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Strong Multiplicity One for the Selberg Class

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Abstract. We investigate the problem of determining elements in the Selberg class by means of their Dirichlet series coefficients at primes.

In [7] A. Selberg axiomatized properties expected of L-functions and introduced the "Selberg class" which is expected to coincide with the class of all arithmetically interesting L-functions. We recall that an element F of the Selberg class $\mathcal{S}$ satisfies the following axioms.

- In the half-plane $\sigma > 1$ the function $F(s)$ is given by a Dirichlet series $\sum_{n=1}^{\infty} a_F(n)n^{-s}$ with $a_F(1) = 1$ and $a_F(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.
- There is a natural number m_F such that $(s-1)^{m_F}F(s)$ extends to an analytic function of finite order in the entire complex plane.
- There is a function $\Phi_F(s) = Q_F^s\Gamma_F(s)F(s)$ where $Q_F > 0$ and

$$\Gamma_F(s) = \prod_{j=1}^{r_F} \Gamma\big(\lambda_j(F)s + \mu_j(F)\big)$$

  with $\lambda_j(F) > 0$ and $\operatorname{Re}\mu_j(F) \geq 0$ such that $\Phi_F(s) = \omega_F\bar{\Phi}_F(1-s)$, where $|\omega_F| = 1$ and for any function f we denote $\bar{f}(s) = \overline{f(\bar{s})}$. We let $d_F := 2\sum_{j=1}^{r}\lambda_j$ denote the "degree" of F.

We may express $\log F(s)$ by a Dirichlet series

$$\log F(s) = \sum_{n=2}^{\infty} b_F(n)\Lambda(n)/(n^s \log n)$$

where $b_F(n) \ll n^\vartheta$ for some $\vartheta < \frac{1}{2}$. We adopt the convention that $b_F(n) = 0$ if n is not a prime power. From the assumption $a_F(n) \ll_\epsilon n^\epsilon$ it follows that $b_F(p^k) \ll_{k,\epsilon} p^\epsilon$.

It is believed that the Selberg class satisfies the following "strong multiplicity one" principle: If F and G are two elements of the Selberg class with $a_F(p) = a_G(p)$ (equivalently $b_F(p) = b_G(p)$) for all but finitely many primes p then $F = G$. In [5]

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R. Murty and K. Murty prove such a result under the additional hypothesis that $a_F(p^2) = a_G(p^2)$ for all but finitely many primes p. Recently J. Kaczorowski and A. Perelli [2] have established this principle under the additional hypothesis that

$$\lim_{\sigma\to 1^+}(\sigma - 1)\sum_p \frac{|a_F(p^2) - a_G(p^2)|}{p^\sigma}\log p < \infty.$$

This criterion is equivalent to saying that $a_F(p^2) - a_G(p^2)$ is bounded on average, that is for all large x

$$\sum_{p\le e^x} \frac{|a_F(p^2) - a_G(p^2)|^2}{p}\log p \ll x. \tag{1}$$

In this note we develop a different method which establishes the strong multiplicity one principle under a much weaker hypothesis than (1), but which fails (barely) to prove the full principle. We use $\log_j$ to denote the j-fold iterated logarithm; thus $\log_2 = \log\log$, $\log_3 = \log\log\log$ and so on.

Theorem *Suppose F and G are elements of the Selberg class with $a_F(p) = a_G(p)$ (equivalently, $b_F(p) = b_G(p)$) for all $p \notin \mathcal{E}$ where $\mathcal{E}$ is a thin set of primes satisfying*

$$\#\{p \in \mathcal{E} \mid p \le x\} \ll x^{\frac{1}{2}-\delta} \tag{2}$$

for some fixed $\delta > 0$. Then F and G have the same degree: $d_F = d_G$. If in addition we have

$$\sum_{p\le e^x} \frac{|a_F(p^2) - a_G(p^2)|^2}{p}\log p \ll \exp\left(\frac{x}{\log x(\log_2 x)^5}\right), \tag{3}$$

then $F = G$.

Although (3) is a considerably weaker restriction than (1) it is still stronger than the bound $\ll e^{\epsilon x}$ which is all we know in general. When combined with the classification of elements of the Selberg class of degree 1 by Kaczorowski and Perelli (see [3] and also the exposition [8]) our Theorem permits the following corollary.

