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Maximal normal subgroups of the integral linear group of countable degree

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This paper continues the second author's investigation of the normal structure of the automorphism group Γ of a free abelian group of countably infinite rank. It is shown firstly that, in contrast with the case of finite degree, for each prime p every linear transformation of the vector space of countably infinite dimension over Z_p, the field of p elements, is induced by an element of Γ. Since by a result of Alex Rosenberg $\mathrm{GL}(\aleph_0, Z_p)$ has a (unique) maximal normal subgroup, it then follows that Γ has maximal normal subgroups, one for each prime.

0. Introduction

Let A be a free abelian group of countably infinite rank. This paper continues the investigation, begun in [2], of the lattice of normal subgroups of the automorphism group of A, which, following [2], we shall denote briefly by Γ rather than, say, $\mathrm{GL}(\aleph_0, Z)$. To be explicit, we prove here that Γ has maximal normal subgroups, one for each prime.

Our proof involves the quotient group A/pA, p prime, which can also be regarded as a vector space of countably infinite dimension over the field of p elements. Every automorphism of A induces an automorphism

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- linear transformation from the other point of view - of A/pA. Using results of Rosenberg [4] we show that, in perhaps surprising contrast with the case of finite dimension, the converse is true:

THEOREM 0.1. *Every automorphism of* A/pA *is induced by an automorphism of* A.

Thus the homomorphism ψ_p, say, from Γ to the automorphism group of A/pA, defined in the obvious way, is onto. Since, again by a result of Rosenberg [4], the automorphism group of A/pA has a unique maximal normal subgroup, it follows immediately that:

COROLLARY 0.2. *The group* Γ *has maximal normal subgroups, one for each prime.*

We remark that while the group $\Gamma/\ker \psi_p$ ($\simeq$ aut A/pA) has modulo its centre only one proper nontrivial normal subgroup (Rosenberg [4]), on the other hand there is a profusion of normal noncongruence subgroups of Γ contained in the principal congruence subgroup $\ker \psi_p$ ([2]). This contrasts also with the situation for $SL(n, Z)$, n finite (see [3]).

An explicit description of the maximal normal subgroups whose existence we show, can be deduced from the description in [2] of the unique maximal normal subgroup of $\Gamma/\ker \psi_p$.

We write briefly A_p for A/pA and $\Gamma(A_p)$ for the automorphism group of A_p. Following [4] we call an element γ of $\Gamma(A_p)$ *locally algebraic* if for all $x \in A_p$, the subspace spanned by $\{x\gamma^i \mid i = 0, 1, 2, \ldots\}$ is finite-dimensional. As a special case of Theorem A of [4] we have that $\Gamma(A_p)$ is generated by the elements of the form $1 + \rho$ where 1 denotes the identity map and ρ is an endomorphism of A_p such that $\rho^2 = 0$. Such automorphisms are easily seen to be locally algebraic, so that certainly $\Gamma(A_p)$ is generated by its locally algebraic elements. Thus to prove Theorem 0.1 it suffices to prove the following result.

THEOREM 0.3. *Every locally algebraic element of* $\Gamma(A_p)$ *is induced by an element of* Γ.

The remainder of the paper is devoted to proving this. In Section 1 we state the two lemmas we need for the proof (Lemmas 1.1, 1.2), prove one of them (Lemma 1.1), and deduce Theorem 0.3. The proof of Lemma 1.2 is relegated to Section 2. This lemma states that any basis of A_p can, after minor modification, be lifted to a free basis of A ; the finite dimensional analogue of this is well-known.

1. Lemmas and proof of Theorem 0.3

We shall say that an automorphism ϕ of A_p is *finitary* if there is a direct decomposition of A_p , $A_p = H_p \oplus K_p$ say, such that H_p has finite dimension, $H_p\phi = H_p$ and ϕ restricted to K_p is the identity map.

LEMMA 1.1. *Every finitary automorphism of A_p is induced by an automorphism of A .*

For the proof of this (and also for the proof of Theorem 0.3) we need the following lemma about lifting a basis of A_p to one of A . See Section 2 for its proof.

