)-\xi(0)\right]+\int_0^{\tau}\mathrm{d}t\,\dot{\xi}^2(t)\,. \tag{1.86}$$

The second term on the RHS vanishes owing to the boundary conditions (1.85) so that the propagator becomes

$$G(x-y) \;=\; \frac{1}{2}\int_0^{\infty} \mathrm{d}\tau\; \mathrm{e}^{-\frac{1}{2}\tau m^2}\,\mathrm{e}^{-(y-x)^2/2\tau} \int\limits_{\xi_{\mu}(0)=\xi_{\mu}(\tau)=0} \mathcal{D}\xi_{\mu}\; \mathrm{e}^{-\frac{1}{2}\int_0^{\tau}\mathrm{d}t\,\dot{\xi}_{\mu}^2(t)}\,. \tag{1.87}$$

The path integral over ξ on the RHS of Eq. (1.87) is a function solely of τ:

$$\int\limits_{\xi_{\mu}(0)=\xi_{\mu}(\tau)=0} \mathcal{D}\xi_{\mu}\; \mathrm{e}^{-\frac{1}{2}\int_0^{\tau}\mathrm{d}t\,\dot{\xi}_{\mu}^2(t)} \;=\; \mathcal{F}(\tau)\,. \tag{1.88}$$

* See, for example, the book by Feynman [Fey72], Chapter 3.

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This expression is to be compared with the proper-time representation of the Euclidean free propagator which reads as

$$\begin{aligned} G(x-y) &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}p(x-y)} \frac{1}{2} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{\tau}{2}(p^2+m^2)} \\ &= \frac{1}{2} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{1}{2}\tau m^2}\, \mathrm{e}^{-(x-y)^2/2\tau} \frac{1}{(2\pi\tau)^{d/2}}. \end{aligned} \tag{1.89}$$

These two expressions coincide provided that

$$\mathcal{F}(\tau) = \frac{1}{(2\pi\tau)^{d/2}}. \tag{1.90}$$

Problem 1.7 Calculate $\mathcal{F}(\tau)$ from the discretized path integral.

Solution The discretized version of the path integral in Eq. (1.70) is

$$\int_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\, \dot{z}_\mu^2(t)} = \frac{1}{(2\pi\epsilon)^{d/2}} \int \prod_{i=1}^{M-1} \frac{\mathrm{d}^d z_i}{(2\pi\epsilon)^{d/2}}\, \mathrm{e}^{-\frac{1}{2\epsilon}\sum_{i=1}^M (z_i-z_{i-1})^2}, \tag{1.91}$$

where $z_0 = x$ and $z_M = y$. The integral can be calculated using the well-known formula for the Gaussian integral

$$\int \frac{\mathrm{d}^d z}{(2\pi)^{d/2}} \exp\left[-\frac{(x-z)^2}{2\tau_1} - \frac{(z-y)^2}{2\tau_2}\right] = \left(\frac{\tau_1\tau_2}{\tau_1+\tau_2}\right)^{d/2} \exp\left[-\frac{(x-y)^2}{2(\tau_1+\tau_2)}\right]. \tag{1.92}$$

After applying this formula $M-1$ times, one arrives at Eq. (1.90). Note that ϵ cancels in the final answer.

Problem 1.8 Which trajectories are essential in the path integral?

Solution It is seen from the discretization on the RHS of Eq. (1.91) that only trajectories with

$$|z_i - z_{i-1}| \sim \sqrt{\epsilon} \tag{1.93}$$

are essential as $\epsilon \to 0$. Such trajectories are typical *Brownian* trajectories. They are continuous as $\epsilon \to 0$ but not smooth ($|z_i - z_{i-1}| \sim \epsilon$ for smooth trajectories). In mathematical language, these functions are said to belong to the Lipshitz class 1/2.

Remark on mathematical structure

The measure (1.66) for integration over functions is sometimes called the Lebesgue measure. It was introduced in mathematics by Wiener [Wie23] in connection with the problem of Brownian motion. With the Gaussian factor incorporated, it is also known as the Wiener measure while the proper path integral is known as the Wiener integral.* The measure (1.66) is defined on the space L_2 (i.e. the space of functions whose square is integrable, in the sense of the Lebesgue integral, $\int \mathrm{d}t\, z^2(t) < \infty$). The integration on L_2 goes over trajectories $z_\mu(t)$, which are generically discontinuous. However, the extra weight factor $\exp\left[-\frac{1}{2}\int_0^\tau \mathrm{d}t\, \dot{z}^2(t)\right]$ restricts the trajectories in the above path integrals to be continuous.

1.6 Transition amplitudes

As is well-known in quantum mechanics, $G(x-y)$ is the probability for a (scalar) particle to propagate from x to y. A convenient notation for a trajectory $z_\mu(t)$ that connects x_μ and y_μ is

$$\Gamma_{yx} \equiv \{z_\mu(t);\ 0 \leq t \leq \tau, \quad z_\mu(0) = x_\mu,\, z_\mu(\tau) = y_\mu\}\,. \tag{1.94}$$

Note that Γ_{yx} denotes a trajectory as a geometric object, while $z_\mu(t)$ is a function that describes a given trajectory in some parametrization t. This function (but not the geometric object itself) depends on the choice of parametrization and changes under the *reparametrization* transformation

$$t \to \sigma(t)\,, \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \geq 0\,, \tag{1.95}$$

with σ being a new parameter.