Corollary *Suppose χ is a primitive Dirichlet character and that F is an element of the Selberg class with $f(p) = \chi(p)$ for all $p \notin \mathcal{E}$ where $\mathcal{E}$ is a set of primes satisfying* (2). *Then $F(s) = L(s, \chi)$.*

To deduce the Corollary we note that our Theorem implies that the degree of F is 1. Since Kaczorowski and Perelli [3] have shown that the only elements of the Selberg class of degree 1 are Dirichlet L-functions the Corollary follows.

We now embark on the proof of our Theorem. Put $c(n) = b_F(n) - b_G(n)$ and observe that in $\operatorname{Re} s > \frac{3}{2}$ we have

$$-\frac{F'}{F}(s) + \frac{G'}{G}(s) = \sum_{n=1}^{\infty}\frac{c(n)\Lambda(n)}{n^s} = \sum_{k=1}^{\infty}\sum_p \frac{c(p^k)\log p}{p^{ks}}. \tag{4}$$

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Since $c(p) = 0$ unless $p \in \mathcal{E}$ we see from (2) that $\sum_p c(p)\log p/p^s$ is entire in $\mathrm{Re}(s) > \frac{1}{2} - \delta$. Since $c(p^2)$ and $c(p^3)$ are $\ll p^\epsilon$ we see that $\sum_p c(p^2)\log p/p^{2s}$ and $\sum_p c(p^3)\log p/p^{3s}$ are entire in $\mathrm{Re}\, s > \frac{1}{2}$ and $\frac{1}{3}$ respectively. Lastly since $c(n) \ll n^\vartheta$ for $\vartheta < \frac{1}{2}$ it follows that $\sum_{k=4}^\infty \sum_p c(p^k)\log p/p^{ks}$ is entire in $\mathrm{Re}\, s > \frac{1}{2}$. Thus the RHS of (4) is an entire function in $\mathrm{Re}\, s > \frac{1}{2}$ furnishing an analytic continuation of $-\frac{F'}{F}(s) + \frac{G'}{G}(s)$ in this region. It follows that the zeros of F and G in this region coincide (including multiplicities), and also that F and G have poles of the same order at 1 (that is, $m_F = m_G$). Arguing similarly with $\bar{F}(s) = \sum_n \overline{a_F(n)}n^{-s}$ and $\bar{G}(s)$ we see that their zeros in $\mathrm{Re}\, s > \frac{1}{2}$ also coincide. Using the functional equation it follows that the zeros of $\Phi_F(s)$ and $\Phi_G(s)$ in $\mathrm{Re}\, s < \frac{1}{2}$ also coincide. Summarizing we see that $\Phi_F(s)$ and $\Phi_G(s)$ have the same zeros except possibly on the critical line $\mathrm{Re}\, s = \frac{1}{2}$, and that they have (possibly) poles of the same order at $s = 1$.

Let $\rho_F = \frac{1}{2} + i\gamma_F$ and $\rho_G = \frac{1}{2} + i\gamma_G$ denote typical zeros of $\Phi_F(s)$ and $\Phi_G(s)$. We do not suppose that γ_F and γ_G are real, although this version of the Riemann hypothesis is expected to be true. A standard application of the argument principle shows that $\#\{\rho_F : |\operatorname{Im} \rho_F| \le T\} = \frac{d_F}{\pi} T\log T + c_F T + O_F(\log T)$ where d_F is the degree and c_F is a constant. Similar estimates apply for the zeros ρ_G up to height T.

We now recall an explicit formula connecting the zeros ρ_F to the prime power values $b_F(n)\Lambda(n)$; for details see for example Z. Rudnick and P. Sarnak [6]. Let g be a smooth compactly supported function and put $h(s) = \int_{-\infty}^\infty g(u)e^{isu}\,du$. We may recover g from h by means of the Fourier inversion formula

