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ON THE n-PARAMETER ABSTRACT CAUCHY PROBLEM

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Let $H_i (i = 1, 2, \dots, n)$, be closed operators in a Banach space X. The generalised initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t_i} u(t_1, t_2, \dots, t_i, \dots, t_n) = H_i u(t_1, \dots, t_n), & t_i \in (0\,, T_i] \quad i = 1, 2, \dots, n \\ u(0) = x, \quad x \in \bigcap\limits_{i=1}^{n} D(H_i), \end{cases}$$

of the abstract Cauchy problem is studied. We show that the uniqueness of solution $u : [0, T_1] \times [0, T_2] \times \cdots \times [0, T_n] \to X$ of this n-abstract Cauchy problem is closely related to C_0-n-parameter semigroups of bounded linear operators on X. Also as another application of C_0-n-parameter semigroups, we prove that many n-parameter initial value problems cannot have a unique solution for some initial values.

1. INTRODUCTION

Suppose X is a Banach space and A is a linear operator from $D(A) \subseteq X$ into X. Given $x \in X$, the abstract Cauchy problem for A with the initial value x, consists of finding a solution $u(t)$ to the initial value problem

$$(1) \qquad \begin{cases} \dfrac{du(t)}{dt} = Au(t) & t \in (0\,, T] \\ u(0) = x \end{cases}$$

where by a solution we mean an X-valued function $u : [0, T] \to X$ which is continuous for $t \geqslant 0$, continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t \in (0, T]$ and (1) is satisfied.

A one-parameter semigroup of operators is a homomorphism $T : (\mathbb{R}_+, +) \to B(X)$ for which $T(0) = I$, where $\mathbb{R}_+ = [0, \infty)$ and $B(X)$ is the Banach space of all bounded linear operators on X. The one-parameter semigroup $\{T(t)\}_{t \geqslant 0}$ is called strongly continuous (or C_0-continuous) if $\lim\limits_{t \to 0} T(t)x = x$, for each $x \in X$ and is called uniformly continuous if $\lim\limits_{t \to 0} T(t) = I$ in $B(X)$. The linear mapping A defined by

$$A(x) = \lim_{t \to 0} \frac{T(t)x - x}{t},$$

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where $D(A) = \left\{x : \lim_{t\to 0}(T(t)x - x)/t \text{ exist}\right\}$, is called the infinitesimal generator of $(T, \mathbb{R}_+, X)$.

The following Theorem which is due to Hille [5], shows the close relation of abstract Cauchy problem with semigroup theory (see also [4]).

Theorem 1.1. *Let A be a closed linear operator in Banach space X, then the following are equivalent:*

(a) *For each $x \in D(A)$ there exists a unique solution for (1).*

(b) *The part $A_1 = A_{|X_1}$ of A in $X_1 := \left(D(A), \|\cdot\|_A\right)$ is the infinitesimal generator of a C_0-one-parameter semigroup of operators on the Banach space X_1, where $\|\cdot\|_A$ is the graph norm on $D(A)$.*

Proof: [2, II.6.6]. □

The previous Theorem has many applications in inhomogenous initial value problems and evaluation systems. One can see some more applications of abstract Cauchy problem in [3, 7].

Let $\mathbb{R}^n_+ = \left\{(t_1, t_2, \dots, t_n) : t_i \geqslant 0, i = 1, 2, \dots, n\right\}$. By an n-parameter semigroup of operators we mean a homomorphism $W : (\mathbb{R}^n_+, +) \to B(X)$ for which $W(0) = I$ and denote it by $(W, \mathbb{R}^n_+, X)$. Suppose H_i is the infinitesimal generator of the one-parameter semigroup $\left\{W(te_i)\right\}_{t\geqslant o}$, where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$, we shall think of $(H_1, H_2, \dots, H_n)$ as the infinitesimal generator of W. As in the one-parameter case, $(W, \mathbb{R}^n_+, X)$ is called strongly continuous (or C_0-continuous) if for each $x \in X$, $\lim_{t\to 0} W(t)x = x$, and is called uniformly continuous if $\lim_{t\to 0} W(t) = I$, where $t \to 0$ in $\mathbb{R}^n_+$. It is not hard to see that $(W, \mathbb{R}^n_+, X)$ is a C_0-continuous (respectively uniformly continuous) if and only if for each $i = 1, 2, \dots, n$, $\left\{W(te_i)\right\}_{t\geqslant o}$ is strongly (respectively uniformly) continuous. The following useful proposition which states some basic Properties of n-parameter semigroups can be found in [1] as is described in [6].