LEMMA 1.2. *Let η be the natural map from A to A_p . Given any basis $v_1, v_2, \ldots$ of the vector space A_p there is a free basis $c_1, c_2, \ldots$ of the free abelian group A , and integers $k_1, k_2, \ldots$ such that*

$$c_i\eta = k_iv_i \quad (i = 1, 2, \ldots) .$$

COROLLARY 1.3. *Let $A_p = H_p \oplus K_p$ be any direct decomposition of A_p . Then there is a direct decomposition of A , $A = H \oplus K$ say, such that $H\eta = H_p$, $K\eta = K_p$, where $\eta : A \to A_p$ is the natural map.*

Proof of Lemma 1.1. Let ϕ be a finitary automorphism of A_p and let $A_p = H_p \oplus K_p$, where H_p, K_p are as in the definition given above of a finitary automorphism (so that, in particular, H_p has finite dimension).

Let $H \oplus K$ be a corresponding direct decomposition of A as in Corollary 1.3. Let $\{c_1, c_2, \ldots\}$ be a free basis of A such that $\{c_1, \ldots, c_n\}$ is a free basis of H, and $\{c_{n+1}, c_{n+2}, \ldots\}$ is a free basis of K. Writing $u_i = c_i\eta$, we then have that $\{u_1, \ldots, u_n\}$ is a basis of H_p, and $\{u_{n+1}, u_{n+2}, \ldots\}$ is a basis of K_p; also, for $i = 1, \ldots, n$, we have

$$u_i\phi = \sum_{j=1}^{n} a_{ij}u_j ,$$

for some $a_{ij} \in Z_p$, while for $i > n$ we have $u_i\phi = u_i$. Since $H_p\phi = H_p$, the matrix $(a_{ij})_{n\times n}$ has nonzero determinant d say, where $0 < d < p$ (taking $Z_p = \{0, 1, \ldots, p-1\}$). Consider the automorphism ϕ_1 of A_p defined by

$$u_i\phi_1 = u_i \quad (i = 1, \ldots, n) ,$$

$$u_i\phi_1 = du_i \quad (i > n) .$$

We shall show that ϕ_1 is induced by an element of Γ.

To this end we write (b_{ij}) for the infinite matrix over Z_p representing ϕ_1 relative to the basis $\{u_1, u_2, \ldots\}$; thus $b_{ij} = a_{ij}$ $(1 \leq i, j \leq n)$, $b_{ii} = d$ $(i > n)$, and all other entries are zero. Let m be a positive integer such that $d^m = 1$ (in Z_p). Writing $m + n - 1 = t$, define the matrix $(c_{ij})_{t\times t}$ to be the $t \times t$ upper left hand corner of (b_{ij}); that is, $c_{ij} = b_{ij}$ $(1 \leq i, j \leq t)$. Clearly $\det(c_{ij})_{t\times t} = d^m = 1$, and therefore (see, for example, [1, Lemma 13]) there is a matrix $(\hat{c}_{ij})_{t\times t}$ over Z such that $\hat{c}_{ij} \equiv c_{ij} \bmod p$, and $\det(\hat{c}_{ij})_{t\times t} = 1$ (in Z). Now let $(d_{ij})_{m\times m}$ be the $m \times m$ scalar matrix over Z_p with all diagonal entries equal to d (and 0's elsewhere). Since $\det(d_{ij})_{m\times m} = d^m = 1$, we have again that there is a matrix

$(\hat{d}_{ij})_{m\times m}$ over Z whose determinant is 1 (in Z), such that $\hat{d}_{ij} \equiv d_{ij} \bmod p$. Clearly the infinite matrix (b_{ij}) is the direct sum of the matrix $(c_{ij})_{t\times t}$ and infinitely many copies of $(d_{ij})_{m\times m}$. Define the infinite integral matrix $(\hat{b}_{ij})$ to be the direct sum of the matrix $(\hat{c}_{ij})_{t\times t}$ and infinitely many copies of $(\hat{d}_{ij})_{m\times m}$. Since these matrices are all unimodular, it follows that relative to the basis $\{c_1, c_2, \ldots\}$, $(\hat{b}_{ij})$ represents an automorphism of A, which we denote by $\hat{\phi}_1$. Clearly $\hat{\phi}_1$ induces ϕ_1.

