TABLES OF THE LOGARITM OF ITERATION OF e^x-1.

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1. Introduction

In the preceding paper [1]) one of us has proposed a definition for the "best" or most regularly growing fractional iterates of logarithmico-exponential type functions. The definition was essentially based on two observations. First, that the functional equation

$$A(e(x)) = A(x) + 1, \quad A(1) = 0, \quad e(x) = e^x - 1 \tag{1}$$

has exactly one solution with the property that

$$a(x) = A'(x) \tag{2}$$

is totally monotonic for every $x > 0$. Secondly, that if $f(x)$ is a logarithmico-exponential function such that

$$f(f(x)) \succ e^x \tag{3}$$

then the Abel equation

$$B(f(x)) = B(x) + 1 \tag{4}$$

has (apart from the arbitrariness of an additive constant) exactly one solution with the property that

$$\lim_{x\to\infty} B'(x)/a(x) = r \tag{5}$$

exists [2]) and the same is true for any reasonably well-behaved function with property (3) whose manner of growth does not transcend Hardy's scale of L-functions. Thus every such function has a uniquely determined family of fractional iterates given by

$$f_\sigma(x) = B_{-1}(B(x) + \sigma) \tag{6}$$

and these $f_\sigma(x)$ may be regarded as the most regularly growing iterates of $f(x)$.

For the actual determination of the regular fractional iterates of a given

[1] This Journal, p. 301. We shall make free use of the definitions and notations of that paper.
[2] r is an integer for all L-functions.

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function it is necessary to have a table of the standard comparison function $A(x)$. The present tables were prepared with precisely this purpose in mind. Although for most purposes a table of $A(x)$ would be sufficient, we found it desirable also to tabulate the derivative $a(x)$ which in some respects has a more fundamental significance than $A(x)$ itself. We conclude with the discussion of some numerical examples in § 3.

The computation of Tables 1 and 2 was carried out on the IBM 7090 computer of the Weapons Research Establishment at Salisbury S.A. We are greatly indebted to the Establishment for the use of their facilities.

2. Construction of the Tables

It is sufficient to construct the tables for the interval $I = (1, e-1)$; outside this interval $A(x)$ and $a(x)$ can be computed from equation (1) and from

$$a(e(x)) = e^{-x}a(x). \tag{7}$$

The construction is based on the asymptotic properties of $A(x)$ near the origin. Let x_0, y_0 be two values in the interval I, $x_0 < y_0$, and write

$$x_n = e_{-n}(x_0),\ y_n = e_{-n}(y_0),\ n = 0, 1, 2, \cdots$$

Then by (1),

$$A(y_0) - A(x_0) = A(y_n) - A(x_n). \tag{8}$$

But $A(y) - A(x) = \sigma$ is equivalent to $y = e_\sigma(x)$ where $e_\sigma(x)$ is the regular fractional iterate of order σ of $e(x)$. Now $e_\sigma(x)$ is characterized by the asymptotic series

$$e_\sigma(x) \simeq x + \tfrac{1}{2}\sigma x^2 + \sum_{n=3}^{\infty} b_n{}^{(\sigma)}x^n,\ x \searrow 0, \tag{9}$$

where the coefficients of the series are determined formally from

$$e(e_\sigma(x)) = e_\sigma(e(x)),\ e(x) = x + \tfrac{1}{2}x^2 + \sum_{n=3}^{\infty} \frac{1}{n!}x^n. \tag{10}$$

Hence we can compute $e_\sigma(x)$ with any desired accuracy provided that x is sufficiently close to 0. But if in (8) n is chosen large enough we can make x_n and y_n as small as we please and therefore $A(y_0) - A(x_0)$ can be determined with any desired accuracy. Similarly we can determine $a(x)$ for small x from the asymptotic series

$$1/a(x) \simeq \tfrac{1}{2}x^2 - \tfrac{1}{12}x^3 + \tfrac{1}{48}x^4 - \tfrac{1}{180}x^5 + \cdots \tag{11}$$

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and $a(x_0)$ is obtained from the relation

$$a(x_0) = a(x_n) \prod_{i=0}^{n-1} (1 + x_i)^{-1} \tag{12}$$

In the actual computation of Tables 1 and 2, $n = 40$ was selected for the number of steps so as to bring x_n, y_n down to about 0.05. In fact if x, y and σ are in the range (0, 0.05) then the series (11) for $a(x)$ and the series

$$\begin{aligned} y = e_\sigma(x) \simeq x &+ \sigma(\tfrac{1}{2}x^2 - \tfrac{1}{12}x^3 + \tfrac{1}{48}x^4 - \tfrac{1}{180}x^5 + \cdots) \\ &+\sigma^2(\tfrac{1}{4}x^3 - \tfrac{5}{48}x^4 + \tfrac{1}{24}x^5 + \cdots) + \sigma^3(\tfrac{1}{8}x^4 + \cdots) \end{aligned} \tag{13}$$

for $e_\sigma(x)$ should not be in error by more than 1 part in 10^8.