Proposition 1.2. *Suppose $(W, \mathbb{R}^n_+, X)$ is a C_0-n-parameter semigroup then*

(a) *If $x \in D(H_i)$, so does $W(t)x$, for each $t \in \mathbb{R}^n_+$ and*

$$H_i W(t)x = W(t)H_i x \qquad (i = 1, 2, \dots, n).$$

(b) *$\bigcap_{i=1}^n D(H_i)$ is dense in X, and $X_1 = \left(\bigcap_{i=1}^n D(H_i), \|\cdot\|_1\right)$ is a Banach space, where for $x \in \bigcap_{i=1}^n D(H_i)$, $\|x\|_1 = \|x\| + \sum_{i=1}^n \|H_i(x)\|$.*

(c) *For each $1 \leqslant i, j \leqslant n$, $D(H_i) \cap D(H_i H_j) \subseteq D(H_j H_i)$, and for $x \in D(H_i) \cap D(H_i H_j)$,*

$$H_i H_j(x) = H_j H_i(x).$$

In the rest of this note we shall state an extension of one-parameter abstract Cauchy problem and establish its relation with C_0-n-parameter semigroups of operators. As another application of C_0-n-parameter semigroups we shall show that some n-parameter initial valued problems cannot have a unique solution. The abstract Cauchy problem also admits another natural generalisation which is discussed in [**5**, **6**, **8**].

2. The main results

Suppose as before X is a Banach space, H_i are closed linear operators from $D(H_i) \subseteq X$ into X and $T_i > 0$, $(i = 1, 2, \ldots, n)$. Then, a continuous X-valued function $u : [0, T_1] \times \cdots \times [0, T_n] \to X$ with continuous partial derivatives which satisfy the following n-parameter abstract Cauchy problem (n-abstract Cauchy problem)

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial}{\partial t_i} u(t_1, t_2, \ldots, t_i, \ldots, t_n) = H_i u(t_1, \ldots, t_n), & i = 1, 2, \ldots, n \quad t_i \in (0\,, T_i] \\ u(0) = x, \quad x \in \bigcap\limits_{i=1}^{n} D(H_i), \end{cases}$$

is called a solution of the initial value problem (2).

For convenience in the rest of this note we denote by I_T the positive n-cell $[0, T_1] \times [0, T_2] \times \cdots \times [0, T_n]$ where $T = (T_1, T_2, \ldots, T_n) \in \mathbb{R}^n_+$ and $T_i > 0$. As mentioned in the previous section, we shall illustrate that (2) is closely related to C_0-n-parameter semigroups of operators. In the following theorem we prove that if I_T is arbitrary and $(H_1, H_2, \ldots, H_n)$ is the infinitesimal generator of a C_0-n-parameter semigroup $(W, \mathbb{R}^n_+, X)$, then (2) has the unique solution $u(t_1, t_2, \ldots, t_n) = W(t_1, t_2, \ldots, t_n)x$, for each $x \in \bigcap\limits_{i=1}^{n} D(H_i)$, where $(t_1, t_2, \ldots, t_n) \in I_T$.

Theorem 2.1. *Suppose I_T is a positive n-cell corresponding to $T \in \mathbb{R}^n_+$, and $(H_1, H_2, \ldots, H_n)$ is the infinitesimal generator of the C_0-n-parameter semigroup $(W, \mathbb{R}^n_+, X)$ of operators, then for each $x \in \bigcap\limits_{i=1}^{n} D(H_i)$ the n-abastract Cauchy problem (2) has a unique solution.*

Proof: Let I_T be arbitrary, $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$ and H_i be the infinitesimal generator of the C_0-n-parameter semigroup $\{W(t_i)\}_{i \geqslant o}$. For $x \in \bigcap\limits_{i=1}^{n} D(H_i)$, define $u : I_T \to X$ by $u(t) = W(t)x$. One can easily see that $u(t)$ is a solution of n-abstract Cauchy problem (2) for the initial value $x \in \bigcap\limits_{i=1}^{n} D(H_i)$. For proving the uniqueness of solution it is enough to show that (2) has no proper (that is, nonzero) solution for the initial value $x = 0$. Theorem 1.1 shows that for each $i = 1, 2, \ldots, n$, the initial value problem

$$\text{(3)} \qquad \begin{cases} \dfrac{du^i(s)}{ds} = H_i u^i(s) & s \in (0\,, T_i] \\ u^i(0) = x & x \in D(H_i) \end{cases}$$