Similarly we define another automorphism ϕ_2 of A_p by

$$u_i\phi_2 = u_i \qquad (i = 1, \ldots, n) ,$$

$$u_i\phi_2 = d^{m-1}u_i \quad (i > n) .$$

We now imitate in part the above procedure to produce an element of Γ inducing ϕ_2. Thus let $(h_{ij})_{m\times m}$ be the $m \times m$ scalar matrix with diagonal entries all d^{m-1}. As before, since $\det(h_{ij})_{m\times m} = (d^{m-1})^m = 1$, there is a matrix $(\hat{h}_{ij})_{m\times m}$ over Z with determinant 1 such that $\hat{h}_{ij} \equiv h_{ij} \bmod p$. Since $\det(\hat{h}_{ij})_{m\times m} = 1$, the infinite matrix over Z obtained by taking the direct sum of the $n \times n$ identity matrix with infinitely many copies of $(\hat{h}_{ij})_{m\times m}$, represents an automorphism $\hat{\phi}_2$ say, of A, relative to the free basis $\{c_1, c_2, \ldots\}$. It follows that $\hat{\phi}_2$ induces ϕ_2, since the infinite matrix over Z_p representing ϕ_2 relative to $\{u_1, u_2, \ldots\}$ is just the direct sum of the $n \times n$ identity matrix over Z_p with infinitely many copies of $(h_{ij})_{m\times m}$.

Finally it is easily verified from their construction that $\phi_1\phi_2 = \phi$. Hence ϕ is induced by the element $\hat{\phi}_1\hat{\phi}_2$ of Γ. This completes the proof.

Proof of Theorem 0.3. Let γ be a locally algebraic element of

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$\Gamma(A_p)$. By imitating the proof of Lemma 2.4 of [4] we shall show that γ can be factorized into a product of two elements of $\Gamma(A_p)$ which are relatively easily seen to be induced by elements of Γ.

Choose any nonzero element a of A_p and write V_1 for the subspace of A_p spanned by $\{a\gamma^i \mid i = 1, 2, \ldots\}$. Since γ is locally algebraic, V_1 has finite dimension. Next choose any element of $A_p - V_1$; the orbit of this element under γ, together with V_1 spans a finite-dimensional subspace $V_1 \oplus V_2$ which is invariant under γ. Continuing in this way we obtain for every $i \geq 1$ a finite dimensional subspace $V_1 \oplus V_2 \oplus \ldots \oplus V_i$, invariant under γ. If we take care to do this in such a way as to ensure that every element of A_p is in some $V_1 \oplus V_2 \oplus \ldots \oplus V_i$, then the union of bases for the V_i is a basis for A_p. By Lemma 1.2 we can choose bases for the V_i so that the union of these bases can be lifted under η to a free basis of A. Let X be a basis for A_p obtained as such a union, and let $\hat{X}$ be a free basis of A such that $\hat{X}\eta = X$. Relative to X, γ is represented by the infinite matrix

$$P = \begin{pmatrix} M_1 & & & \\ * & M_2 & & \\ * & * & M_3 & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where the blocks M_i $(i = 1, 2, \ldots)$ are finite, square, nonsingular matrices over Z_p, and all entries above these blocks are zero. If we denote by M the direct sum of the M_i, it is clear that M represents an automorphism of A_p, and that

$$P = MQ,$$

where Q is obtained from P by replacing the M_i by identity matrices of the appropriate size (and leaving the blocks indicated by asterisks as they are). It is clear that Q is invertible over Z, since if we regard A as the group of all finitary sequences of integers, then the rows of Q generate A. Hence the element of $\Gamma(A_p)$ represented by Q relative to the basis X is induced by the element of Γ represented by Q relative to $\hat{X}$. Hence the proof will be complete if we can show that the element μ say, of $\Gamma(A_p)$, represented by M, is induced by an element of Γ.

