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PERMUTATIONS RELATED TO SECANT, TANGENT AND EULERIAN NUMBERS

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1. Introduction. It is well known that

$$\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sec x + \tan x \tag{1}$$

where A_n denotes the number of "up-down" or alternating permutations

$$e_1 < e_2 > e_3 < e_4 > \cdots \tag{2}$$

of $1, 2, \ldots, n$. The numbers A_{2n} and A_{2n+1} are known as the secant and tangent numbers respectively and $A_{2n} = (-1)^n E_{2n}$, where E_n is the Euler number.

A brief history of the numbers A_n is given by Gould [63] who notes that alternating permutations were enumerated in closed form by D. André [7] as long ago as 1879 and that a detailed exposition of the method is given by E. Netto [69, pp. 105–112] where various references in the literature are given. By first obtaining a recurrence relation and then solving a differential equation, Blundon [9] gives a short self-contained derivation of (1).

The familiar Eulerian numbers $A_r(n)$ given explicitly by

$$A_r(n) = \sum_{i=0}^{r} (-1)^i \binom{n+1}{i} (r+1-i)^n, \tag{3}$$

with the generating function

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n-1} A_r(n) y^r = \frac{1-y}{e^{(y-1)x} - y}, \tag{4}$$

enumerate those permutations $e_1 e_2 \cdots e_n$ of $1, 2, \ldots, n$ containing precisely r rises, a rise being a pair $e_i < e_{i+1}$, $i = 1, \ldots, n-1$. (Often an initial rise is considered to occur at e_1 but this is not done here.) The Eulerian numbers were discussed in 1755 by Euler [54]. Discussions of the Eulerian numbers are also given by Carlitz and Scoville [38], Comtet [45], Foata and Schützenberger [56] and Riordan [72].

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The sec-tan and Eulerian numbers, their properties, combinatorial interpretations as permutation problems, related numbers and problems, generalizations, extensions and refinements have been studied extensively in the literature. Papers involving both numbers, include [22, 23, 26, 58]. Papers relating more to the sec-tan numbers include [7]–[10] incl., [17, 19, 20, 24, 25, 27, 29, 31, 32, 36, 37, 46, 49, 51, 52, 53, 55, 57, 59, 64, 74, 78]. Papers relating more to Eulerian numbers include [1] to [6] incl., [11]–[16] incl., [18, 21, 24, 28, 29, 30, 33, 34, 35], [38]–[44] incl., [47, 48, 50, 60, 62, 65, 66, 67, 68, 71, 73, 75, 79, 80, 81].

Stanley [76], [77] and Gessel [61] in his Ph.D. thesis under the supervision of Stanley unify many disparate topics in combinatorics. Included in their works are generalizations of the sec-tan and Eulerian numbers which contain as special cases many results obtained by others. More precisely, a general theory for the enumeration of order-reversing maps of finite ordered sets into chains is developed in [76], a unified method involving Möbuis functions associated with binomial posets for enumerating permutations of sets and multisets is given in [77] and a general theory of enumeration of finite sequences by generating functions is developed in [61].

In this paper, we give a simple, elementary and purely combinatorial derivation of the following generalization of the sec-tan and Eulerian numbers. This is done by noting a natural relationship between the restricted permutations considered and corresponding restricted distributions of distinct objects into distinct cells.

In a permutation $e_1e_2\cdots e_n$ of $1, 2, \ldots, n$ we define a run of length $l \geq 1$ to be a sequence

$$e_{i+1} < e_{i+2} < \cdots < e_{i+l}$$

satisfying $e_i > e_{i+1}$, $e_{i+l} > e_{i+l+1}$ with $e_0 = n+1$, $e_{n+1} = 0$. For example, the permutation 3 4 1 6 2 7 9 8 5 contains precisely five runs,

$$3<4, 1<6, 2<7<9, 8, 5,$$

of lengths 2, 2, 3, 1, 1 respectively.

Note that a permutation of $1, 2, \ldots, n$ with precisely r runs contains precisely $n-r$ rises and precisely $r-1$ falls, a fall being a pair $e_i > e_{i+1}$, $i = 1, 2, \ldots, n-1$.