The value of $a(x_0)$ was obtained directly from (11) and (12) and $A(y_0) - A(x_0)$ was calculated from (8), by solving (13) for σ. The selection of the values for x_0, y_0 was as follows: Starting from $x_0 = 1$, y_0 was increased by steps of 0.001 until the computed value of $A(y_0) - A(x_0)$ has reached 0.05. Then x_0 was replaced by the last value of y_0 and the process repeated.

Values of $A(x)$ were checked against a similarly constructed table for the interval (log 2,1) by means of the relation

$$A(x) - A(\log(1 + x)) = 1. \tag{14}$$

Numerical integration of the table of $a(x)$ at intervals of 0.06 has afforded another independent check, through the formula

$$A(y) = \int_1^y a(x)dx. \tag{15}$$

These tests revealed a systematic error in both tables, presumably due to the accumulation of truncation errors by the machine. However when a correction factor of 1/1.00000100 was applied to all $A(x)$ values and a factor of 1/1.00000200 to all $a(x)$ values, a complete agreement was reached and Tables 1 and 2 should not be in error by more than 1 figure in the last tabulated decimal place.

3. Examples

We demonstrate the use of the tables by some numerical examples.

Problem 1. To determine the (regular) fractional iterates of a given function.

Example: Calculate $\exp_{\frac{1}{2}} 0$. From tables of exp and log

$$\exp_3 0 = 15.1542623, \quad e_{-3}(\exp 15.1542623) = 1.3303016$$

and from Table 1, $A(1.3303016) = 0.5839356$.

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Therefore $A(e_{-3-\frac{1}{2}}(\exp 15.1542623)) = 0.0839356$ and from Table 1 $e_{-3-\frac{1}{2}}(\exp 15.1542623) = 1.0376284$, hence

$$\exp_{\frac{1}{2}} 15.1542623 \simeq e_{\frac{1}{2}}(\exp 15.1542623) = e_4(1.0376284) = \exp.\ 178.00275;$$

$$\exp_{\frac{1}{2}} 0 = \log_3 178.00275 = 0.4978330.$$

The calculation gives immediately the values

$$\exp_{\frac{1}{2}} 0.4978330 = 1,\ \exp_{\frac{1}{2}} 1 = 1.6451523,$$

$$\exp_{\frac{1}{2}} 1.6451523 = 2.7182818 \text{ etc., and also}$$

$$\exp_{\frac{1}{2}}(-\infty) = \log 0.4978330 = -0.6974906.$$

Similarly we obtain

$$\begin{aligned} \exp_{\frac{1}{2}} 10 &= 61.3841 = \exp 4.11715,\\ \exp_{\frac{1}{2}} 100 &= \exp 12.1873,\\ \exp_{\frac{1}{2}} 1000 &= \exp 26.8538,\\ \exp_{\frac{1}{2}} 10000 &= \exp 50.6144. \end{aligned}$$

The graph of the function $\exp_{\frac{1}{2}} x$ is shown in Fig. 1.