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has a unique solution for each $x \in D(H_i)$. By definition of solution we know that for $t \in I_T$, $u(t)$ which is a solution of (2) for $x = 0$, is in $\bigcap_{i=1}^{n} D(H_i)$, so for the initial value $x = u(t_1, t_2, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \in D(H_i)$, $u^i(s) = u(t_1, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n)$ and $v^i(s) = W(se_i)u(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$ is a solution of (3) for x. Uniqueness of solution of (3) implies that

$$\begin{aligned} W(se_i)u(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) &= v^i(s) \\ &= u^i(s) = u(t_1, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n), \end{aligned} \tag{4}$$

for each $i = 1, 2, \dots, n$, $0 \leqslant s \leqslant T_i$ and $0 \leqslant t_j \leqslant T_j$, $i \neq j = 1, 2, \dots, n$. Using (4) for $t = \sum_{i=1}^{n} t_i e_i \in I_T$, shows that

$$\begin{aligned} u(t) = u(t_1, t_2, \dots, t_n) &= W(t_1 e_1)u(0, t_2, \dots, t_n) && (i = 1, s = t_1) \\ &= W(t_1 e_1)(W(t_2 e_2)u(0, 0, t_3, \dots, t_n) && (i = 2, s = t_2) \\ (W\text{in } n\text{-parameter}) &= W\left(\sum_{i=1}^{n} t_i e_i\right)u(0, 0, \dots, 0) = W(t)(0) = 0 \end{aligned}$$

Hence $u(t) = 0$ and (2) cannot have a proper solution for $x = 0$, or equivalently (2) has a unique solution for each $x \in \bigcap_{i=1}^{n} D(H_i)$. ∎

Now let H_i's $(i = 1, 2, \dots, n)$ from $D(H_i) \subseteq X$ into X be closed operators. Similarly to Proposition 1.2 (b) one can see $X_1 = \left(\bigcap_{i=1}^{n} D(H_i), \|\cdot\|_1\right)$, where $\|x\|_1 = \|x\| + \sum_{i=1}^{n} \|H_i(x)\|$, $\left(x \in \bigcap_{i=1}^{n} D(H_i)\right)$ is a Banach space. In the next theorem we are going to show that for positive n-cells I_T and $I_{T'}$, where $I_T \subsetneq I_{T'}$, if (2) has a unique solution for each $x \in X_1$ then there exist a C_0-n-parameter semigroup $(W, \mathbb{R}_+^n, X_1)$ with the infinitesimal generator $(K_1, K_2, \dots, K_n)$ for which $W(t)x = u(t; x)$, the unique solution of (2) for $x \in X_1$ and $t \in I_T$, also for $x \in D(K_i)$, $K_i(x) = H_i(x)$.

Theorem 2.2. *Suppose H_i's $(i = 1, 2, \dots, n)$ are closed linear operators and for positive n-cells I_T and $I_{T'}$, where $I_T \subsetneq I_{T'}$, the n-abstract Cauchy problem (2) has a unique solution for each $x \in X_1$, then there exist a C_0-n-parameter semigroup $(W, \mathbb{R}_+^n, X_1)$ of linear bounded operators with the infinitesimal generator $(K_1, K_2, \dots, K_n)$ such that for $t \in I_T$ and $x \in X_1$, $W(t)x = u(t; x)$ where $u(t; x)$ is the unique solution of (2) for the initial value x, and for $x \in D(K_i)$, $K_i(x) = H_i(x)$.*

Proof: Let $u(t; x)$ be the unique solution of (2) for $x \in X_1$. For $t \in I_T$, we define the operator $W_1(t) : X_1 \to X_1$ by $W_1(t)x = u(t; x)$. Trivially $W_1(t)$ is well-defined and a linear operator, since the solution is unique. We are going to show that $W_1(t)$ is bounded. Define the mapping $\Phi : X_1 \to C^1(I_T, X_1)$ by $\Phi(x)(t) = W_1(t)(x)$, where $C^1(I_T, X_1)$ is the Banach space of all continuous X_1-valued functions on I_T with continuous partial derivative, equiped with the supremum norm. Φ is linear, we prove it is closed. Suppose $x_m \to x$ in X_1 and $\Phi(x_m) \to f$ in $C^1(I_T, X_1)$, integrating of (2) implies that for each $i = 1, 2, \ldots, n$, $m \in \mathbb{N}$ and $t = (t_1, \ldots, t_n) \in I_T$,

$$(5)\quad W_1(t_1,\ldots,t_n)x_m = W_1(t_1,t_2,\ldots,t_{i-1},0,t_{i+1},\ldots,t_n)x_m + \int_0^{t_i} H_i W_1(t_1,t_2,\ldots,t_{i-1},s,t_{i+1},\ldots,t_n)x_m\,ds.$$