Denote by $A_r(k, n, t_1, t_2)$, $0 \leq r \leq n$, k, t_1, $t_2 \geq 1$, the number of permutations of $1, 2, \ldots, kn+t_1+t_2$ with (a) precisely $n+2-r$ runs, (b) the lengths of the first and last runs not less than t_1 and t_2 respectively and congruent to $t_1 \pmod k$ and $t_2 \pmod k$ respectively and (c) the lengths of all the other $n-r$ runs congruent to 0 (mod k). That is, $A_r(k, n, t_1, t_2)$ is the number of permutations of $1, 2, \ldots, kn+t_1+t_2$ satisfying

$$(5) \qquad e_1 < e_2 < \cdots < e_{m_1} > e_{m_1+1} < e_{m_1+2} < \cdots < e_{m_1+m_2} > \cdots > \cdots$$

with

(6) $t_1 \le m_1 \equiv t_1 (\mathrm{mod}\ k),\ t_2 \le m_{n+2-r} \equiv t_2\ (\mathrm{mod}\ k),\ 0 < m_i \equiv 0 (\mathrm{mod}\ k),$

$i = 2, \ldots, n+1-r$ where m_i is the length of the ith run, $i = 1, 2, \ldots, n+2-r$. We define $A_{n+1}(k, n, t_1, t_2) = 1$.

In §2 we give an explicit formula (19) for $A_r(k, n, t_1, t_2)$ and also show that

$$(7)\quad \sum_{n=0}^{\infty} \sum_{r=0}^{n+1} A_r(k, n, t_1, t_2) y^r \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!} = (y-1)\left[\frac{\varphi_{k,t_1}(x, y)\varphi_{k,t_2}(x, y)}{y-\varphi_{k,0}(x, y)} + \varphi_{k,t_1+t_2}(x, y)\right]$$

where

$$(8)\qquad \varphi_{k,t}(x, y) = \sum_{j=0}^{\infty} \frac{(y-1)^j x^{t+jk}}{(t+jk)!}, \qquad t = 0, 1, 2, \ldots.$$

The special cases $A_0(k, n-1, k, t)$ and $A_0(k, n-1, k, k)$ noted in §4 are results of Carlitz [25]. These special cases involve the Olivier functions $\varphi_{k,t}(x, 0)$ defined by (8) with $y = 0$. See Carlitz [10] for some arithmetic properties of these functions.

2. **Distributions of** $1, 2, \ldots, nk+t_1+t_2$ **into distinct cells.** Denote by $g(k, n, t_1, t_2, p)$, $n \ge p-2 \ge 0$, $t_1 \ge 1$, $t_2 \ge 1$, the number of distributions.

$$(9)\qquad \underbrace{e_1 < e_2 < \cdots < e_{m_1}}_{\text{cell } 1} \mid \underbrace{e_{m_1+1} < \cdots < e_{m_1+m_2}}_{\text{cell } 2} \mid \cdots \mid \underbrace{\cdots < e_{kn+t_1+t_2}}_{\text{cell } p} \mid$$

of the integers $1, 2, \ldots, kn+t_1+t_2$ into p distinct cells such that

(10) $t_1 \le m_1 \equiv t_1\ (\mathrm{mod}\ k),\ t_2 \le m_p \equiv t_2\ (\mathrm{mod}\ k)$ and $0 < m_i \equiv 0\ (\mathrm{mod}\ k)$

for $i = 2, \ldots, p-1$, where m_i denotes the number of objects in cell i. Integers in a cell are arranged in rising order. Then

$$(11)\quad g(k, n, t_1, t_2, p) = \sum (kn+t_1+t_2)!/(i_1k+t_1)!\,(i_2k)!\,(i_3k)! \cdots (i_{p-1}k)!\,(i_pk+t_2)!$$

where the summation is over all $i_1 \ge 0$, $i_p \ge 0$, $i_2 > 0$, $i_3 > 0, \ldots, i_{p-1} > 0$ satisfying $i_1 + \cdots + i_p = n$ and the exponential generating function is

$$(12)\quad \sum_{n=p-2}^{\infty} g(k, n, t_1, t_2, p) \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!} = \left(\sum_{i=1}^{\infty} \frac{x^{ik}}{(ik)!}\right)^{p-2} \sum_{j=0}^{\infty} \frac{x^{t_1+jk}}{(t_1+jk)!} \sum_{l=0}^{\infty} \frac{x^{t_2+lk}}{(t_2+lk)!}.$$

We define $g(k, n, t_1, t_2, 1) = 1$.