A convenient parametrization is via the proper length of Γ_{yx} which is given by

$$s = \int_{\Gamma_{yx}} \mathrm{d}s\,, \tag{1.96}$$

where

$$\mathrm{d}s = \sqrt{\dot{z}^2(\sigma)}\,\mathrm{d}\sigma \tag{1.97}$$

and $\sigma \in [\sigma_0, \sigma_1]$ is some parametrization. For obvious reasons the parametrization

$$t = \frac{1}{m}s \tag{1.98}$$

with s given by Eq. (1.96) is called the *proper-time* parametrization. Note that the dimension of t is $[\text{length}]^2$ according to Eq. (1.98).

* See, for example, the books [Kac59, Sch81, Wie86, Roe94] for a description of the path-integral approach to Brownian motion.

Let us denote*

$$S[\Gamma_{yx}] \equiv \frac{m^2\tau}{2} + \frac{1}{2}\int_0^\tau \mathrm{d}t\, \dot{z}^2(t)\,. \tag{1.99}$$

The sense of this notation is that the RHS coincides with the classical action of a relativistic free (scalar) particle in the proper-time parametrization (1.98) when

$$\int_0^\tau \mathrm{d}t\, \dot{z}^2(t) = m\int_0^\tau \mathrm{d}s = m\,\mathrm{Length}[\Gamma] \tag{1.100}$$

since

$$\left(\frac{\mathrm{d}z_\mu(s)}{\mathrm{d}s}\right)^2 = 1 \tag{1.101}$$

and $m\tau = \mathrm{Length}[\Gamma]$ by the definition of the proper time.

Therefore, the path-integral representation (1.70) is nothing but the sum over trajectories with the weight being an exponential of (minus) the classical action:

$$G(x-y) = \sum_{\Gamma_{yx}} \mathrm{e}^{-S[\Gamma_{yx}]}. \tag{1.102}$$

This sum is split in Eq. (1.70) into the trajectories along which the particle propagates during the proper time τ and the integral over τ.

Equation (1.102) implies that the transition amplitude in quantum mechanics is a sum over all paths which connects x and y. In other words, a particle propagates from x to y along all paths Γ_{yx}, including the ones which are forbidden by the free classical equation of motion

$$\ddot{z}_\mu(t) = 0\,. \tag{1.103}$$

Only the classical trajectory (1.103) survives the path integral in the classical limit $\hbar \to 0$. The reason for this is that if the dependence on Planck's constant is restored, it appears in the exponent:

$$G(x-y) = \sum_{\Gamma_{yx}} \mathrm{e}^{-S[\Gamma_{yx}]/\hbar}. \tag{1.104}$$

As $\hbar \to 0$ the path integral is dominated by a saddle point, which is given in the free case by the classical equation of motion (1.103).

* The notation $S[\Gamma]$ with square brackets means that S is a functional of Γ, while $f(x)$ with parentheses stands for functions.

It is worth noting that the sum-over-path representation (1.102) is written entirely in terms of trajectories as geometric objects and does not refer to a concrete parametrization. For the free theory $S[\Gamma]$ is proportional to the length of the trajectory Γ:

$$S_{\text{free}}[\Gamma] \;=\; m\,\text{Length}[\Gamma]\,, \tag{1.105}$$

where the length is given for some parametrization σ of the trajectory Γ by

$$\text{Length}[\Gamma] \;=\; \int\limits_{\sigma_0}^{\sigma_1} \mathrm{d}\sigma \sqrt{\dot{z}^2(\sigma)}\,. \tag{1.106}$$

The sum-over-path representation (1.102) with $S[\Gamma]$ given by the classical action (Eq. (1.105) in the free case) is often considered as a first principle of constructing quantum mechanics given the classical action $S[\Gamma]$.

Problem 1.9 Represent the matrix element of the (Euclidean) evolution operator $\langle y \,|\, \exp\left(-\boldsymbol{H}\tau\right) |\, x\rangle$ for the nonrelativistic Hamiltonian

$$\boldsymbol{H} \;=\; -\frac{\partial^2}{2m} + V(x) \tag{1.107}$$

as a path integral.

Solution The calculation is similar to that already done in Sect. 1.4. It is most convenient to use the path integral over velocity which was considered in Problem 1.6 on p. 16. The appropriate disentangling formula is given as

$$\left\langle y \,\middle|\, \mathrm{e}^{-\boldsymbol{H}\tau} \,\middle|\, x \right\rangle = \int \mathcal{D}v_\mu(t)\, \mathrm{e}^{-\frac{m}{2}\int_0^\tau \mathrm{d}t\, v_\mu^2(t)}\, \boldsymbol{P}\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, v^\mu(t)\partial_\mu(t) - \int_0^\tau \mathrm{d}t\, V(x;t)}\, \delta^{(d)}(x-y)\,. \tag{1.108}$$

Here the argument t in $V(x;t)$ is just the ordering parameter, while the same formula holds when the potential is explicitly time-dependent.

In contrast to Eq. (1.71), we have put the minus sign in front of the linear-in-v term in the exponent in Eq. (1.108), so that it agrees with Appendix B of Feynman's paper [Fey51]. In fact, it does not matter what sign is used since the integral over $v(t)$ is Gaussian, so only even powers of v survive after the integration.