Let $m \to \infty$, so $\operatorname{Sup}_{t\in I_T} \left\|\Phi(x_m)(t) - f(t)\right\|_1 \to 0$, this, (5), together with the closedness of H_i, imply that for each $i = 1, 2, \ldots, n$,

$$(6)\quad f(t_1,\ldots,t_n) = f(t_1,t_2,\ldots,t_{i-1},0,t_{i+1},\ldots,t_n) + \int_0^{t_i} H_i f(t_1,t_2,\ldots,t_{i-1},s,t_{i+1},\ldots,t_n)\,ds.$$

Thus (6) and the fact that $f \in C^1(I_T, X_1)$ show that

$$\begin{cases} \dfrac{\partial}{\partial t_i} f(t_1,t_2,\ldots,t_i,\ldots,t_n) = H_i f(t_1,\ldots,t_n), & i = 1,2,\ldots,n \quad t_i \in (0\,,\,T_i] \\ f(0) = \lim\limits_{m\to\infty} \Phi(x_m)(0) = \lim\limits_{m\to\infty} W_1(0)(x_m) = x. \end{cases}$$

Hence f is a solution of (2) for the initial value x, the uniqueness of solution gives

$$f(t) = W_1(t)x = \Phi(x)t, \qquad t \in I_T,$$

it means that Φ is closed operator from the Banach space X_1 into the Banach space $C^1(I_T, X_1)$ and the closed graph theorem tell us

$$\operatorname*{Sup}_{\|x\|_1\leqslant 1} \|W_1(\cdot)x\|_\infty = \operatorname*{Sup}_{\|x\|_1\leqslant 1} \left(\operatorname*{Sup}_{t\in I_T} \left\|W_1(t)x\right\|_1\right) = M < \infty.$$

Thus for each $t \in I_T$, $W_1(t)$ is a bounded operator on X_1. Now let $T'' = (T_1'', T_2'', \ldots, T_n'')$, where $T_i'' = min\{T_i', T_i - T_i'\}$. We are going to show that for each $t, t' \in I_{T''}$, $W_1(t+t') = W_1(t)W_1(t')$. First we notice that for $t \in I_{T'}$ and $t' \in I_{T''}$, $t_i \leqslant T_i'$ and $t_i' \leqslant T_i - T_i'$ so $t_i + t_i' \leqslant T_i$ and $t + t' \in I_T$. Let t' be fixed, for $x \in X_1$ define $v(\cdot) : I_{T'} \to X_1$ by $v(t) = W_1(t+t')x$. Trivially $v(t)$ and $u(t) = W_1(t)W_1(t')x$ are solutions of (2) in $I_{T'}$ for the initial value $W_1(t')x$, by the uniqueness of solution of (2) in $I_{T'}$ we have

$$W_1(t+t')x = v(t) = u(t) = W_1(t)(W_1(t')x).$$

Now we can extend W_1 to an n-parameter semigroup of operators. Let $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}_+^n$, and choose $m_i \in \mathbb{N}$ and $r_i \in [0, T_i'']$ so that $s_i = m_i T_i'' + r_i$, $i = 1, 2, \ldots, n$. Suppose

$$W(s)x = W_1(r)\left[\Pi_{i=1}^n \left(W_1(T_i''e_i)\right)^{m_i}\right](x)$$

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where $r = (r_1, r_2, \ldots, r_n) \in I_{T''}$. By the previous parts of proof the operators in the right hand side of the last equality commute and are bounded linear operators on X_1. One can easily see that $(W, \mathbb{R}_+^n, X_1)$ is an n-parameter semigroup of operators and the fact that $\lim_{t\to 0} W(t)x = x$ (by continuity of $u(t;x)$) show that W is strongly continuous. Also for $s \in I_T$, $W_1(s)x = W(s)x$, since

$$\begin{aligned}\frac{\partial}{\partial s_i} W(s)x &= \frac{\partial}{\partial r_i} W_1(r)\left[\Pi_{i=1}^n \big(W_1(T_i''e_i)\big)^{m_i}\right](x)\\ &= H_i W_1(r)\left[\Pi_{i=1}^n \big(W_1(T_i''e_i)\big)^{m_i}\right](x) = H_i W(s)x\end{aligned}$$