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In a distribution (9) satisfying (10) define the distribution property P_i to be, "e_{t_1+ik} and e_{t_1+ik+1} are in the same cell", $i=0,1,2,\ldots,n$.

LEMMA. *The number of distributions* (9) *satisfying* (10) *and satisfying precisely j of the $n+1$ distribution properties $P_0, P_1, \ldots, P_n$ is $g(k,n,t_1,t_2,n+2-j)$.*

Proof. Suppose a distribution (9) satisfying (10) satisfies precisely j of the $n+1$ properties with cell i satisfying precisely $j_i(\geq 0)$ of the properties. Then cell i contains $(j_i+1)k$ integers, $i=2,\ldots,p-1$, cell 1 contains j_1k+t_1 integers and cell p contains j_pk+t_2 integers. Since $j_1+\cdots+j_p=j$ and $(j_1k+t_1)+(j_2+1)k+(j_3+1)k+\cdots+(j_{p-1}+1)k+(j_pk+t_2)=kn+t_1+t_2$ it follows that $p=n+2-j$.

3. **The numbers** $A_r(k,n,t_1,t_2)$. There are $(kn+t_1+t_2)!/(k!)^n t_1!\,t_2!$ linear arrangements

$$e_1e_2\cdots e_{kn+t_1+t_2} \tag{13}$$

of $1,2,\ldots,kn+t_1+t_2$, $n\geq 0$, $k,t_1,t_2\geq 1$ satisfying

$$\begin{aligned} &e_1<e_2<\cdots<e_{t_1},\\ &e_{t_1+1}<e_{t_1+2}<\cdots<e_{t_1+k}\\ &e_{t_1+k+1}<e_{t_1+k+2}<\cdots<e_{t_1+k2}\\ &\vdots\\ &e_{t_1+k(n-1)+1}<e_{t_1+k(n-1)+2}<\cdots<e_{t_1+kn}\\ &e_{t_1+kn+1}<e_{t_1+kn+2}<\cdots<e_{t_1+kn+t_2}. \end{aligned} \tag{14}$$

Denote by $A_r(k,n,t_1,t_2)$ the number of arrangements (13) satisfying (14) and satisfying precisely r of the $n+1$ arrangement properties

$$e_{t_1}<e_{t_1+1},\ e_{t_1+k}<e_{t_1+k+1},\ e_{t_1+k2}<e_{t_1+k2+1},\ldots,\ e_{t_1+kn}<e_{t_1+kn+1} \tag{15}$$

denoted by P_0, $P_1,\ldots,$ respectively.

Denote by $N(P_{u_1},P_{u_2},\ldots,P_{u_j})$ the number of arrangements (13) satisfying (14) and satisfying the j arrangement properties $P_{u_1},\ldots,P_{u_j}$ (and possibly other of the properties). Let

$$s(j)=\sum N(P_{u_1},P_{u_2},\ldots,P_{u_j}) \tag{16}$$

where the summation is taken over all j-combinations $0\leq u_1<u_2<\cdots<u_j\leq n$. By the general principle of inclusion and exclusion we have

$$A_r(k,n,t_1,t_2)=\sum_{j=0}^{n+1-r}(-1)^j\binom{r+j}{j}s(r+j) \tag{17}$$

with $s(0)=(kn+t_1+t_2)!/(k!)^n t_1!\,t_2!$. We now show that

$$s(j)=g(k,n,t_1,t_2,n+2-j). \tag{18}$$

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Proof of (18). Upon removing the bars of a distribution (9) satisfying (10) and also satisfying precisely the j distribution properties $P_{u_1}, \ldots, P_{u_j}$ we obtain an arrangement counted by $N(P_{u_1}, \ldots, P_{u_j})$ where the arrangement properties $P_{u_1}, \ldots, P_{u_j}$ correspond to the distribution properties $P_{u_1}, \ldots, P_{u_j}$. Further every arrangement counted by $N(P_{u_1}, \ldots, P_{u_j})$ can be obtained this way. Using the Lemma in §2, (18) follows.