The path-ordered exponential in Eq. (1.108) reads explicitly as

$$\boldsymbol{P}\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, v^\mu(t)\partial_\mu(t) - \int_0^\tau \mathrm{d}t\, V(x;t)} \;=\; \lim_{\epsilon\to 0} \prod_{i=1}^{M} \left[1 - \epsilon v_i^\mu \frac{\partial}{\partial x^\mu} - \epsilon V(x; i\epsilon)\right], \tag{1.109}$$

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which can be rewritten as

$$\boldsymbol{P}\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\, v^\mu(t)\partial_\mu(t) - \int_0^\tau \mathrm{d}t\, V(x;t)} = \lim_{\epsilon\to 0}\prod_{i=1}^{M}\left[1-\epsilon v_i^\mu \frac{\partial}{\partial x^\mu}\right]\left[1-\epsilon V(x;i\epsilon)\right], \tag{1.110}$$

if terms which vanish as $\epsilon \to 0$ are neglected, or equivalently as

$$\boldsymbol{P}\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\, v^\mu(t)\partial_\mu(t) - \int_0^\tau \mathrm{d}t\, V(x;t)} = \prod_{t=0}^{\tau}\left[1-\mathrm{d}t\, v^\mu(t) \frac{\partial}{\partial x^\mu}\right]\left[1-\mathrm{d}t\, V(x;t)\right]. \tag{1.111}$$

There is no need to write down the t-dependence of $\partial_\mu(t)$ in these formulas since the order of the operators is explicit.

To disentangle the operator expression (1.111), let us note that

$$\left[1-\mathrm{d}t\, v^\mu(t)\partial_\mu\right] = U^{-1}(t+\mathrm{d}t)\, U(t) \tag{1.112}$$

with

$$U(t) = \exp\left[\int_0^t \mathrm{d}t' v^\mu(t')\partial_\mu\right] \tag{1.113}$$

being the shift operator. It obviously obeys the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = v^\mu(t)\,\partial_\mu\, U(t)\,. \tag{1.114}$$

Now since

$$U(t)\left[1-\mathrm{d}t\, V(x;t)\right]U^{-1}(t) = \left[1-\mathrm{d}t\, V\left(x+\int_0^t \mathrm{d}t' v(t');t\right)\right], \tag{1.115}$$

the RHS of Eq. (1.111) can be written in the form

$$\begin{aligned}
&\prod_{t=0}^{\tau}\left[1-\mathrm{d}t\, v^\mu(t)\frac{\partial}{\partial x^\mu}\right]\left[1-\mathrm{d}t\, V(x;t)\right] \\
&\quad = U^{-1}(\tau)\prod_{t=0}^{\tau}\left[1-\mathrm{d}t\, V\left(x+\int_0^t \mathrm{d}t' v(t');t\right)\right] \\
&\quad = U^{-1}(\tau)\,\exp\left[-\int_0^\tau \mathrm{d}t\, V\left(x+\int_0^t \mathrm{d}t' v(t');t\right)\right],
\end{aligned} \tag{1.116}$$

which is completely disentangled.

The operator $U^{-1}(\tau)$ is now in the proper order to be applied to the variable y in the argument of the delta-function, which results in the shift

$$\delta^{(d)}(x-y) \Longrightarrow \delta^{(d)}\left(x+\int_0^\tau \mathrm{d}t\, v(t) - y\right). \tag{1.117}$$

This will be explained in more detail in the next paragraphs.

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Passing to the variable (1.76), we get finally

$$\left\langle y \,\middle|\, \mathrm{e}^{-\boldsymbol{H}\tau} \middle|\, x \right\rangle = \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \mathcal{L}(t)}, \tag{1.118}$$

where

$$\mathcal{L}(t) = \frac{m}{2}\dot{z}_\mu^2(t) + V(z(t)) \tag{1.119}$$

is the Lagrangian associated with the Hamiltonian $\boldsymbol{H}$. The unusual plus sign in this formula is due to the Euclidean-space formalism. It is clear from the derivation that Eq. (1.118) holds for time-dependent potentials as well.

Notice that the path integral in Eq. (1.118) is now over trajectories along which the particle propagates in the fixed proper time τ with no integration over τ.

A special comment about the operator $U^{-1}(\tau)$ in Eq. (1.116) is required. In the Schrödinger representation of quantum mechanics, one is interested in the matrix elements of the evolution operator between some vectors $\langle g|$ and $|f\rangle$ in the Hilbert space. According to Eq. (1.23), in the coordinate representation one has

$$\left\langle g \,\middle|\, \mathrm{e}^{-\boldsymbol{H}\tau} \middle|\, f \right\rangle = \int \mathrm{d}^d x\, g(x)\, \mathrm{e}^{-\boldsymbol{H}\tau} f(x)\,. \tag{1.120}$$

Integrating by parts, the operator $U^{-1}(\tau)$ can then be applied to $g(x)$ which results in the shift

$$g(x) \implies U(\tau)\, g(x)\, U^{-1}(\tau) = g\Bigg(x + \int\limits_0^\tau \mathrm{d}t\, v(t)\Bigg). \tag{1.121}$$

Passing to the variable (1.76), Eq. (1.120) becomes

$$\left\langle g \,\middle|\, \mathrm{e}^{-\boldsymbol{H}\tau} \middle|\, f \right\rangle = \int \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \mathcal{L}(t)} g(z(\tau)) f(z(0))\,. \tag{1.122}$$

There are no restrictions on the initial and final points of the trajectories $z_\mu(t)$ in this formula.