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^\infty h(u)e^{-ixu}\,du.$$

The explicit formula now reads

$$\sum_{\gamma_F} h(\gamma_F) = m_F\Big(h\Big(-\frac{i}{2}\Big) + h\Big(\frac{i}{2}\Big)\Big)$$
$$+ \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\Big(2\log Q_F + \frac{\Gamma_F'}{\Gamma_F}\Big(\frac{1}{2}+ir\Big) + \frac{\overline{\Gamma_F}'}{\overline{\Gamma_F}}\Big(\frac{1}{2}-ir\Big)\Big)\,dr$$
$$- \sum_{n=1}^\infty \Big(\frac{b_F(n)\Lambda(n)}{\sqrt{n}}g(\log n) + \frac{\overline{b_F(n)}\Lambda(n)}{\sqrt{n}}g(-\log n)\Big). \tag{5}$$

We now choose g to be a smooth non-negative function in $[-1,1]$ such that $g(0) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\,dr \gg 1$ and such that

$$|h(t)| \ll \exp(-|t|/\log^2|t|), \tag{6}$$

for large $|t|$. A result of A. E. Ingham [1] shows that such a choice of g is possible. For example, following Ingham we may take

$$h(t) = \prod_{n=N}^\infty \left(\frac{\sin\Big(2\pi t/\big(n(\log n)^{\frac{3}{2}}\big)\Big)}{2\pi t/\big(n(\log n)^{\frac{3}{2}}\big)}\right)^2$$

for some large N and then one could check that h and its Fourier transform g are a permissible choice.

Let T be a large positive number and let t be in $(T, 2T)$. Let $\log^2 T \geq L \geq \log T$ be a large parameter to be chosen later. Then an application of (5) gives that

$$\begin{aligned}\sum_{\gamma_F} h\big(L(\gamma_F - t)\big) &= m_F\Big(h\Big(L\Big(-\frac{i}{2} - t\Big)\Big) + h\Big(L\Big(\frac{i}{2} + t\Big)\Big)\Big) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} h\big(L(r-t)\big)\Big(2\log Q_F + \frac{\Gamma_F'}{\Gamma_F}\Big(\frac{1}{2} + ir\Big) + \frac{\overline{\Gamma_F}'}{\overline{\Gamma_F}}\Big(\frac{1}{2} - ir\Big)\Big)\, dr \\ &\quad - \frac{1}{L}\sum_{n=1}^{\infty}\Big(\frac{b_F(n)\Lambda(n)}{n^{\frac{1}{2}+it}} g\Big(\frac{\log n}{L}\Big) + \frac{\overline{b_F(n)}\Lambda(n)}{n^{\frac{1}{2}-it}} g\Big(-\frac{\log n}{L}\Big)\Big). \qquad (7)\end{aligned}$$

We call the middle term on the RHS above $H_F(t, L)$ and the third term there $D_F(t, L)$. Stirling's formula shows that

$$H_F(t, L) = \frac{d_F \log T + O(1)}{L} \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\, dr = g(0)\frac{d_F \log T}{L} + O(1/L). \qquad (8)$$

We apply the explicit formula (7) for G as well and subtract the two formulae. From our remarks on the poles of F and G and their zeros not on the critical line we conclude that

$$Z_F(t, L) - Z_G(t, L) = H_F(t, L) - H_G(t, L) - D_F(t, L) + D_G(t, L), \qquad (9a)$$

where

$$Z_F(t, L) = \sum_{\gamma_F \in \mathbb{R}} h\big(L(\gamma_F - t)\big) \quad \text{and} \quad Z_G(t, L) = \sum_{\gamma_G \in \mathbb{R}} h\big(L(\gamma_G - t)\big). \qquad (9b)$$

We now record a mean-value estimate for the $D_F(t, L) - D_G(t, L)$ terms. Since $|c(p)| \ll p^{\epsilon}$ we see from (2) that $\sum_p |c(p)g(\log p/L)| \log p/\sqrt{p} = O(1)$. Further as noted earlier $\sum_{k\geq 3}\sum_p |c(p^k)\log p g(k\log p/L)|/p^{k/2} = O(1)$. Thus

$$\begin{aligned}L\big(D_F(t, L) - D_G(t, L)\big) \ll 1 &+ \Big|\sum_p \frac{a_F(p^2) - a_G(p^2)}{p^{1+2it}} \log p\, g\Big(\frac{2\log p}{L}\Big)\Big| \\ &+ \Big|\sum_p \frac{\overline{a_F(p^2)} - \overline{a_G(p^2)}}{p^{1-2it}} \log p\, g\Big(-\frac{2\log p}{L}\Big)\Big|.\end{aligned}$$