and the equality holds from the uniqueness of solution in I_T. If $(K_1, K_2, \ldots, K_n)$ is the generator of W and $x \in D(K_i) \subseteq X_1 = \bigcap_{i=1}^n D(H_i)$, then

$$\|\cdot\|_1\text{-}\lim_{t\to 0} \frac{W(te_i)x - x}{t} = K_i(x)$$

which implies $\lim_{t\to 0}\big(W(te_i)x - x\big)/t = K_i(x)$, but $x \in D(H_i)$ and so

$$\begin{aligned}\lim_{t\to 0} \frac{W(te_i)x - x}{t} &= \frac{\partial}{\partial t_i} W(0, 0, \ldots, 0)x\\ &= H_i W(0)x = H_i(x)\end{aligned}$$

Thus $K_i(x) = H_i(x)$ and this complete the proof of theorem. ∎

In the previous Theorem we could replace the assumption of existence of a unique solution for (2) in I_T and $I_{T'}$, by the assumption that (2) has a unique solution in I_T and whole of $\mathbb{R}_+^n$, which seems stronger than our hypothesis. As another application of C_0-n-parameter semigroups, we shall show that for a closed linear operator $A : D(A) \subseteq X \to X$, the n-parameter initial value problem

$$(7)\qquad \begin{cases} \sum_{i=1}^n \dfrac{\partial}{\partial ti} u(t_1, t_2, \ldots, t_n) = Au(t_1, t_2, \ldots, t_n), & t = (t_1, t_2, \ldots, t_n) \in I_T\\ u(0) = x, \quad x \in D(A) \end{cases}$$

does not have a unique solution in both I_T and $I_{T'}$ for each $x \in D(A)$, for which $I_{T'} \subseteq I_T$.

The initial value problem (7) can have a solution, for example if $(H_1, H_2, \ldots, H_n)$ is generator of a C_0-n-parameter semigroup $(W, \mathbb{R}_+^n, X)$ and $A = H_1 + H_2 + \cdots + H_n$, then obviously $u(t) = W(t)x$ is a solution of (7) in any positive n-cell I_T, for the initial value $x \in \bigcap_{i=1}^n D(H_i) \subseteq D(A)$.

Before proving our claim we need the following lemmas.

Lemma 2.3. *Suppose $\{T(t)\}_{t\geqslant 0}$ is a C_0-one parameter semigroup of operators with the infinitesimal generator A, and $B \in B(X)$, then $A + B$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on X satisfying*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds\,, \quad x \in X.$$

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PROOF: See [7, III.1.1 and III.1.2]. ∎

Also the next lemma which provide a necessary and sufficient condition for the composition of C_0-one parameter semigroups to be a C_0-n-parameter semigroup, has a principal role in the next theorem.

Recall that for linear operator H in Banach space X, $\rho(H)$ denotes the resorvent set of H and for $\lambda \in \rho(H)$, $R(\lambda; H)$ is used for $(\lambda I - H)^{-1}$.

LEMMA 2.4. *Suppose $\{U^i(s)\}_{s\geqslant 0}$ is a C_0-one-parameter semigroup of operators on Banach space X with the infinitesimal generator H_i, $(i = 1, 2, \dots, n)$, then $W(t_1, t_2, \dots, t_n) = U^1(t_1)U^2(t_2)\dots U^n(t_n)$ is a C_0-n-parameter semigroup of operators if and only if there is an $\omega > 0$ such that for each $i = 1, 2, \dots, n$, $[\omega, \infty) \subseteq \rho(H_i)$ and for each integers $0 \leqslant i, j \leqslant n$ and λ, $\lambda' \geqslant \omega$, we have*

$$R(\lambda'; H_j)R(\lambda; H_i) = R(\lambda; H_i)R(\lambda'; H_j).$$

PROOF: First suppose W is a C_0-n-parameter semigroup of operators. Since H_i is the infinitesimal generator of $\{u^i(t)\}_{t\geqslant 0}$, by the Hille-Yosida Theorem ([7, I.5.3]), there is an $\omega_i > 0$ such that for each $\lambda \geqslant \omega_i$, $R(\lambda; H_i)$ exist and are bounded operators. Let $\omega = \max\{\omega_i : i = 1, 2, \dots, n\}$. If $\lambda \geqslant \omega$, from [7, I.5.4]