Problem 1.10 Calculate the diagonal resolvent of the Schrödinger operator in the potential $V(x)$:

$$R_\omega(x,x;V) = \left\langle x \,\middle|\, \frac{1}{-\mathcal{G}\partial^2 + \omega^2 + V} \,\middle|\, x \right\rangle, \tag{1.123}$$

in the limit $\mathcal{G} \to 0$ for $d = 1$.

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Solution Using the formula of the type (1.118), we represent $R_\omega(x,x;V)$ as the path integral

$$R_\omega(x,x;V) \;=\; \frac{1}{2}\int\limits_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{1}{2}\tau\omega^2} \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=x_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2\mathcal{G}}\int_0^\tau \mathrm{d}t\, \dot{z}_\mu^2(t) - \int_0^\tau \mathrm{d}t\, V(z(t))}. \tag{1.124}$$

As $\mathcal{G} \to 0$ this path integral is dominated by the t-independent saddle-point trajectory

$$z(t) \;=\; x\,, \tag{1.125}$$

which is associated with a particle standing at the point x. Substituting V at this saddle point, i.e. replacing $V(z(t))$ by $V(x)$, and calculating the Gaussian integral over quantum fluctuations around the trajectory (1.125) using Eqs. (1.88) and (1.90), one finds

$$R_\omega(x,x;V) \;=\; \frac{1}{2\sqrt{\omega^2+V(x)}} \tag{1.126}$$

in $d=1$.

Equation (1.126) can be alternatively derived by applying the Gel'fand–Dikii technique [GD75] which says that $R_\omega(x,x;V)$ obeys the third-order linear differential equation

$$\frac{1}{2}\left[\frac{\mathcal{G}}{2}\partial^3 - \partial V(x) - V(x)\partial\right] R_\omega(x,x;V) \;=\; \omega^2 \partial R_\omega(x,x;V)\,. \tag{1.127}$$

$R_\omega(x,x;V)$ given by Eq. (1.126) obviously satisfies this equation as $\mathcal{G} \to 0$.

One more way to derive Eq. (1.126) is to perform a semiclassical Wantzel–Kramers–Brillouin (WKB) expansion of $R_\omega(x,y;V)$ in the parameter $\mathcal{G}$. This is explained in Chapter 7 of the book [LL74].

Problem 1.11 Derive Eq. (1.127).

Solution The resolvent

$$R_\omega(x,y;V) \;=\; \left\langle y \left| \frac{1}{-\mathcal{G}\partial^2+\omega^2+V} \right| x \right\rangle \tag{1.128}$$

obeys the equations

$$\left.\begin{aligned} \left[-\mathcal{G}\frac{\partial^2}{\partial x^2}+\omega^2+V(x)\right] R_\omega(x,y;V) &= \delta^{(1)}(x-y)\,, \\ \left[-\mathcal{G}\frac{\partial^2}{\partial y^2}+\omega^2+V(y)\right] R_\omega(x,y;V) &= \delta^{(1)}(x-y)\,. \end{aligned}\right\} \tag{1.129}$$

It can be expressed via the two solutions $f_\pm(x)$ of the homogeneous equation

$$\left[-\mathcal{G}\frac{\partial^2}{\partial x^2}+\omega^2+V(x)\right] f_\pm(x) \;=\; 0\,, \tag{1.130}$$

where f_+ or f_- are regular at $+\infty$ or $-\infty$, respectively. Then the full solution is

$$R_\omega(x,y;V) = \frac{f_+(x)f_-(y)\theta(x-y)+f_-(x)f_+(y)\theta(y-x)}{\mathcal{G}\,W_\omega} \tag{1.131}$$

with

$$W_\omega = f_+(x)f_-'(x) - f_+'(x)f_-(x) \tag{1.132}$$

being the Wronskian of these solutions. Applying $\partial/\partial x$ to Eq. (1.132), it is easy to show that W_ω is an x-independent function of ω.

The simplest way to prove Eq. (1.127) is to differentiate

$$R_\omega(x,x;V) = \frac{f_+(x)f_-(x)}{\mathcal{G}\,W_\omega} \tag{1.133}$$

using Eq. (1.130), in order to verify that it satisfies the nonlinear differential equation

$$-2\mathcal{G}R_\omega R_\omega'' + \mathcal{G}\left(R_\omega'\right)^2 + 4\left(\omega^2+V\right)R_\omega^2 = 1\,. \tag{1.134}$$

One more differentiation of Eq. (1.134) with respect to x results in Eq. (1.127).

It is worth noting that Eq. (1.134) is very convenient for calculating the semiclassical expansion of $R_\omega(x,x;V)$ in $\mathcal{G}$. In particular, the leading order (1.126) is obvious.