We proceed to show this. Let $d_i \in Z_p$ be the determinant of M_i $(i = 1, 2, \ldots)$. It is not too difficult to show that the sequence $\{d_1, d_2, \ldots\}$ of elements of Z_p (in fact any sequence of such elements) can be reordered to give a sequence $\{d'_1, d'_2, \ldots\}$ say, with the property that

$$\prod_{j=n_i+1}^{n_{i+1}} d'_j = 1 \quad (i = 1, 2, \ldots)$$

for some strictly increasing sequence $\{n_1, n_2, \ldots\}$ of positive integers. (Note that no condition is placed on the product of $d'_1, \ldots, d'_{n_1}$.) It follows from this that there is an automorphism π of A_p which simply permutes X, such that the matrix N representing $\pi\mu$ is the direct sum of the blocks M_i in such an order, say $\{M'_i \mid i = 1, 2, \ldots\}$, that $\det M'_i = d'_i$. In view of the special property of the sequence $\{d'_i\}$, it follows by grouping finite numbers of consecutive M'_i, that N is the direct sum of finite matrices $N_1, N_2, \ldots$, where N_i is the direct sum of finitely many of the M'_i, $\det N_1 = \prod_{j=1}^{n_1} d'_j \neq 0$, and $\det N_i = 1$ $(i \geq 2)$. Thus $\pi\mu$ can be factorized as $\phi\nu$, where ϕ is a finitary automorphism of A_p (represented relative to X by the direct sum of N_1 and the infinite identity matrix), and ν is represented by the direct sum D say of the $n_1 \times n_1$ identity matrix and the matrices $N_2, N_3, \ldots$. Since, for $i \geq 2$, $\det N_i = 1$ in Z_p, we can find (as in the proof of

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Lemma 1.1) integer matrices $\hat{N}_2, \hat{N}_3, \ldots$ of determinant 1 (in Z) such that the entries of $\hat{N}_i$ are congruent modulo p to the corresponding entries of N_i . Define $\hat{D}$ to be the integer matrix obtained as the direct sum of the $n_1 \times n_1$ identity matrix and $\hat{N}_2, \hat{N}_3, \ldots$. Clearly $\hat{D}$ represents an automorphism of A relative to $\hat{X}$, and since $\hat{X}\eta = X$, this automorphism induces ν .

Since, by Lemma 1.1, ϕ is induced by an element of Γ , and since π^{-1} is clearly induced (by the automorphism in Γ which permutes $\hat{X}$ appropriately), it follows that $\pi^{-1}\phi\nu = \mu$ is induced by an element of Γ , as required.

2. Proof of Lemma 1.2

We have to show that, given any basis $\{v_1, v_2, \ldots\}$ of A_p , there is a free basis $\{c_1, c_2, \ldots\}$ of A and integers $k_1, k_2, \ldots$ such that $c_i\eta = k_iv_i$ $(i = 1, 2, \ldots)$. For convenience we shall understand A in this section to be the group of all finitary sequences of integers (that is, infinite sequences with finite support), A_p to be the vector space of all finitary sequences of elements of Z_p , and $\eta : A \to A_p$ to be the map replacing members of each sequence in A by their images under the natural map from Z to Z_p .

We shall need some further notation. For each nonzero $x = \{x_i \mid i = 1, 2, \ldots\} \in A$, write $\lambda(x)$ for the "length" of x ; that is, for the largest integer i such that $x_i \neq 0$. Write also $\mu(x)$ for $x_{\lambda(x)}$; that is, $\mu(x)$ is the last nonzero member of the sequence x . Define λ, μ for nonzero sequences in A_p similarly.

We isolate part of the proof as a lemma. This lemma says that, given $m > 1$ sequences from A such that:

(1) their images under η are linearly independent;

(2) they can be ordered so that their lengths are strictly

increasing except for the last two, which have the same length; and

(3) the last entries of the first $(m-1)$ sequences are all 1,

then the mth sequence can be replaced by a linear combination of the m sequences plus an element in $\ker \eta$, which:

(1) is of smaller length;

(2) differs in length from the first $(m-1)$ sequences; and

(3) has last entry 1.

The notation used in the following statement of the lemma is chosen for ease of application to the proof of Lemma 1.2.