$$R(\lambda; H_i)(x) = \int_0^\infty e^{-\lambda s} U^i(s)(x)\, ds.$$

Also we know that for each integers $0 \leqslant i, j \leqslant n$,

$$U^i(s)U^j(t) = W(se_i)W(te_j) = W(te_j)W(se_i) = U^j(t)U^i(s),$$

so

$$\begin{aligned} R(\lambda; H_i)(U^j(t)x) &= \int_0^\infty e^{-\lambda t} U^i(s)U^j(t)x\, ds \\ &= \int_0^\infty e^{-\lambda t} U^j(t)U^i(s)x\, ds = U^j(t)\int_0^\infty e^{-\lambda t} U^i(s)x\, ds \\ &= u^j(t)R(\lambda; H_i)x. \end{aligned}$$

Now let $\lambda' \geqslant \omega$, we know $R(\lambda; H_i)$ is bounded so

$$\begin{aligned} R(\lambda; H_i)R(\lambda'; H_j)x &= R(\lambda; H_i)\int_0^\infty e^{-\lambda' t} U^j(t)x\, dt \\ &= \int_0^\infty e^{-\lambda' t} U^j(t)R(\lambda; H_i)x\, dt \\ &= R(\lambda'; H_j)R(\lambda; H_i)x \end{aligned}$$

and this prove the necessary part of lemma.

For the converse suppose there is an $\omega > 0$ such that for each $\lambda, \lambda' > 0$, $R(\lambda; H_i)$ and $R(\lambda'; H_j)$ exist and commute. So we have $H^i_\lambda H^j_{\lambda'} = H^j_{\lambda'} H^i_\lambda$ where $H^i_\lambda = \lambda^2 R(\lambda; H_i) - \lambda I$

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and $H^j_{\lambda'} = \lambda'^2 R(\lambda; H_j) - \lambda' I$ are the Yosida approximation of H_i and H_j respectively. Applying [7, I.3.5] we have $U^i(s)x = \lim_{\lambda\to\infty} e^{sH^i_\lambda}x$ and $U^j(t)x = \lim_{\lambda'\to\infty} e^{tH^j_{\lambda'}}x$, thus

$$\begin{aligned} U^i(s)U^j(t)x &= \lim_{\lambda\to\infty} e^{sH^i_\lambda}U^j(t)x \\ &= \lim_{\lambda\to\infty} e^{sH^i_\lambda}(\lim_{\lambda'\to\infty} e^{tH^j_{\lambda'}}x) \\ &= \lim_{\lambda\to\infty}\lim_{\lambda'\to\infty} e^{sH^i_\lambda}e^{tH^j_{\lambda'}}x \qquad (e^{sH^i_\lambda} \text{ is continuous}) \\ &= \lim_{\lambda\to\infty}\lim_{\lambda'\to\infty} e^{tH^j_{\lambda'}}e^{sH^i_\lambda}x \qquad (\text{since } H^i_\lambda H^j_{\lambda'} = H^j_{\lambda'}H^i_\lambda) \\ &= \lim_{\lambda\to\infty} U^j(t)e^{sH^i_\lambda}x \\ &= U^j(t)U^i(s)x \qquad (U^j(t) \text{ is continuous}). \end{aligned}$$

Hence $W(t_1, t_2, \ldots, t_n) = U^1(t_1)U^2(t_2)\ldots U^n(t_n)$ is a C_0-n-parameter semigroup of operators. ∎

Now we are ready for this theorem.

Theorem 2.5. *Suppose A is a closed operator from $D(A) \subseteq X$ into X and I_T and $I_{T'}$, $I_{T'} \subsetneq I_T$, is given. Then the initial value problem (7) cannot have a unique solution for each $x \in D(A)$ in both I_T and $I_{T'}$.*

Proof: Suppose to the contrary (7) has a unique solution for each $x \in D(A)$ in both I_T and $I_{T'}$. As in Theorem 2.2 we are going to show that if $u(t;x)$ is the unique solution of (7) for $x \in D(A)$ and $t \in I_T$, then $W_1(t)x = u(t;x)$ can be extended to a C_0-n-parameter semigroup of operators, and using previous lemma we shall get a contradiction.