Remark on parametric invariant representation

The Green function $G(x-y)$ can alternatively be calculated from the parametric invariant representation

$$G(x-y) \propto \int\limits_{\substack{z_\mu(\sigma_0)=x_\mu \\ z_\mu(\sigma_1)=y_\mu}} \mathcal{D}z_\mu(\sigma)\,\mathrm{e}^{-m_0\int_{\sigma_0}^{\sigma_1}\mathrm{d}\sigma\sqrt{\dot{z}^2(\sigma)}} \tag{1.135}$$

as prescribed by Eqs. (1.105) and (1.106). In contrast to (1.70), this path integral is not easy to calculate. The integration over $\mathcal{D}z_\mu(\sigma)$ in Eq. (1.135) involves integration over the reparametrization group, which gives the proper group-volume factor since the exponent is parametric invariant. Eq. (1.70) is recovered after fixing parametrization to be proper time. How this calculation can be performed is explained in Chapter 9 of the book by Polyakov [Pol87].

If one makes a naive discretization of the parameter σ using equidistant intervals, the exponent in Eq. (1.135) is highly nonlinear in the variables z_i, leading to complicated integrals. In contrast, the discretization (1.91) of the path integral in Eq. (1.70), where the parametric invariance is fixed, results in a Gaussian integral which is easily calculable.

Problem 1.12 Calculate the path integral in Eq. (1.135), discretizing the measure by

$$\mathcal{D}z_\mu \quad \to \quad \sum_{M=1}^{\infty}\prod_{i=1}^{M}\frac{\mathrm{d}^d z_i}{(2\pi\epsilon)^{d/2}} \tag{1.136}$$

and applying the central limit theorem as $M \to \infty$.

Solution By making the discretization, we represent the RHS of Eq. (1.135) as the probability integral

$$\begin{aligned} G_\epsilon(x-y) \quad &= \quad \frac{1}{(2\pi\epsilon)^{d/2}}\sum_{M=1}^{\infty}\int\prod_{i=1}^{M-1}\frac{\mathrm{d}^d z_i}{(2\pi\epsilon)^{d/2}} \\ &\quad \times \rho\,(x \to z_1)\,\rho\,(z_1 \to z_2)\cdots\rho\,(z_{M-1} \to y) \end{aligned} \tag{1.137}$$

with

$$\rho(z_{i-1} \to z_i) \quad = \quad \mathrm{e}^{-m_0|z_i - z_{i-1}|} \tag{1.138}$$

being an (unnormalized) probability function and ϵ is a parameter with the dimension of $[\text{length}]^2$. The probability interpretation of each term in the sum is standard for random walk models, and means, as usual, that a particle propagates via independent intermediate steps. The discretization of the measure given by Eq. (1.137) looks like that in Eq. (1.66), but the summation over M is now added.

Since the integral in Eq. (1.137) is a convolution, the central limit theorem states that

$$\begin{aligned} G_\epsilon(x-y) \quad &= \quad \frac{1}{(2\pi\epsilon)^{d/2}}\sum_{M}\left[\frac{c_0}{(2\pi\epsilon m_0^2)^{d/2}}\right]^M \\ &\quad \times \frac{1}{(2\pi\sigma^2 M)^{d/2}}\,\mathrm{e}^{-m_0^2(x-y)^2/(2\sigma^2 M)+\mathcal{O}(M^{-2})} \end{aligned} \tag{1.139}$$

at large M, where c_0 and σ^2 are the zeroth and (normalized) second moments of ρ:

$$\left.\begin{aligned} c_0 \quad &= \quad \int \mathrm{d}^d x\,\mathrm{e}^{-|x|} \;=\; 2\pi^{d/2}\frac{\Gamma\,(d)}{\Gamma\,(d/2)}\,, \\ \sigma^2 \quad &= \quad \frac{1}{c_0}\int \mathrm{d}^d x\,x^2\,\mathrm{e}^{-|x|} \;=\; d(d+1)\,. \end{aligned}\right\} \tag{1.140}$$

The sum over M in Eq. (1.139) is convergent for

$$m_0 \quad > \quad m_\mathrm{c} \;=\; \frac{c_0^{1/d}}{\sqrt{2\pi\epsilon}} \tag{1.141}$$

and is divergent for $m_0 < m_\mathrm{c}$. Choosing $m_0 > m_\mathrm{c}$, but $m_0^2 - m_\mathrm{c}^2 \sim 1$ in the limit $\varepsilon \to 0$, the sum over M will be convergent, while dominated by terms with large

$$M \quad \sim \quad m_\mathrm{c}^2 \;\sim\; \frac{1}{\epsilon}\,. \tag{1.142}$$

This is easily seen by rewriting Eq. (1.139) as

$$G_\epsilon(x-y) = \sum_M \left(\frac{m_c^2}{2\pi\sigma^2 M}\right)^{d/2} \times \mathrm{e}^{-dM\ln(m_0/m_c)-m_0^2(x-y)^2/(2\sigma^2 M)+\mathcal{O}(M^{-2})}. \tag{1.143}$$

Each term with $M \sim m_c^2$ contributes $\mathcal{O}(1)$ to the sum, so that

$$G_\epsilon(x-y) \sim m_c^2. \tag{1.144}$$

This justifies the using of the central limit theorem in this case. The typical distances between the z_i, which are essential in the integral on the RHS of Eq. (1.137), are

$$|z_i - z_{i-1}| \sim \frac{1}{m_0} \sim \sqrt{\epsilon} \tag{1.145}$$

as in Eq. (1.93). The relation (1.142) between the essential values of M and ϵ is also similar to what we had in Sect. 1.4.