LEMMA 2.1. *Let* $a_1, \ldots, a_m$ *be* $m > 1$ *sequences from* A *satisfying the following three conditions:*

$$a_1\eta, \ldots, a_m\eta \text{ are linearly independent;}$$

$$\lambda(a_1) < \ldots < \lambda(a_{m-1}) = \lambda(a_m) ;$$

$$\mu(a_i) = 1 \quad (i = 1, \ldots, m-1) .$$

Then there exist integers $l_1, \ldots, l_{m-1}, k$ *, a sequence* a *in* $\ker \eta$ *, and an integer* r *,* $1 \le r \le m$ *, such that the sequences* $a_i^{(1)}$ $(i = 1, \ldots, m)$ *given by*

$$\text{(1)} \qquad \begin{cases} a_i^{(1)} = a_i \quad (i = 1, \ldots, r-1) ; \\ a_r^{(1)} = ka_m + a + \sum_{j=1}^{m-1} l_j a_j ; \\ a_i^{(1)} = a_{i-1} \quad (i = r+1, \ldots, m) , \end{cases}$$

satisfy the following two conditions

$$\text{(2)} \qquad \lambda\left(a_1^{(1)}\right) < \ldots < \lambda\left(a_m^{(1)}\right) ;$$

$$\text{(3)} \qquad \mu\left(a_i^{(1)}\right) = 1 \quad (i = 1, \ldots, m) .$$

Proof. Write briefly λ_i for $\lambda(a_i)$ $(i = 1, \ldots, m)$. There are

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integers $n_1, \ldots, n_{m-1}$ such that

$$b = a_m + n_1a_1 + n_2a_2 + \ldots + n_{m-1}a_{m-1}$$

has its λ_ith components $(i = 1, \ldots, m)$ all zero. Not every component of b is divisible by p since the $a_i\eta$ $(i = 1, \ldots, m)$ are linearly independent. Let j be the largest integer such that the jth component of b is not divisible by p, and let k be an integer such that the jth component of kb is congruent to 1 modulo p. Define $a \in A$ to be such that $a \in \ker \eta$ (that is, its components are all divisible by p), and $kb + a$ has its λ_ith components $(i = 1, \ldots, m)$ all zero, all its components after the jth zero, and its jth component 1. There is an integer r with $1 \leq r < m$ such that

$$\lambda_{r-1} < j < \lambda_r$$

(where λ_0 is defined to be 0). Set $l_i = kn_i$ $(i = 1, \ldots, m-1)$. With these definitions of $k, l_1, \ldots, l_{m-1}, a, r$, the conclusion of the lemma is readily verified.

Proof of Lemma 1.2. We may assume that the v_i have been ordered so that $\lambda(v_i) \leq \lambda(v_{i+1})$ $(i = 1, 2, \ldots)$. We define a sequence $\{m_1, m_2, \ldots\}$ (which may be finite or even empty) of integers as follows: let m_1 be the smallest integer such that $\lambda(v_{m_1-1}) = \lambda(v_{m_1})$, and for $i > 1$, define m_i inductively to be the smallest integer greater than m_{i-1} such that $\lambda(v_{m_i-1}) = \lambda(v_{m_i})$. For each $i \geq 1$ which is not an m_j for any j, define k_i to be an integer such that $k_i\mu(v_i)$ is the identity element of Z_p. Corresponding to each such v_i choose $a_i \in A$ such that $a_i\eta = k_iv_i$ and $\mu(a_i) = 1$. For each $i \geq 1$ that *is* an m_j for some j, choose $a_i \in A$ such that $a_i\eta = v_i$ and $\lambda(a_i) = \lambda(v_i)$. (Thus for these i, although $\mu(a_i) \not\equiv 0 \bmod p$, we may not have that $\mu(a_i) = 1$.)