Hence, (17) and (18) give the formula

$$A_r(k, n, t_1, t_2) = \sum_{j=0}^{n+1-r} (-1)^j \binom{r+j}{j} g(k, n, t_1, t_2, n+2-r-j), \tag{19}$$

and the ordinary generating function is

$$\sum_{r=0}^{n+1} A_r(k, n, t_1, t_2) y^r = \sum_{j=0}^{n+1} g(k, n, t_1, t_2, n+2-j)(y-1)^j. \tag{20}$$

Proof of (7). From (20), the left side of (7)

$$= \sum_{n=0}^{\infty} (y-1)^{n+1} \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!} + \sum_{n=0}^{\infty} \sum_{m=2}^{n+2} g(k, n, t_1, t_2, m)(y-1)^{n+2-m} \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!}$$

$$= (y-1)\varphi_{k,t_1+t_2}(x, y) + \sum_{m=2}^{\infty} \sum_{n=m-2}^{\infty} g(k, n, t_1, t_2, m) \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!} (y-1)^{n+2-m}$$

$$= (y-1)\varphi_{k,t_1+t_2}(x, y) + \sum_{m=2}^{\infty} (\varphi_{k,0}(x, y) - 1)^{m-2}$$

$$\times \varphi_{k,t_1}(x, y)\varphi_{k,t_2}(x, y)(y-1)^{2-m} \quad \text{(by (12))}$$

$$= (y-1)\varphi_{k,t_1+t_2}(x, y) + \frac{(y-1)}{y - \varphi_{k,0}(x, y)} \varphi_{k,t_1}(x, y)\varphi_{k,t_2}(x, y)$$

$=$ right side of (7).

We also note the relation

$$A_r(k, n, t_1, t_2) - A_{r-1}(k, n-1, t_1, k+t_2)$$
$$= \binom{kn+t_1+t_2}{t_2} A_r(k, n-1, t_1, k) - A_r(k, n-1, t_1, k+t_2)$$

where both sides are equal to the number of permutations counted by $A_r(k, n, t_1, t_2)$ with the last run of length exactly t_2.

4. **Special Cases.** The special case $A_0(k, n, t_1, t_2)$ counts those permutations of $1, 2, \ldots, kn+t_1+t_2$ with the first and last runs of lengths t_1 and t_2 respectively and each of the other runs of length k. By (7), with $y = 0$, we have

$$\sum_{n=0}^{\infty} A_0(k, n, t_1, t_2) \frac{x^{kn+t_1+t_2}}{(kn+t_1+t_2)!} = \frac{\varphi_{k,t_1}(x, 0)\varphi_{k,t_2}(x, 0)}{\varphi_{k,0}(x, 0)} + \varphi_{k,t_1+t_2}(x, 0). \tag{21}$$

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Eulerian numbers. The number of permutations of $1, 2, \ldots, n$ containing precisely r rises (or precisely $n-r$ runs) is the well-known Eulerian number $A_r(n) = A_r(1, n-2, 1, 1)$. Noting that

$$\varphi_{1,1}(x, y) = \frac{e^{(y-1)x} - 1}{y-1}, \tag{22}$$

and by (7), we obtain the known generating function (4). From (12) it follows that $g(1, n-2, 1, 1, p) = p!S(n, p)$ where

$$S(n, p) = \sum_{i=0}^{p-1} (-1)^i \binom{p}{i} (p-i)^n / p!$$

is the Stirling number of the second kind. By (19), use of a simple identity, and the fact that $A_r(n)$ has the symmetric property $A_r(n) = A_{n-1-r}(n)$ we obtain the familiar explicit formula

$$\begin{aligned} A_r(n) &= \sum_{i=0}^{n-1-r} (-1^i \binom{r+i}{r} (n-r-i)!\, S(n, n-r-i) \\ &= \sum_{i=0}^{n-1-r} (-1)^i \binom{n+1}{i} (n-r-i)^n. \\ &= \sum_{i=0}^{r} (-1)^i \binom{n+1}{i} (r+1-i)^n. \end{aligned} \tag{23}$$

The numbers $A_r(k, n, k, t)$. Using notation similar to that of Carlitz [25] we let $A_{r,k,t}(kn+t) = A_r(k, n-1, k, t)$ and define $A_{0,k,t}(k0+t) = 1$. Then, for $0 \le r \le n-1$, $A_{r,k,t}(kn+t)$ is the number of permutations of $1, 2, \ldots, kn+t$ with (a) exactly $n+1-r$ runs and (b) the length of each of the first $n-r$ runs congruent to $0 \pmod k$ and the length of the last run not less than t and congruent to $t \pmod k$. Let $A_{n,k,t}(kn+t) = 1$. Letting $t_1 = k$ and $t_2 = t$ in (7) and noting that

$$\begin{aligned} \varphi_{k,k}(x, y) &= \frac{\varphi_{k,0}(x, y) - 1}{y-1} \\ \varphi_{k,k+t}(x, y) &= \frac{1}{y-1}\left(\varphi_{k,t}(x, y) - \frac{x^t}{t!}\right) \end{aligned} \tag{24}$$

we obtain

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} A_{r,k,t}(kn+t) y^r \frac{x^{kn+t}}{(kn+t)!} = \frac{(y-1)\varphi_{k,t}(x, y)}{y - \varphi_{k,0}(x, y)}. \tag{25}$$