The sum over M in Eq. (1.143) can be replaced by a continuous integral over the variable

$$\tau = \frac{\sigma^2 M}{m_c^2}, \tag{1.146}$$

which is $\mathcal{O}(1)$ for $M \sim m_c^2$. Also introducing the variable m by

$$m^2 \equiv \frac{d}{\sigma^2}\left(m_0^2 - m_c^2\right) > 0, \tag{1.147}$$

we rewrite Eq. (1.143) as

$$G_\epsilon(x-y) \stackrel{\epsilon\to 0}{\to} \frac{m_c^2}{\sigma^2}\int_0^\infty \mathrm{d}\tau \frac{1}{(2\pi\tau)^{d/2}}\,\mathrm{e}^{-\frac{1}{2}m^2\tau-(x-y)^2/2\tau}, \tag{1.148}$$

the RHS of which is proportional to that in Eq. (1.89) for the Euclidean propagator.

Remark on discretized path-ordered exponential

As is discussed in Sect. 1.3, the order of operators A_i and B_i with the same index i is not essential in the path-ordered exponential (1.52) as $M \to \infty$. If Eq. (1.52) is promoted to be valid at finite M (or at least to the order of $\mathcal{O}(M^{-1})$), this specifies the commutator of A_i and B_i. Analogously, a discretization of Eq. (1.118) specifies in which order the product of $x_i p_i$ in the classical theory should be substituted by the operators $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$ in the operator formalism. For details see the books by Berezin [Ber86] (Chapter 1 of Part II) and Sakita [Sak85] (Chapter 6).

1.7 Propagators in external field

Let us now consider a (quantum) particle in a classical electromagnetic field. The standard way of introducing an external electromagnetic field is to substitute the (operator of the) four-momentum $\boldsymbol{p}^\mu$ by

$$\boldsymbol{p}^\mu \quad \longrightarrow \quad \boldsymbol{p}^\mu - eA^\mu(x)\,. \tag{1.149}$$

Recalling the definition (1.26) of the Euclidean four-momentum, ∂_μ needs to be replaced by the covariant derivative

$$\partial_\mu \quad \longrightarrow \quad \nabla_\mu \;=\; \partial_\mu - \mathrm{i}eA_\mu(x)\,. \tag{1.150}$$

Inverting the operator ∇_μ^2 using the disentangling procedure, one finds

$$\begin{aligned} G(x,y;A) & \\ &\equiv \left\langle y \left| \frac{1}{-\nabla_\mu^2 + m^2} \right| x \right\rangle \\ &= \frac{1}{2}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\frac{1}{2}\tau m^2} \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\, \dot{z}_\mu^2(t) + \mathrm{i}e\int_0^\tau \mathrm{d}t\, \dot{z}^\mu(t) A_\mu(z(t))}. \end{aligned} \tag{1.151}$$

Note that the exponent is just the classical (Euclidean) action of a particle in an external electromagnetic field. Therefore, this expression is again of the type in Eq. (1.102).

The path-integral representation (1.151) for the propagator of a scalar particle in an external electromagnetic field is due to Feynman [Fey50] (Appendix A).

Problem 1.13 Derive Eq. (1.151) using Eq. (1.71) with $D_\mu = -\nabla_\mu$.

Solution The calculation is analogous to that of Problem 1.9 on p. 22. We have

$$D_\mu(t) \;=\; -\nabla_\mu(t) \;\equiv\; -\,\partial_\mu(t) + \mathrm{i}eA_\mu(x;t) \tag{1.152}$$

so that explicitly

$$\begin{aligned} \boldsymbol{P}\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\, v^\mu(t)\nabla_\mu(t)} &= \prod_{t=0}^{\tau}\left[1 - \mathrm{d}t\, v^\mu(t)\frac{\partial}{\partial x^\mu} + \mathrm{i}e\,\mathrm{d}t\, v^\mu(t)A_\mu(x;t)\right] \\ &= \prod_{t=0}^{\tau}\left[1 - \mathrm{d}t\, v^\mu(t)\frac{\partial}{\partial x^\mu}\right]\left[1 + \mathrm{i}e\,\mathrm{d}t\, v^\mu(t)A_\mu(x;t)\right]. \end{aligned} \tag{1.153}$$

This looks exactly like the expression (1.111) with

$$V(x;t) \;=\; -\mathrm{i}e\, v^\mu(t)A_\mu(x;t)\,. \tag{1.154}$$

Substituting this potential into Eq. (1.118) and remembering the additional integration over τ, we obtain the path-integral representation (1.151).

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We can alternatively rewrite Eq. (1.151) in the spirit of Sect. 1.6 as

$$G(x,y;A) \;=\; \sum_{\Gamma_{yx}}{}' \, \mathrm{e}^{\mathrm{i}e\int_{\Gamma_{yx}} \mathrm{d}z^\mu A_\mu(z)}, \tag{1.155}$$

where we have included the free action in the definition of the sum over trajectories:

$$\sum_{\Gamma_{yx}}{}' \;\stackrel{\text{def}}{=}\; \sum_{\Gamma_{yx}} \mathrm{e}^{-S_{\text{free}}[\Gamma_{yx}]}, \tag{1.156}$$

and represented the (parametric invariant) integral over $\mathrm{d}t$ as the contour integral over

$$\mathrm{d}z^\mu \;=\; \mathrm{d}t\,\dot{z}^\mu(t) \tag{1.157}$$

along the trajectory Γ_{yx}.