We now define a sequence

$$a^{(1)},\ a^{(2)},\ \ldots \tag{4}$$

(possibly finite or even empty) of elements of A, all in $\ker \eta$, and a sequence

$$k_{m_1},\ k_{m_2},\ \ldots \tag{5}$$

of integers, as follows: the elements $a_1, \ldots, a_{m_1}$ of A satisfy the hypotheses of Lemma 2.1 (with m_1 replacing m), so that there is a sequence $a^{(1)}$ (corresponding to a in Lemma 2.1), an integer k_{m_1} (corresponding to k), and sequences $a_1^{(1)}, \ldots, a_{m_1}^{(1)}$, satisfying the conclusion. Then $a_1^{(1)}, \ldots, a_{m_1}^{(1)}, a_{m_1+1}, \ldots, a_{m_2}$ again satisfy the hypotheses of Lemma 2.1. Suppose inductively that $a_1^{(i)}, \ldots, a_{m_i}^{(i)}$ have been defined so that

$$a_1^{(i)}, \ \ldots, \ a_{m_i}^{(i)}, \ a_{m_i+1}, \ \ldots, \ a_{m_{i+1}}$$

satisfy the hypotheses of Lemma 2.1; we define $a^{(i+1)}$, $k_{m_{i+1}}$, and $a_1^{(i+1)}, \ldots, a_{m_{i+1}}^{(i+1)}$ as in the conclusion. It then follows that

$$a_1^{(i+1)}, \ \ldots, \ a_{m_{i+1}}^{(i+1)}, \ a_{m_{i+1}+1}, \ \ldots, \ a_{m_{i+1}}$$

again satisfy the hypotheses of Lemma 2.1. This completes the definition of the sequences (4) and (5) and of the elements $a_i^{(j)}$ (for sufficiently large j).

For each $i \geq 1$ let $r(i)$ be the least of the integers j for which $a_i^{(j)}$ is defined. (Thus $m_{r(i)-1} < i \leq m_{r(i)}$.) From the definition of the $a_i^{(j+1)}$ $(1 \leq i \leq m_{j+1})$ in terms of both the $a_i^{(j)}$ $(1 \leq i \leq m_j)$ and the a_i $(m_j < i \le m_{j+1})$, as in (1), and the fact (corresponding to (2)) that

$$\lambda\left(a_1^{(j)}\right) < \lambda\left(a_2^{(j)}\right) < \ldots < \lambda\left(a_{m_j}^{(j)}\right) \quad (j = 1, 2, \ldots) ,$$

it follows that the sequence

$$a_i^{(r(i))}, a_i^{(r(i)+1)}, \ldots ,$$

becomes constant after at most $\lambda\left(a_i^{(r(i)+1)}\right)$ steps. Hence for each $i \ge 1$ there is a j such that for all $s \ge 1$, $a_i^{(j)} = a_i^{(j+s)}$ $(= b_i$ say$)$. This together with (1) implies that the sets

$$B = \{b_i \mid i = 1, 2, \ldots\}$$

and

$$C = \{a_i \mid i = 1, 2, \ldots, i \ne m_j \text{ for any } j\} \cup$$
$$\cup \left\{k_{m_i} a_{m_i} + a^{(i)} \mid i = 1, 2, \ldots\right\}$$

generate the same subgroup of A. Now the set C has one of the properties demanded in Lemma 1.2, namely, if we define $c_i = a_i$ $(i \ne m_j$ for any $j)$, $c_{m_i} = k_{m_i} a_{m_i} + a^{(i)}$, then for all i, $c_i \eta = k_i v_i$. To complete the proof we shall show that B, and hence C, generates the whole of A.

Since $C\eta$ spans A_p, and B and C generate the same subgroup of A, we have that $B\eta$ also spans A_p. Now (2), (3) imply that

(6) $$\lambda(b_i) < \lambda(b_{i+1}) \quad (i = 1, 2, \ldots) ,$$

and

(7) $$\mu(b_i) = 1 \quad (i = 1, 2, \ldots) .$$

From (6) and (7) it follows that the vectors $b_1\eta, b_2\eta, \ldots$, are also "in echelon"; that is, that $\lambda(b_i\eta) < \lambda(b_{i+1}\eta)$ $(i = 1, 2, \ldots)$. This and the fact that the $b_i\eta$ span A_p together imply that $\lambda(b_i\eta) = i$ $(i = 1, 2, \ldots)$, whence by (7), also $\lambda(b_i) = i$. This and (7) imply that B is a free basis for A , as required.

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