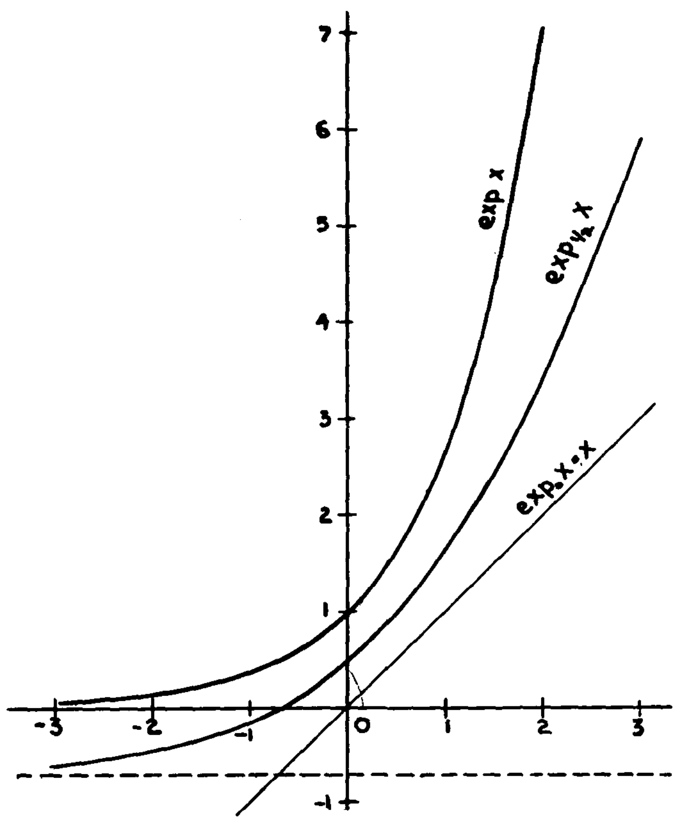

Fig. 1.

PROBLEM 2. To determine the logarithm of iteration $\sigma = B(y_0) - B(x_0)$ of a given $f(x)$.

EXAMPLE: Calculate $B(10000) - B(100)$ for $f(x) = \exp x^{\frac{1}{2}}$.

We have $f_4(100) = \exp_3 75.51343$, $e_{-5}(f_4(100)) = 1.6697762$ and $A(1.6697762) = 0.9577071$ from Table 1. Similarly $f_3(10000) = \exp_3 49.306853$, $e_{-5}(f_3(10000)) = 1.5929307$, $A(1.5929307) = 0.8859201$, $A(e_{-4}(f_3(10000)) = 1.8859201$ and $B(10000) - B(100) \simeq 1.8859201 - 0.9577071 = 0.928213$. Or,

$$f_{0.928213}(100) = 10000.$$

PROBLEM 3. To determine the derivative of the logarithm of iteration of a given $f(x)$.

EXAMPLE: Calculate $b(4) = B'(4)$ for $f(x) = \Gamma(x)$. We have $\Gamma_4(4) = \exp_2 459.1386$, $e_{-4}(\exp_2 459.1386) = 1.0867170$ and from Table 2, $A'(1.0867170) = 1.976464$. By repeated application of

$$e^x A'(e(x)) = A'(x)$$

we obtain

$$\exp_2 459.1386 \times A'(\exp_2 459.1386) = 0.2031713 \times 10^{-3}. \tag{16}$$

Since $\Gamma(x)$ is of the type $\exp\ (x \log x)$, we have

$$\lim_{x\to\infty} B'(x)/A'(x) = 1$$

and we can set

$$B'(\Gamma_4(4)) = B'(\exp_2 459.1386) \simeq A'(\exp_2 459.1386)$$

where $A'(\exp_2 459.1386)$ is given by (16).

To go back to $B'(4)$ we make use of the functional equation $b(x) = \Gamma'(x)/\Gamma(x)) \cdot \Gamma(x) \cdot b(\Gamma(x))$ where $b(x) = B'(x)$. From a table of $\Gamma'(x)/\Gamma(x)$[3] we get $b(4) = 1.502877$.

It is interesting to compare this value with the one calculated from the formula

$$\lim_{x\to\xi} (x - \xi) b^*(x) = 1/\log \Gamma'(\xi) = 0.72101723 \tag{17}$$

where $\xi = 3.56238229$ is the largest real fixpoint $\Gamma(\xi) = \xi$ of $\Gamma(x)$.[4] Now $\Gamma_{-6}(4) = 3.56247894$ and $b^*(3.56247894) \sim 0.721017/0.00009665 = 7460$ from (17), which gives $b^*(4) = 1.4987$ as compared with $b(4) = 1.5029$. The first value of course refers to the logarithm of iteration relative to the finite fixpoint ξ, and the slight difference between the two values merely illustrates the general phenomenon that iterates which are "best" near a finite fixpoint are not necessarily the best at infinity.

[3] British Association Tables, London, vol. 1 (1931), 42—46.

[4] (17) follows from a well known formula of Koenigs.