The meaning of Eq. (1.155) is that the transition amplitude of a quantum particle in a classical electromagnetic field is the sum over paths of the Abelian *phase factor*

$$U[\Gamma_{yx}] \;=\; \mathrm{e}^{\mathrm{i}e\int_{\Gamma_{yx}} \mathrm{d}z^\mu A_\mu(z)}. \tag{1.158}$$

Under the gauge transformation

$$A_\mu(z) \;\stackrel{\text{g.t.}}{\longrightarrow}\; A_\mu(z) + \frac{1}{e}\partial_\mu\alpha(z)\,, \tag{1.159}$$

the Abelian phase factor transforms as

$$U[\Gamma_{yx}] \;\stackrel{\text{g.t.}}{\longrightarrow}\; \mathrm{e}^{\mathrm{i}\alpha(y)}\,U[\Gamma_{yx}]\,\mathrm{e}^{-\mathrm{i}\alpha(x)}. \tag{1.160}$$

Noting that a wave function at the point x is transformed under the gauge transformation (1.159) as

$$\varphi(x) \;\stackrel{\text{g.t.}}{\longrightarrow}\; \mathrm{e}^{\mathrm{i}\alpha(x)}\,\varphi(x)\,, \tag{1.161}$$

we conclude that the phase factor is transformed as the product $\varphi(y)\varphi^\dagger(x)$:

$$U[\Gamma_{yx}] \;\stackrel{\text{g.t.}}{\sim}\; \text{“}\varphi(y)\,\varphi^\dagger(x)\text{”}, \tag{1.162}$$

where “$\cdots$” means literally “transforms as ...”.

As a consequence of Eqs. (1.160) and (1.161), a wave function at the point x transforms like one at the point y after multiplication by the phase factor:

$$U[\Gamma_{yx}]\,\varphi(x) \;\stackrel{\text{g.t.}}{\sim}\; \text{“}\varphi(y)\text{”}, \tag{1.163}$$

and analogously

$$\varphi^\dagger(y)\, U[\Gamma_{yx}] \stackrel{\text{g.t.}}{\sim} \text{“}\varphi^\dagger(x)\text{”}. \tag{1.164}$$

Equations (1.163) and (1.164) show that the phase factor plays the role of a *parallel transporter* in an electromagnetic field, and that in order to compare phases of a wave function at points x and y, one should first make a parallel transport along some contour Γ_{yx}. The result is, generally speaking, Γ-dependent except when $A_\mu(z)$ is a pure gauge. The sufficient and necessary condition for the phase factor to be Γ-independent is the vanishing of the field strength, $F_{\mu\nu}(z)$, which is a consequence of the Stokes theorem when applied to the Abelian phase factor.*

Below we shall deal with determinants of various operators. Analogous to Eq. (1.151), one finds

$$\begin{aligned} \ln\det\nabla_\mu^2 &= \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\,\mathrm{Tr}\;\mathrm{e}^{\frac{1}{2}\tau\nabla_\mu^2} \\ &= \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\int\limits_{z_\mu(0)=z_\mu(\tau)} \mathcal{D}z_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^\tau \mathrm{d}t\,\dot{z}_\mu^2(t)+\mathrm{i}e\oint_\Gamma \mathrm{d}z^\mu A_\mu(z)}, \end{aligned} \tag{1.165}$$

where the path integral goes over trajectories which are closed owing to the periodic boundary condition $z_\mu(0) = z_\mu(\tau)$. To derive Eq. (1.165), we have used the formula

$$\ln\det\boldsymbol{D} \;=\; \mathrm{Tr}\ln\boldsymbol{D}\,, \tag{1.166}$$

which relates the determinant and the trace of a Hermitian operator (or a matrix) $\boldsymbol{D}$.

Problem 1.14 Prove Eq. (1.166).

Solution Let $\boldsymbol{D}$ be positive definite. We first reduce $\boldsymbol{D}$ to a diagonal form by a unitary transformation and denote (positive) eigenvalues as D_i. Then Eq. (1.166) can be written as

$$\ln\prod_i D_i \;=\; \sum_i \ln D_i \tag{1.167}$$

which is obviously true.

* Strictly speaking, this statement holds for the case when Γ can be chosen everywhere in space-time, i.e. which is simply connected. However, there exist situations when Γ cannot penetrate into some regions of space as for the Aharonov–Bohm experiment which is discussed below in Sect. 5.4.

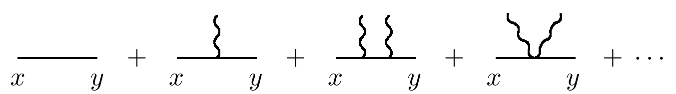
The phase factor for a closed contour Γ enters Eq. (1.165). It describes parallel transportation along a closed loop, and is gauge invariant as a consequence of Eq. (1.160):

$$\mathrm{e}^{\mathrm{i}e \oint_\Gamma \mathrm{d}z^\mu A_\mu(z)} \quad \xrightarrow{\text{g.t.}} \quad \mathrm{e}^{\mathrm{i}e \oint_\Gamma \mathrm{d}z^\mu A_\mu(z)}. \tag{1.168}$$

This quantity, which plays a crucial role in modern formulations of gauge theories, will be discussed in more detail in Chapter 5.