Using now a familiar mean-value estimate of Montgomery and Vaughan ([4], see Corollary 3) we see that

$$\int_T^{2T} \Big| \sum_p \frac{a_F(p^2) - a_G(p^2)}{p^{1+2it}} \log p g\Big(\frac{2\log p}{L}\Big)\Big|^2 dt$$
$$\ll \sum_{p \le e^{L/2}} \frac{|a_F(p^2) - a_G(p^2)|^2}{p^2} (T+p) \log^2 p$$
$$\ll T + \sum_{p\le e^{L/2}} \frac{|a_F(p^2) - a_G(p^2)|^2}{p} \log^2 p.$$

We conclude unconditionally that

$$\text{(10a)} \qquad \int_T^{2T} \big(L|D_F(t,L) - D_G(t,L)|\big)^2 dt \ll T + e^{L\epsilon},$$

and if we assume the condition (3) that

$$\text{(10b)} \qquad \int_T^{2T} \big(L|D_F(t,L) - D_G(t,L)|\big)^2 dt \ll T + \exp\Big(\frac{L}{\log L(\log_2 L)^5}\Big).$$

Let $W \ge 1$ be a real parameter and let $\mathcal{L} = \mathcal{L}(W)$ denote the set of $t \in [T, 2T]$ such that there exists either γ_F or γ_G in $\big(t - 1/(W\log T), t + 1/(W \log T)\big)$. Let $\bar{\mathcal{L}}$ denote the complementary set $[T, 2T] \setminus \mathcal{L}$. Since there are $\ll T \log T$ ordinates γ_F or γ_G in $[T, 2T]$ we see that $\text{meas}(\mathcal{L}) \ll T/W$. Now

$$L \int_{t\in\bar{\mathcal{L}}} |Z_F(t,L)|\, dt \ll L \int_T^{2T} \sum_{\substack{\gamma_F \in \mathbb{R} \\ |\gamma_F - t| \ge 1/(W\log T)}} \big| h\big(L(\gamma_F - t)\big)\big|\, dt.$$

If the distance of γ_F from $(T, 2T)$ exceeds n then by (6) the contribution of such a γ_F to the RHS above is $\ll \exp\big(-nL/(\log nL)^2\big)$. Further the number of ordinates γ_F whose distance from $(T, 2T)$ is between n and $n+1$ is $\ll \log\big(T(n+1)\big)$ and so we conclude that the contribution to the RHS above from zeros not in $(T-1, 2T+1)$ is $\ll 1$. Thus

$$L \int_{t\in\bar{\mathcal{L}}} |Z_F(t,L)|\, dt \ll 1 + \sum_{\gamma_F \in (T-1, 2T+1)} \int_{|y| \ge L/(W\log T)} |h(y)|\, dy$$
$$\ll 1 + T\log T \int_{|y|\ge L/(W\log T)} |h(y)|\, dy.$$

A similar estimate applies for $|Z_G(t,L)|$ so that

$$\text{(11)} \qquad L\int_{t\in\bar{\mathcal{L}}} |Z_F(t,L) - Z_G(t,L)|\, dt \ll 1 + T\log T \int_{|y|\ge L/(W\log T)} |h(y)|\, dy.$$

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With these preliminaries in place we are now ready to finish the proof of our Theorem. Suppose first that $d_F \neq d_G$. Then by (8) we know that $L\big(H_F(t,L) - H_G(t,L)\big) \gg \log T$. Integrating (9a) over $t \in \bar{\mathcal{L}}$ we find that

$$\begin{aligned} T \log T &\ll \int_{t\in\bar{\mathcal{L}}} L\big| H_F\big((t,L) - H_G(t,L)\big)\big| \, dt \\ &\ll \int_{t\in\bar{\mathcal{L}}} L|Z_F(t,L) - Z_G(t,L)| \, dt + \int_{t\in\bar{\mathcal{L}}} L|D_F(t,L) - D_G(t,L)| \, dt \\ &\ll 1 + T\log T \int_{|y|\geq L/(W\log T)} |h(y)| \, dy + (T + e^{L\epsilon}), \end{aligned}$$

using (11) and (10a) above. We now choose $L = W^2 \log T$ and choose W to be large but smaller than $1/\sqrt{\epsilon}$. Then for large T the above gives that $1 \ll \int_{|y|>W} |h(y)| \, dy$ which is impossible for sufficiently large W in view of (6).