TABLE 1

$A(x)$

	0	1	2	3	4
1.00	0.0000000	0.0023071	0.0046098	0.0069082	0.0092024
1.01	0.0228778	0.0251423	0.0274026	0.0296586	0.0319105
1.02	0.0453351	0.0475581	0.0497771	0.0519920	0.0542029
1.03	0.0673836	0.0695665	0.0717453	0.0739203	0.0760912
1.04	0.0890350	0.0911788	0.0933186	0.0954547	0.0975869
1.05	0.1103003	0.1124060	0.1145080	0.1166062	0.1187007
1.06	0.1311901	0.1332589	0.1353239	0.1373854	0.1394432
1.07	0.1517147	0.1537475	0.1557767	0.1578024	0.1598245
1.08	0.1718840	0.1738818	0.1758761	0.1778669	0.1798544
1.09	0.1917075	0.1936711	0.1956315	0.1975885	0.1995421
1.10	0.2111943	0.2131248	0.2150521	0.2169761	0.2188968
1.11	0.2303534	0.2322516	0.2341466	0.2360385	0.2379272
1.12	0.2491932	0.2510600	0.2529236	0.2547842	0.2566416
1.13	0.2677221	0.2695582	0.2713913	0.2732213	0.2750484
1.14	0.2859480	0.2877542	0.2895574	0.2913578	0.2931552
1.15	0.3038785	0.3056556	0.3074298	0.3092012	0.3109697
1.16	0.3215211	0.3232698	0.3250157	0.3267588	0.3284992
1.17	0.3388830	0.3406040	0.3423223	0.3440379	0.3457507
1.18	0.3559711	0.3576651	0.3593564	0.3610451	0.3627312
1.19	0.3727921	0.3744598	0.3761248	0.3777873	0.3794472
1.20	0.3893525	0.3909945	0.3926339	0.3942708	0.3959051
1.21	0.4056586	0.4072755	0.4088898	0.4105018	0.4121112
1.22	0.4217165	0.4233088	0.4248988	0.4264863	0.4280714
1.23	0.4375319	0.4391004	0.4406664	0.4422302	0.4437915
1.24	0.4531107	0.4546558	0.4561986	0.4577390	0.4592772
1.25	0.4684584	0.4699806	0.4715006	0.4730184	0.4745339
1.26	0.4835802	0.4850801	0.4865779	0.4880734	0.4895668
1.27	0.4984813	0.4999595	0.5014355	0.5029094	0.5043811
1.28	0.5131668	0.5146237	0.5160785	0.5175311	0.5189817
1.29	0.5276415	0.5290776	0.5305116	0.5319436	0.5333735
1.30	0.5419102	0.5433259	0.5447396	0.5461513	0.5475610
1.31	0.5559774	0.5573732	0.5587671	0.5601590	0.5615489
1.32	0.5698476	0.5712239	0.5725983	0.5739708	0.5753414
1.33	0.5835250	0.5848823	0.5862377	0.5875912	0.5889429
1.34	0.5970138	0.5983525	0.5996893	0.6010243	0.6023575
1.35	0.6103181	0.6116386	0.6129572	0.6142741	0.6155891

Table 1

	5	6	7	8	9
1.00	0.0114923	0.0137779	0.0160592	0.0183363	0.0206092
1.01	0.0341583	0.0364019	0.0386414	0.0408767	0.0431079
1.02	0.0564097	0.0586125	0.0608113	0.0630061	0.0651968
1.03	0.0782583	0.0804214	0.0825807	0.0847360	0.0868874
1.04	0.0997153	0.1018398	0.1039606	0.1060776	0.1081908
1.04	0.1207915	0.1228786	0.1249620	0.1270417	0.1291177
1.06	0.1414975	0.1435481	0.1455951	0.1476385	0.1496784
1.07	0.1618432	0.1638583	0.1658700	0.1678781	0.1698828
1.08	0.1818384	0.1838190	0.1857962	0.1877700	0.1897404
1.09	0.2014924	0.2034394	0.2053831	0.2073234	0.2092605
1.10	0.2208143	0.2227285	0.2246396	0.2265474	0.2284520
1.11	0.2398127	0.2416951	0.2435743	0.2454504	0.2473234
1.12	0.2584960	0.2603474	0.2621956	0.2640408	0.2658830
1.13	0.2768724	0.2786935	0.2805116	0.2823267	0.2841388
1.14	0.2949497	0.2967412	0.2985299	0.3003156	0.3020985
1.15	0.3127353	0.3144981	0.3162581	0.3180153	0.3197696
1.16	0.3302367	0.3319715	0.3337035	0.3354327	0.3371592
1.17	0.3474608	0.3491683	0.3508730	0.3525750	0.3542744
1.18	0.3644146	0.3660953	0.3677734	0.3694489	0.3711218
1.19	0.3811045	0.3827592	0.3844114	0.3860610	0.3877080
1.20	0.3975370	0.3991663	0.4007931	0.4024174	0.4040393
1.21	0.4137182	0.4153227	0.4169248	0.4185245	0.4201217
1.22	0.4296541	0.4312345	0.4328124	0.4343879	0.4359611
1.23	0.4453506	0.4469072	0.4484616	0.4500136	0.4515633
1.24	0.4608131	0.4623467	0.4638780	0.4654071	0.4669338
1.25	0.4760472	0.4775582	0.4790670	0.4805736	0.4820780
1.26	0.4910580	0.4925470	0.4940338	0.4955185	0.4970010
1.27	0.5058507	0.5073182	0.5087835	0.5102467	0.5117078
1.28	0.5204302	0.5218766	0.5233210	0.5247632	0.5262034
1.29	0.5348014	0.5362272	0.5376510	0.5390728	0.5404925
1.30	0.5489687	0.5503744	0.5517782	0.5531799	0.5545796
1.31	0.5629368	0.5643229	0.5657069	0.5670891	0.5684693
1.32	0.5767101	0.5780769	0.5794417	0.5808047	0.5821658
1.33	0.5902927	0.5916406	0.5929867	0.5943309	0.5956733
1.34	0.6036888	0.6050183	0.6063460	0.6076718	0.6089959
1.35	0.6169024	0.6182138	0.6195235	0.6208314	0.6221375