Problem 1.15 Show how the path-integral representation (1.151) recovers for $G(x, y; A)$ the diagrammatic expansion of propagator in an external field A_μ:

$G(x, y; A) =$



(1.169)

Solution Let us expand the phase factor in Eq. (1.155) in e. The linear-in-e term can be transformed using the formula

$$\sum_{\Gamma_{yx}}{}' \int_{\Gamma_{yx}} \mathrm{d}\xi_\mu \, \delta^{(d)}(\xi - z) \cdots \quad = \quad \sum_{\Gamma_{yz}}{}' \frac{\overleftrightarrow{\partial}}{\partial z_\mu} \sum_{\Gamma_{zx}}{}' \cdots \,, \tag{1.170}$$

where

$$\overleftrightarrow{\partial}_\mu \quad = \quad -\overleftarrow{\partial}_\mu + \partial_\mu \,, \tag{1.171}$$

to reproduce the second diagram on the RHS of Eq. (1.169). Equation (1.170) can be proven by varying both sides of Eq. (1.155) with respect to $A_\mu(z)$.

Equation (1.170) can be rewritten using the formula

$$\partial^x_\mu \sum_{\Gamma_{zx}}{}' \cdots \quad = \quad -\sum_{\Gamma_{zx}}{}' v_\mu(x) \cdots \,, \tag{1.172}$$

where $v_\mu(x) = \dot{\xi}_\mu(0)$ is the velocity at the point x of the trajectory Γ. Using Eq. (1.172), we find

$$\sum_{\Gamma_{yx}}{}' \int_{\Gamma_{yx}} \mathrm{d}\xi_\mu \, \delta^{(d)}(\xi - z) \cdots \quad = \quad \sum_{\Gamma_{yz}}{}' v_\mu(z) \sum_{\Gamma_{zx}}{}' \cdots + \sum_{\Gamma_{yz}}{}' \sum_{\Gamma_{zx}}{}' v_\mu(z) \cdots \,. \tag{1.173}$$

Equation (1.172) can be proven by shifting variable in the path integral, while Eq. (1.173) holds, strictly speaking, only if an integrand (denoted by $\cdots$) does not include velocities. Otherwise, additional contact terms might appear.

They can be obtained by noting that the velocity $v_\nu(x)$ corresponds to the covariant derivative (1.150), where $A_\nu(x)$ is also to be varied. Doing so, we arrive

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at

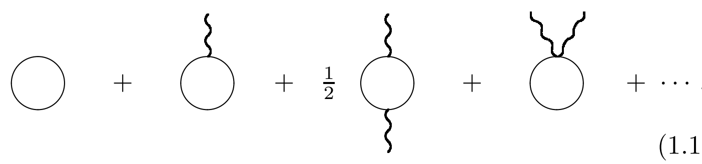
$$\sum_{\Gamma_{yx}}{}' v_\nu(x) \int_{\Gamma_{yx}} \mathrm{d}\xi_\mu \, \delta^{(d)}(\xi - z) \cdots$$
$$= \sum_{\Gamma_{yz}}{}' v_\nu(x)\, v_\mu(z) \sum_{\Gamma_{zx}}{}' \cdots + \sum_{\Gamma_{yz}}{}' v_\nu(x) \sum_{\Gamma_{zx}}{}' v_\mu(z) \cdots + \delta_{\mu\nu} \sum_{\Gamma_{yx}}{}' \cdots . \tag{1.174}$$

For the case of a more complicated integrand, each velocity produces the same type of contact terms since the variation $\delta/\delta A_\mu(z)$ acts linearly. This reproduces the contact terms as in the fourth term on the RHS of Eq. (1.169).

This Problem is based on Appendix A of the paper [MM81].

Problem 1.16 Establish the equivalence of the path-integral representation (1.165) of $\ln\det\nabla_\mu^2$ and the sum of one-loop diagrams in an external field A_μ:

$\ln\det\nabla_\mu^2 =$



(1.175)

Solution The derivation is the same as in the previous Problem. The combinatoric factor of 1/2 in the third diagram on the RHS of Eq. (1.175) is associated with a symmetry factor.

Remark on analogy with statistical mechanics

A formula of the type (1.165), which represents the trace of an operator via a path integral over closed trajectories, is known as the Feynman–Kac formula. The terminology comes from statistical mechanics where the partition function (or equivalently the statistical sum) is given by the Boltzmann formula

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta \boldsymbol{H}} \tag{1.176}$$

(with β being the inverse temperature) whose path-integral representation is of the type given in Eq. (1.165). The expression which is integrated on the RHS of Eq. (1.165) over $\mathrm{d}\tau/\tau$ is associated, in statistical-mechanical language, with the partition function of a closed elastic string, the energy of which is proportional to its length, that interacts with an external electromagnetic field. This shows an analogy between Euclidean quantum mechanics in d dimensions and statistical mechanics in d (spatial) and one (temporal) dimensions whose time-dependence disappears, since nothing depends on time at equilibrium. We shall explain this analogy in more detail in Part 2 (Chapter 9) when discussing quantum field theory at finite temperature.