We may now suppose that $d_F = d_G$. In this case we will require the additional hypothesis (3) which permits the bound (10b) above. Let m be a fixed integer with $c(m) \neq 0$. Naturally if no such m exists then $F = G$. We multiply (9a) by Lm^{it} and integrate for $t \in \bar{\mathcal{L}}$. Using (11) we see that the LHS gives

$$\text{(12a)} \qquad \int_{t\in\bar{\mathcal{L}}} Lm^{it}\big(Z_F(t,L) - Z_G(t,L)\big) \, dt \ll 1 + T\log T \int_{|y|\geq L/(W\log T)} |h(y)| \, dy.$$

On the other hand using the RHS of (9a) this is also equal to

$$\int_{t\in\bar{\mathcal{L}}} Lm^{it}\big(H_F(t,L) - H_G(t,L) - D_F(t,L) + D_G(t,L)\big) \, dt.$$

Now

$$\begin{aligned} &\int_{t\in\bar{\mathcal{L}}} Lm^{it}\big(H_F(t,L) - H_G(t,L)\big) \, dt \\ &\qquad = \int_T^{2T} Lm^{it}\big(H_F(t,L) - H_G(t,L)\big) \, dt + O\Big(\int_{t\in\mathcal{L}} L|H_F(t,L) - H_G(t,L)| \, dt\Big). \end{aligned}$$

Since $d_F = d_G$ we see by (8) that the second term above is $\ll \operatorname{meas}(\mathcal{L}) \ll T/W$. Further using integration by parts, and since $L\frac{d}{dt}\big(H_F(t,L) - H_G(t,L)\big) \ll 1/T$ by Stirling's formula, we see that the first term above is $\ll 1$. Thus

$$\text{(12b)} \qquad \int_{t\in\bar{\mathcal{L}}} Lm^{it}\big(H_F(t,L) - H_G(t,L)\big) \, dt \ll 1 + T/W.$$

Further

$$\begin{aligned} &\int_{t\in\bar{\mathcal{L}}} Lm^{it}\big(D_F(t,L) - D_G(t,L)\big) \, dt \\ &\qquad = \int_T^{2T} Lm^{it}\big(D_F(t,L) - D_G(t,L)\big) \, dt + O\Big(\int_{t\in\mathcal{L}} L|D_F(t,L) - D_G(t,L)| \, dt\Big). \end{aligned}$$

The second term above is $\ll (T + \exp(\frac{L}{\log L(\log_2 L)^5})/\sqrt{W}$ upon using (10b) and Cauchy's inequality. Integrating term by term we see that the first term is

$$T\frac{c(m)}{\sqrt{m}}\Lambda(m) + O\Big(\sum_{n\le e^L}\frac{|c(n)|\log n}{\sqrt{n}}\Big) = T\frac{c(m)}{\sqrt{m}}\Lambda(m) + O\Big(\exp\Big(\frac{L}{\log L(\log_2 L)^5}\Big)\Big)$$

using (2) and (3). Thus we conclude that

$$\int_{t\in\tilde{\mathcal{L}}} Lm^{it}\big(D_F(t,L) - D_G(t,L)\big)\,dt = T\frac{c(m)}{\sqrt{m}}\Lambda(m) + O\Big(\frac{T}{\sqrt{W}} + \exp\Big(\frac{L}{\log L(\log_2 L)^5}\Big)\Big). \tag{12c}$$

Combining (12a, b, c) we get that

$$1 \ll \frac{c(m)}{\sqrt{m}}\Lambda(m) \ll \frac{1}{T} + \log T\int_{|y|\ge L/(W\log T)}|h(y)|\,dy + \frac{1}{\sqrt{W}} + \frac{\exp(\frac{L}{\log L(\log_2 L)^5})}{T}.$$

We now choose $W = \log_3 T$, $L = \log T\log_2 T(\log_3 T)^4$ and use (6) to obtain a contradiction.

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