TABLE 1

	0	1	2	3	4
1.36	0.6234419	0.6247445	0.6260453	0.6273444	0.6286417
1.37	0.6363889	0.6376741	0.6389575	0.6402392	0.6415191
1.38	0.6491630	0.6504310	0.6516973	0.6529620	0.6542249
1.39	0.6617676	0.6630188	0.6642685	0.6655164	0.6667628
1.40	0.6742063	0.6754412	0.6766744	0.6779061	0.6791361
1.41	0.6864825	0.6877013	0.6889186	0.6901342	0.6913483
1.42	0.6985996	0.6998027	0.7010042	0.7022042	0.7034026
1.43	0.7105607	0.7117484	0.7129345	0.7141192	0.7153023
1.44	0.7223691	0.7235417	0.7247127	0.7258823	0.7270503
1.45	0.7340277	0.7351855	0.7363417	0.7374966	0.7386499
1.46	0.7455396	0.7466828	0.7478246	0.7489650	0.7501039
1.47	0.7569076	0.7580366	0.7591641	0.7602903	0.7614151
1.48	0.7681345	0.7692495	0.7703632	0.7714755	0.7725864
1.49	0.7792231	0.7803245	0.7814245	0.7825232	0.7836205
1.50	0.7901761	0.7912641	0.7923507	0.7934360	0.7945199
1.51	0.8009961	0.8020709	0.8031443	0.8042165	0.8052874
1.52	0.8116855	0.8127474	0.8138080	0.8148673	0.8159253
1.53	0.8222469	0.8232961	0.8243440	0.8253907	0.8264362
1.54	0.8326827	0.8337195	0.8347550	0.8357893	0.8368224
1.55	0.8429952	0.8440198	0.8450431	0.8460653	0.8470862
1.56	0.8531867	0.8541993	0.8552107	0.8562209	0.8572299
1.57	0.8632594	0.8642602	0.8652599	0.8662584	0.8672558
1.58	0.8732156	0.8742048	0.8751930	0.8761800	0.8771659
1.59	0.8830572	0.8840352	0.8850120	0.8859877	0.8869623
1.60	0.8927865	0.8937533	0.8947190	0.8956836	0.8966472
1.61	0.9024053	0.9033612	0.9043161	0.9052698	0.9062224
1.62	0.9119158	0.9128610	0.9138051	0.9147481	0.9156901
1.63	0.9213198	0.9222544	0.9231879	0.9241204	0.9250519
1.64	0.9306191	0.9315433	0.9324665	0.9333887	0.9343099
1.65	0.9398156	0.9407297	0.9416427	0.9425548	0.9434658
1.66	0.9489112	0.9498152	0.9507183	0.9516204	0.9525215
1.67	0.9579074	0.9588017	0.9596949	0.9605872	0.9614785
1.68	0.9668061	0.9676907	0.9685743	0.9694570	0.9703387
1.69	0.9756090	0.9764841	0.9773582	0.9782314	0.9791036
1.70	0.9843176	0.9851833	0.9860481	0.9869120	0.9877750
1.71	0.9929335	0.9937900	0.9946457	0.9955004	0.9963542