Letting $t = k$ in (25) we obtain

$$1 + \sum_{n=0}^{\infty} \sum_{r=0}^{n} A_{r,k,k}(kn+k) y^r \frac{x^{kn+k}}{(kn+k)!} = \frac{y-1}{y - \varphi_{k,0}(x, y)} \tag{26}$$

with $A_{r,k,k}(kn+k)$ being the number of permutations of $1, 2, \ldots, k(n+1)$ having exactly $n+1-r$ runs and with the length of each run a positive multiple of k.

In the case $r=0$, by (21) or (25), we obtain

$$\sum_{n=0}^{\infty} A_{0,k,t}(kn+t)\frac{x^{kn+t}}{(kn+t)!}=\frac{\varphi_{k,t}(x,0)}{\varphi_{k,0}(x,0)} \tag{27}$$

and letting $t=k$ in (27) we have

$$1+\sum_{n=0}^{\infty} A_{0,k,k}(kn+k)\frac{x^{kn+k}}{(kn+k)!}=\frac{1}{\varphi_{k,0}(x,0)}, \tag{28}$$

where $A_{0,k,t}(kn+t)$ is the number of permutation of $1, 2, \ldots, kn+t$ with the last run of length t and each of the other runs of length k. Both (27) and (28) are results of Carlitz [22] and [25] where he notes that $\varphi_{k,t}(x, 0)$ are the Olivier functions. See also Gessel [16, p. 51, example 2] and Stanley [74].

Permutations with runs of even length. The number of permutations of $1, 2, \ldots, 2n+2$ with precisely $n+1-r$ runs and with each of the runs of even length is $A_{r,2,2}(2n+2)$ and by (26),

$$1+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A_{r,2,2}(2n+2)y^r\frac{x^{2n+2}}{(2n+2)!}=\frac{y-1}{y-\varphi_{2,0}(x,y)} =\frac{y-1}{y-\cos(x\sqrt{(1-y)})}. \tag{29}$$

The number of permutations of $1, 2, \ldots, 2n+1$ with precisely $n+1-r$ runs and with each of the first $n-r$ runs of even length and the last run of odd length is $A_{r,2,1}(2n+1)$ with $A_{n,2,1}(2n+1)=1$ and by (25) it follows that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} A_{r,2,1}(2n+1)y^r\frac{x^{2n+1}}{(2n+1)!}=\frac{(y-1)\varphi_{2,1}(x,y)}{y-\varphi_{2,0}(x,y)} =\frac{\sqrt{(1-y)}\sin(x\sqrt{(1-y)})}{\cos(x\sqrt{(1-y)})-y}. \tag{30}$$

Let $A_{r,n}$ denote the number of permutations of $1, 2, \ldots, n$ with precisely $[(n+1)/2]-r$ runs and with each of even length if n is even and all but the last run even if n is odd. By combining (29) and (30) we obtain

$$1+\sum_{n=1}^{\infty}\sum_{r=0}^{(n-1)/2} A_{r,n}y^r\frac{x^n}{n!}=(y-1)\frac{(1+\varphi_{2,1}(x,y))}{(y-\varphi_{2,0}(x,y))} \tag{31}$$

Up-down permutations. The cases $A_{0,2,2}(2n+2)$ and $A_{0,2,1}(2n+1)$ are the numbers of up-down permutations for the even and odd cases respectively and

(29) and (30) reduce, respectively, to

(32) $$1+\sum_{n=0}^{\infty} A_{0,2,2}(2n+2)\frac{x^{2n+2}}{(2n+2)!}=\text{secant } x$$

and

(33) $$\sum_{n=0}^{\infty} A_{0,2,1}(2n+1)\frac{x^{2n+1}}{(2n+1)!}=\text{tangent } x.$$

Combining (32) and (33) and defining $A_0=1$ we obtain (1).

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