Obviously uniqueness of solution shows that $W_1(t)x = u(t;x)$ is a well-defined linear operator on Banach space $X_1 = \big(D(A), \|\cdot\|_A\big)$ where $\|\cdot\|_A$ is the graph norm on X_1. Before proving the boundedness of $W_1(t)$ we notice that $Y = \big(C^1(I_T, X_1), \|\cdot\|'\big)$, where $\|f\|' = \|f\|_\infty + \sum_{i=1}^n \left\|\frac{\partial}{\partial t_i}f\right\|_\infty$ is a Banach space. Next we show that the mapping $\Phi : X_1 \to Y$ defined by $\Phi(x)(t) = W_1(t)$ is closed, for; suppose $x_m \to x$ in X_1 and $\Phi(x_m) \to f$ in Y. Integrating of (7) for initial value x_m, we have

$$W_1(t_1, t_2, \ldots, t_n)x_m = W_1(0, t_2, \ldots, t_n)x_m - \sum_{i=2}^n \int_0^{t_1} \frac{\partial}{\partial t_i}W_1(s, t_2, \ldots, t_n)x_m\, ds + \int_0^{t_1} AW_1(s, t_2, \ldots, t_n)x_m\, ds.$$

As $m \to \infty$ by our choosing of the norm and the closeness of A we get

$$\left\|\frac{\partial}{\partial t_i}W_1(\cdot)x_m - \frac{\partial}{\partial t_i}f(\cdot)\right\|_\infty \to 0, \quad as\ m \to \infty, \quad i = 1, 2, \ldots, n$$

and

$$\|W_1(\cdot)x_m - f(\cdot)\|_\infty = \operatorname*{Sup}_{t\in I_T}\Big(\|W_1(t)x_m - f(t)\|_A\Big)$$
$$= \operatorname*{Sup}_{t\in I_T}\Big(\|W_1(t)x_m - f(t)\| + \|AW_1(t)x_m - Af(t)\|\Big) \to 0$$

as $m \to \infty$. Hence

$$f(t_1, t_2, \ldots, t_n) = f(0, t_2, \ldots, t_n) - \sum_{i=2}^{n} \int_0^{t_1} \frac{\partial}{\partial t_i} f(s, t_2, \ldots, t_n)\, ds + \int_0^{t_1} Af(s, t_2, \ldots, t_n)\, ds.$$

It gives

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial t_i} f(t_1, t_2, \ldots, t_n) = Au(t_1, t_2, \ldots, t_n) \\ f(0) = \lim_{m\to\infty} W_1(0)x_m = x. \end{cases}$$

So f is a solution of (7) and by the uniqueness of solution we conclude $f(t) = W_1(t)x$, equivalently f is closed and by closed graph theorem Φ is bounded, thus $\operatorname*{Sup}_{t\in I_T}\|W_1(t)\| < \infty$.

As in Theorem 2.2 $W_1(t)$ can be extended to a C_0-n-parameter semigroup $(W, \mathbb{R}_+^n, X_1)$. Let $(H_1, H_2, \ldots, H_n)$ be the infinitesimal generator of W, for $x \in \bigcap_{i=1}^{n} D(H_i) \subseteq D(A)$ we have

$$\frac{\partial}{\partial t_i} W(t)x = H_i W(t)x.$$

Thus

$$\sum_{i=1}^{n} \frac{\partial}{\partial t_i} W(t)x = \Big(\sum_{i=1}^{n} H_i\Big) W(t)x = AW(t)x.$$

From the continuity of $\frac{\partial}{\partial t_i} W(t)x$ and strong continuity of $W(t)x$, the fact that $\sum_{i=1}^{n} H_i W(t)x = W(t) \sum_{i=1}^{n} H_i x$ (Proposition 1.2), and the closedness of A as $t \to 0$, the last equality yields

(8) $$\sum_{i=1}^{n} H_i(x) = A(x)\ , \text{ for each } x \in \bigcap_{i=1}^{n} D(H_i).$$

Applying Lemma 2.4 shows that there is $\omega > 0$ such that for each λ, $\lambda' \geqslant \omega$, we have

$$R(\lambda'; H_j)R(\lambda; H_i) = R(\lambda; H_i)R(\lambda'; H_j).$$

Now let $H_1' = H_1 + I$ and $H_2' = H_2 - I$, if $\omega' = \omega + 1$ and $\lambda,\ \lambda' \geqslant \omega'$, we have $\lambda + 1,\ \lambda' - 1 \geqslant \omega$ and

$$\begin{aligned} R(\lambda'; H_1')R(\lambda; H_2') &= R(\lambda' - 1; H_1)R(\lambda + 1; H_2) \\ &= R(\lambda + 1; H_2)R(\lambda' - 1; H_1) \\ &= R(\lambda; H_2')R(\lambda'; H_1'). \end{aligned}$$