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TABLE 1

	5	6	7	8	9
1.36	0.6299373	0.6312311	0.6325232	0.6338135	0.6351021
1.37	0.6427974	0.6440739	0.6453487	0.6466218	0.6478932
1.38	0.6554862	0.6567458	0.6580038	0.6592600	0.6605146
1.39	0.6680074	0.6692505	0.6704919	0.6717316	0.6729698
1.40	0.6803654	0.6815913	0.6828165	0.6840401	0.6852621
1.41	0.6925608	0.6937717	0.6949810	0.6961888	0.6973950
1.42	0.7045995	0.7057948	0.7069886	0.7081808	0.7093716
1.43	0.7164838	0.7176639	0.7188425	0.7200195	0.7211951
1.44	0.7282169	0.7293821	0.7305457	0.7317078	0.7328685
1.45	0.7398018	0.7409523	0.7421013	0.7432488	0.7443949
1.46	0.7512414	0.7523775	0.7535121	0.7546453	0.7557772
1.47	0.7625385	0.7636605	0.7647811	0.7659003	0.7670181
1.48	0.7736959	0.7748041	0.7759109	0.7770163	0.7781204
1.49	0.7847164	0.7858110	0.7869043	0.7879963	0.7890869
1.50	0.7956026	0.7966839	0.7977639	0.7988426	0.7999200
1.51	0.8063569	0.8074252	0.8084922	0.8095579	0.8106224
1.52	0.8169821	0.8180376	0.8190918	0.8201448	0.8211965
1.53	0.8274804	0.8285233	0.8295650	0.8306055	0.8316447
1.54	0.8378542	0.8388849	0.8399143	0.8409425	0.8419694
1.55	0.8481059	0.8491245	0.8501418	0.8511580	0.8521729
1.56	0.8582378	0.8592444	0.8602499	0.8612543	0.8622574
1.57	0.8682519	0.8692470	0.8702408	0.8712336	0.8722251
1.58	0.8781506	0.8791342	0.8801166	0.8810980	0.8820782
1.59	0.8879358	0.8889081	0.8898794	0.8908495	0.8918186
1.60	0.8976096	0.8985709	0.8995312	0.9004903	0.9014484
1.61	0.9071740	0.9081245	0.9090739	0.9100223	0.9109696
1.62	0.9166310	0.9175708	0.9185096	0.9194474	0.9203841
1.63	0.9259824	0.9269118	0.9278402	0.9287675	0.9296938
1.64	0.9352301	0.9361492	0.9370673	0.9379844	0.9389005
1.65	0.9443759	0.9452849	0.9461930	0.9471000	0.9480061
1.66	0.9534216	0.9543207	0.9552189	0.9561160	0.9570122
1.67	0.9623689	0.9632583	0.9641467	0.9650341	0.9659206
1.68	0.9712195	0.9720993	0.9729781	0.9738560	0.9747330
1.69	0.9799750	0.9808454	0.9817148	0.9825833	0.9834509
1.70	0.9886370	0.9894981	0.9903584	0.9912177	0.9920760
1.71	0.9972072	0.9980592	0.9989103	0.9997605	1.0006099

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TABLE 2

$a(z)$

	0	1	2	3	4
1.00	2.309225	2.304901	2.300590	2.296292	2.292006
1.01	2.266550	2.262351	2.258164	2.253988	2.249824
1.02	2.225094	2.221014	2.216945	2.212888	2.208843
1.03	2.184810	2.180844	2.176889	2.172946	2.169014
1.04	2.145653	2.141797	2.137953	2.134119	2.130296
1.05	2.107581	2.103832	2.100093	2.096365	2.092647
1.06	2.070553	2.066906	2.063269	2.059643	2.056026
1.07	2.034532	2.030984	2.027446	2.023917	2.020397
1.08	1.999481	1.996028	1.992584	1.989150	1.985725
1.09	1.965365	1.962004	1.958651	1.955308	1.951973
1.10	1.932151	1.928878	1.925614	1.922358	1.919111
1.11	1.899807	1.896619	1.893439	1.890268	1.887105
1.12	1.868301	1.865196	1.862098	1.859009	1.855928
1.13	1.837606	1.834580	1.831561	1.828551	1.825548
1.14	1.807693	1.804744	1.801802	1.798868	1.795941
1.15	1.778535	1.775660	1.772792	1.769931	1.767078
1.16	1.750108	1.747304	1.744508	1.741718	1.738936
1.17	1.722386	1.719652	1.716924	1.714203	1.711489
1.18	1.695346	1.692679	1.690018	1.687364	1.684716
1.19	1.668965	1.666363	1.663767	1.661177	1.658593
1.20	1.643223	1.640683	1.638149	1.635621	1.633099
1.21	1.618097	1.615618	1.613145	1.610677	1.608215
1.22	1.593570	1.591149	1.588734	1.586325	1.583921
1.23	1.569620	1.567256	1.564898	1.562545	1.560198
1.24	1.546231	1.543922	1.541618	1.539320	1.537028
1.25	1.523384	1.521128	1.518878	1.516633	1.514393
1.26	1.501063	1.498859	1.496660	1.494466	1.492278
1.27	1.479251	1.477097	1.474948	1.472804	1.470665
1.28	1.457933	1.455827	1.453727	1.451631	1.449540
1.29	1.437093	1.435035	1.432982	1.430933	1.428888
1.30	1.416718	1.414706	1.412698	1.410694	1.408695
1.31	1.396794	1.394826	1.392862	1.390902	1.388947
1.32	1.377306	1.375381	1.374360	1.371543	1.369631
1.33	1.358243	1.356360	1.354481	1.352605	1.350734
1.34	1.339592	1.337749	1.335910	1.334075	1.332244
1.35	1.321341	1.319537	1.317737	1.315941	1.314149

