PURE-COMPLETE SUBGROUPS OF DIRECT SUMS OF PRÜFER GROUPS

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Suppose that G is a p-primary abelian group. The subgroup $G[p] = \{x \in G: px = 0\}$ is called the socle of G and any subgroup S of $G[p]$ is called a subsocle of G. If each subsocle of G supports a pure subgroup, then G is said to be pure-complete [**1**]. It is well known that, if G is a direct sum of cyclic groups, then G is necessarily pure-complete. Further results about pure-complete groups are contained in [**1**] and [**3**].

By the Prüfer group (associated with the prime p), we mean the p-primary group generated by $a_0, a_1, \ldots, a_n, \ldots$ with defining relations: $pa_0 = 0$ and $p^n a_n = a_0$ for $n \geqq 1$. Suppose that G is a subgroup of an arbitrary direct sum of Prüfer groups. Then G is not too far removed from a direct sum of cyclic groups, for G is an extension of an elementary group by a direct sum of cyclic groups. Thus it would be natural to ask the following question. Must G be pure-complete (if it has no elements of infinite height)? The answer is in the negative; in fact, I proved in [**2**] that there exists a subgroup G of a direct sum of Prüfer groups such that:

(1) G has no elements of infinite height,
(2) G is not a direct sum of cyclic groups,
(3) G is not pure-complete.

At the time, little did I suspect that (2) *implies* (3) (whenever G is a subgroup of a direct sum of Prüfer groups). The object of this paper is to prove that this is the case.

THEOREM. *Suppose that G is a subgroup of a direct sum of Prüfer groups. If G is pure-complete, then G is a direct sum of cyclic groups.*

Proof. Let G be a subgroup of a direct sum K of Prüfer groups and suppose that G is pure-complete. Letting $S = G \cap p^\omega K$, we have that $pS = 0$ and that $G/S \cong \{G, p^\omega K\}/p^\omega K \subseteq K/p^\omega K$ is a direct sum of cyclic groups. We know [**4**] that $G[p]/S \subseteq (G/S)[p]$ is a summable subsocle of G/S. Hence $G[p]/S = \sum_{n<\omega} T_n$, where the nonzero elements of T_n have height exactly n in G/S. We can write $T_n = \sum_{i \in I(n)} \{x_i + S\}$, and we can choose x_i to have height n in G for each $i \in I(n)$. We now observe that

$$(D) \qquad G[p] = S + \sum_{n<\omega} \Big(\sum_{i \in I(n)} \{x_i\} \Big)$$

is a natural decomposition of $G[p]$; see [**4**]. Since G is pure-complete, there exists a pure subgroup H of G such that $H[p] = S$. And there is a pure subgroup A of G such that $A[p] = \sum \{x_i\}$ (whether G is pure-complete or not). Since (D) is a natural decomposition, $A + H$ is pure in G and we have $G = A + H$. However, $G/S \cong A + pH$, which implies that pH, as well as A, is a direct sum of cyclic groups. It follows that H is a direct sum of cyclic groups and so is G; see, for example, [**6**, p. 92].

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An immediate corollary to the theorem is the following.

COROLLARY. *If G_1 and G_2 are subgroups of a direct sum of Prüfer groups, then $G = G_1 + G_2$ is pure-complete if and only if G_1 and G_2 are pure-complete.*

The corollary is false if the hypothesis that G_1 and G_2 are subgroups of a direct sum of Prüfer groups is deleted [**1**].

In conclusion, we remark that the above theorem points out the preponderance of groups without elements of infinite height having fairly simple structure that are not pure-complete. The reader may wish to compare this result with the first example [**5**] of a group without elements of infinite height and not pure-complete, as well as with the results of [**3**] on pure-complete groups.

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