Similarly $R(\lambda; H_i')R(\lambda'; H_j) = R(\lambda'; H_j)R(\lambda; H_i')$, for $\lambda,\ \lambda' \geqslant \omega'$ $, i = 1, 2$, and $j = 3, 4, \dots, n$. By Lemma 2.3 H_1' and H_2' are the infinitesimal generators of two C_0-one-parameter semigroups of operators. With the above equalities and Lemma 2.4, this shows that $(H_1', H_2', H_3, \dots, H_n)$ is the infinitesimal generator of a C_0-n-parameter semigroup, say $(W', \mathbb{R}_+^n, X_1)$. So by Lemma 2.3, for each $x \in X_1$,

$$W'(te_1)x = W(te_1)x + \int_0^t W\big((t - \mu)e_1\big)W'(\mu e_1)x d\mu,$$

and

$$W'(te_2)x = W(te_2)x - \int_0^t W\big((t - \nu)e_2\big)W'(\nu e_2)x d\nu.$$

Also $W'(te_i) = W(te_i)$, for $i > 2$. We conclude that for $x \in \bigcap_{i=1}^{n} D(H_i)$,

$$\frac{\partial}{\partial t_i} W'(t_1, t_2, \dots, t_n)x = \begin{cases} H_i' W'(t_1, t_2, \dots, t_n) & i = 1, 2 \\ H_i W'(t_1, t_2, \dots, t_n) & i > 2. \end{cases}$$

Hence by (8)

$$\begin{cases} \dfrac{\partial}{\partial ti} W'(t) = (H_1' + H_2' + H_3 + \dots + H_n)W'(t) = \sum_{i=1}^{n} H_i W'(t) = AW'(t)x \\ W'(0) = x. \end{cases}$$

But the solution of (7) is unique, and so for $i = 1, \dots, n$ and $0 \leqslant t \leqslant T_i$,

$$W'(te_i) = W(te_i).$$

This implies that

$$\begin{aligned} W(te_1)x = W'(te_1)x &= W(te_1)x + \int_0^t W\big((t - \mu)e_1\big)W'(\mu e_1)x d\mu \\ &= W(te_1)x + \int_0^t W(te_1)x d\mu \\ &= W(te_1)x + tW(te_1)x. \end{aligned}$$

So $tW(te_1)x = 0$ or $W(te_1)x = 0$. This is a contradiction, because $0 = \lim_{t \to 0} W(te_1)x = x \neq 0$. Thus (7) cannot have a unique solution for each $x \in D(A)$. ∎

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REMARK 2.6. Our technique for proving Theorem 2.2 and a part of Theorem 2.5 is based on Hille's technique for one-parameter case [5]. C_0-n-parameter semigroups are solutions of many initial value problems contain partial derivative and as in previous Theorem, C_0-n-parameter semigroups can be used for showing that these initial value problems cannot have a unique solution. As another example for second order initial value problems, consider the two-parameter initial value problem

$$\begin{cases} \dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t}u(s,t) = Au(s,t) \\ (s,t) \in [0,S] \times [0,T] \\ u(0,0) = x, \qquad x \in D(A) \end{cases}$$

where A is a closed operator. If this problem has a unique solution for each $x \in D(A)$ in both $I_{(S,T)}$ and $I_{(S',T')}$ for which $I_{(S',T')} \subsetneq I_{(S,T)}$, then $W_1(s,t) = u(s,t;x)$ can be extended to a C_0-two-parameter semigroup on Banach space $X_1 = \big(D(A), \|\cdot\|_A\big)$, with the infinitesimal generator (H,K). We know $\overline{D(HK) \cap D(KH)}^{\|\cdot\|_A} = D(A)$, (it can be proved completely similarly to the proof of Proposition 1.2 (b)), so $D(HK) \cap D(KH) \neq 0$. Now for $x \in D(HK) \cap D(KH)$, by Proposition 1.2 $HK(x) = KH(x)$ and one can see that this is equal to $A(x)$. Also it can be checked that $(H/2, 2K)$ is the generator of $W'(s,t) = W(s/2, 2t) \neq W(s,t)$. So for $x \in D(HK) \cap D(KH)$,

$$\begin{cases} \dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t}W'(s,t) = \big(\tfrac{1}{2}H\big)(2K)W'(s,t)x = HKW(s/2,t)x = AW'(s,t) \\ W'(0,0)x = x \end{cases}$$

and this is a contradiction with the uniqueness of solution.

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