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TABLE 2

	5	6	7	8	9
1.00	2.287733	2.283472	2.279223	2.274986	2.270762
1.01	2.245673	2.241534	2.237406	2.233290	2.229186
1.02	2.204809	2.200786	2.196775	2.192775	2.188787
1.03	2.165093	2.161183	2.157284	2.153396	2.149519
1.04	2.126483	2.122681	2.118890	2.115110	2.111340
1.05	2.088939	2.085241	2.081554	2.077877	2.074210
1.06	2.052419	2.048822	2.045235	2.041658	2.038090
1.07	2.016887	2.013387	2.009896	2.006415	2.002943
1.08	1.982308	1.978901	1.975504	1.972115	1.968736
1.09	1.948647	1.945330	1.942022	1.938723	1.935433
1.10	1.915872	1.912642	1.909420	1.906207	1.903003
1.11	1.883950	1.880804	1.877666	1.874536	1.871415
1.12	1.852854	1.849788	1.846731	1.843681	1.840640
1.13	1.822553	1.819566	1.816586	1.813614	1.810650
1.14	1.793021	1.790109	1.787205	1.784308	1.781418
1.15	1.764232	1.761393	1.758561	1.755736	1.752919
1.16	1.736160	1.733392	1.730630	1.727875	1.725127
1.17	1.708782	1.706081	1.703387	1.700700	1.698020
1.18	1.682075	1.679440	1.676812	1.674190	1.671574
1.19	1.656016	1.653445	1.650880	1.648321	1.645769
1.20	1.630584	1.628075	1.625571	1.623074	1.620583
1.21	1.605760	1.603310	1.600866	1.598428	1.595996
1.22	1.581523	1.579131	1.576745	1.574364	1.571989
1.23	1.557856	1.555520	1.553190	1.550865	1.548545
1.24	1.534741	1.532459	1.530182	1.527911	1.525645
1.25	1.512158	1.509929	1.507705	1.505486	1.503272
1.26	1.490094	1.487915	1.485742	1.483573	1.481409
1.27	1.468531	1.466402	1.464277	1.462157	1.460043
1.28	1.447454	1.445373	1.443296	1.441224	1.439156
1.29	1.426849	1.424814	1.422783	1.420757	1.418735
1.30	1.406701	1.404711	1.402725	1.400744	1.398767
1.31	1.386996	1.385050	1.383108	1.381170	1.379236
1.32	1.367723	1.365819	1.363919	1.362023	1.360131
1.33	1.348867	1.347004	1.345145	1.343290	1.341439
1.34	1.330417	1.328594	1.326775	1.324959	1.323148
1.35	1.312361	1.310577	1.308796	1.307119	1.305246

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TABLE 2

	0	1	2	3	4
1.36	1.303477	1.301712	1.299950	1.298192	1.296438
1.37	1.285991	1.284263	1.282538	1.280817	1.279100
1.38	1.268872	1.267180	1.265491	1.263806	1.262124
1.39	1.252109	1.250452	1.248798	1.247148	1.245501
1.40	1.235692	1.234069	1.232449	1.230833	1.229220
1.41	1.219612	1.218022	1.216435	1.214852	1.213272
1.42	1.203858	1.202300	1.200746	1.199195	1.197647
1.43	1.188424	1.186897	1.185374	1.183854	1.182337
1.44	1.173299	1.171803	1.170310	1.168820	1.167334
1.45	1.158475	1.157009	1.155546	1.154086	1.152628
1.46	1.143945	1.142507	1.141073	1.139642	1.138213
1.47	1.129700	1.128291	1.126884	1.125481	1.124080
1.48	1.115733	1.114351	1.112972	1.111596	1.110223
1.49	1.102037	1.100682	1.099329	1.097979	1.096632
1.50	1.088604	1.087275	1.085948	1.084624	1.083303
1.51	1.075428	1.074124	1.072823	1.071524	1.070228
1.52	1.062502	1.061223	1.059946	1.058672	1.057400
1.53	1.049820	1.048565	1.047312	1.046062	1.044814
1.54	1.037375	1.036143	1.034914	1.033687	1.032462
1.55	1.025162	1.023953	1.022747	1.021543	1.020341
1.56	1.013175	1.011988	1.010804	1.009622	1.008442
1.57	1.001408	1.000243	0.999080	0.997919	0.996761
1.58	0.989855	0.988711	0.987570	0.986431	0.985293
1.59	0.978512	0.977389	0.976268	0.975149	0.974032
1.60	0.967373	0.966270	0.965169	0.964071	0.962974
1.61	0.956434	0.955350	0.954269	0.953190	0.952112
1.62	0.945688	0.944624	0.943562	0.942502	0.941444
1.63	0.935133	0.934088	0.933045	0.932003	0.930963
1.64	0.924763	0.923736	0.922711	0.921688	0.920666
1.65	0.914574	0.913565	0.912558	0.911552	0.910548
1.66	0.904562	0.903570	0.902580	0.901592	0.900606
1.67	0.894722	0.893747	0.892774	0.891803	0.890834
1.68	*0.885051*	*0.884093*	*0.883137*	*0.882182*	*0.881229*
1.69	0.875545	0.874603	0.873663	0.872724	0.871787
1.70	0.866199	0.865273	0.864349	0.863426	0.862505
1.71	0.857010	0.856100	0.855191	0.854284	0.853378

TABLE 2

	5	6	7	8	9
1.36	1.294688	1.292941	1.291198	1.289459	1.287723
1.37	1.277386	1.275676	1.273970	1.272267	1.270568
1.38	1.260446	1.258772	1.257101	1.255433	1.253769
1.39	1.243858	1.242218	1.240581	1.238948	1.237318
1.40	1.227610	1.226004	1.224401	1.222801	1.221205
1.41	1.211695	1.210121	1.208550	1.206983	1.205419
1.42	1.196102	1.194560	1.193021	1.191485	1.189953
1.43	1.180823	1.179312	1.177804	1.176299	1.174798
1.44	1.165850	1.164369	1.162891	1.161416	1.159944
1.45	1.151174	1.149722	1.148273	1.146828	1.145385
1.46	1.136787	1.135364	1.133944	1.132526	1.131112
1.47	1.122682	1.121287	1.119894	1.118505	1.117118
1.48	1.108852	1.107483	1.106118	1.104755	1.103394
1.49	1.095288	1.093946	1.092607	1.091270	1.089936
1.50	1.081984	1.080668	1.079354	1.078043	1.076734
1.51	1.068934	1.067643	1.066354	1.065067	1.063783
1.52	1.056131	1.054864	1.053599	1.052337	1.051077
1.53	1.043568	1.042325	1.041084	1.039845	1.038609
1.54	1.031240	1.030020	1.028802	1.027586	1.026373
1.55	1.019141	1.017943	1.016748	1.015555	1.014364
1.56	1.007264	1.006088	1.004915	1.003744	1.002575
1.57	0.995605	0.994451	0.993299	0.992149	0.991001
1.58	0.984158	0.983025	0.981893	0.980764	0.979637
1.59	0.972917	0.971805	0.970694	0.969585	0.968478
1.60	0.961879	0.960786	0.959695	0.958606	0.957519
1.61	0.951037	0.949963	0.948892	0.947822	0.946754
1.62	0.940387	0.939333	0.938280	0.937229	0.936180
1.63	0.929925	0.928889	0.927855	0.926823	0.925792
1.64	0.919646	0.918628	0.917612	0.916598	0.915585
1.65	0.909546	0.908546	0.907547	0.906550	0.905555
1.66	0.899621	0.898638	0.897656	0.896676	0.895698
1.67	0.889866	0.888900	0.887935	0.886972	0.886011
1.68	0.880277	0.879327	0.878379	0.877433	0.876488
1.69	0.870852	0.869918	0.868986	0.868055	0.867126
1.70	0.861585	0.860667	0.859750	0.858835	0.857922
1.71	0.852474	0.851571	0.850670	0.849770	0.848872

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