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ON THE NORMAL VERSION OF THE SIMPLICIAL COHOMOLOGY OF OPERATOR ALGEBRAS

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We show that the normal version $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*)$ of the Banach simplicial cohomology of operator algebras can be expressed in terms of the functor Ext on the category of Banach $\mathcal{R}$-bimodules. As an application, we prove that $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*) = \mathcal{H}^n(\mathcal{R}, \mathcal{R}_*)$ and thus the latter space vanishes for some von Heumann algebras for positive r.

1. INTRODUCTION

The present paper concerns a new variety of the cohomology of a von Neumann algebra $\mathcal{R}$ with coefficients in its predual bimodule $\mathcal{R}_*$. It was introduced by Christensen and Sinclair [1] and was denoted by $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*)$. Christensen and Sinclair used this cohomology as a means for computing the (usual simplicial) cohomology $\mathcal{H}^n(A, A^*)$ of a C^*-algebra A with coefficients in its dual bimodule A^*, and they have shown [1] that the vanishing of $\mathcal{H}_\sigma^n(A^{**}, A^*)$ implies the vanishing of $\mathcal{H}^n(A, A^*)$. In [5] one of us has asked the question: is it possible to express the cohomology in terms of some suitable Ext? Now we give the desired expression in terms of an Ext for some Banach $\mathcal{R}$-bimodules. This, in particular, enables us to prove that the space $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*)$ in fact coincides, up to a topological isomorphism, with $\mathcal{H}^n(\mathcal{R}, \mathcal{R}_*)$; thus we obtain a "simplicial version" of a well-known result of Johnson, Kadison and Ringrose [6] on the coincidence of the "usual" and the normal cohomology. In its turn, this enables us to establish the vanishing of $\mathcal{H}^n(\mathcal{R}, \mathcal{R}_*)$ for some von Neumann algebras and all $n \geq 1$. (As to $n = 1$, Haagerup [2, Theorem 4.1] has proved such a vanishing for all von Neumann algebras).

Recall the definition of the cohomology of Banach algebras. Let A be a Banach algebra, and let X be a Banach A-bimodule. We consider a sequence

$$(\mathcal{C}(A, X)) \qquad 0 \longrightarrow C^0(A, X) \xrightarrow{\delta^0} \dots \xrightarrow{\delta^{n-1}} C^n(A, X) \xrightarrow{\delta^n} C^{r+1}(A, X) \xrightarrow{\delta^{n+1}} \dots,$$

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where $C^0(A, X)$ is just X and, for $n > 0$, $C^n(A, X)$ is the Banach space of all bounded n-linear operators $F\colon \underbrace{A \times A \times \ldots \times A}_{n} \to X$; as to δ^n; $n = 0, 1, \ldots$, it acts by the formulae $\delta^0 x(a) = a \cdot x - x \cdot a$, and, for $n > 0$,

$$\delta^n f(a_1, \ldots, a_{n+1}) = a_i f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}.$$

It is easy to show that $\mathcal{C}(A, X)$ is in fact a complex. Its n-th cohomology is called the n-dimensional (Banach) cohomology group of A with coefficients in X, and it is denoted by $\mathcal{H}^n(A, X)$.

Following [**1**], we denote the Banach space of all continuous n-linear forms on a von Neumann algebra $\mathcal{R}$ by $C^n(\mathcal{R}, \mathbb{C})$, and we denote its closed subspace consisting of normal (that is, separately ultraweakly continuous) forms by $C^n_\sigma(\mathcal{R}, \mathbb{C})$. For every $\varphi \in C^n(\mathcal{R}, \mathcal{R}^*)$, we define a certain $\omega_\varphi \in C^{r+1}(\mathcal{R}, \mathbb{C})$ by $\omega_\varphi(a_0, \ldots, a_n) = \langle a_0, \varphi(a_1, \ldots, a_n)\rangle$. We denote the closed subspace in $C^n(\mathcal{R}, \mathcal{R}^*)$ consisting of those φ for which $\omega_\varphi \in C^{r+1}_\sigma(\mathcal{R}, \mathbb{C})$, by $C^n_\sigma(\mathcal{R}, \mathcal{R}_*)$. Being considered for all $n \geqslant 0$, these subspaces $C^n_\sigma(\mathcal{R}, \mathcal{R}_*)$ constitute a subcomplex in $\mathcal{C}(\mathcal{R}, \mathcal{R}^*)$ whose cohomology is denoted by $\mathcal{H}^n_\sigma(\mathcal{R}, \mathcal{R}_*)$ [**1**].

Theorem. *Let $\mathcal{R}$ be a von Neumann algebra, and let $\mathcal{R}_*$ be its predual bimodule. Then, for any $n \geqslant 0$, we have*

$$\mathcal{H}^n_\sigma(\mathcal{R}, \mathcal{R}_*) = \operatorname{Ext}^n_{\mathcal{R}-\mathcal{R}}(\mathcal{R}, \mathcal{R}_*) = \mathcal{H}^n(\mathcal{R}, \mathcal{R}_*)$$

up to a topological isomorphism.

The proof will use some ideas of the paper [**3**] of one of the authors. Let us consider the space $C^n_\sigma(\mathcal{R}, \mathbb{C})$; $n \geqslant 1$ as an $\mathcal{R}$-bimodule with outer multiplications $(a \cdot f)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n a)$ and $(f \cdot a)(a_1, \ldots, a_n) = f(a a_1, \ldots, a_n)$. It was shown in [**3**, Lemma 4] that the complex

$$(St_*\mathcal{R}_*) \qquad \begin{gathered} 0 \longrightarrow \mathcal{R}_* \xrightarrow{\varepsilon_*} C^2_\sigma(\mathcal{R}, \mathbb{C}) \xrightarrow{\varepsilon^0_*} C^3_\sigma(\mathcal{R}, \mathbb{C}) \longrightarrow \ldots \\ \ldots \longrightarrow C^{n+2}_\sigma(\mathcal{R}, \mathbb{C}) \xrightarrow{\varepsilon^n_*} C^{n+3}_\sigma(\mathcal{R}, \mathbb{C}) \longrightarrow \ldots \end{gathered}$$

where $\varepsilon_* f(a, b) = f(ab)$ and $\varepsilon^n_* f(a_1, \ldots, a_{n+3}) = \sum_{i=1}^{n+2} (-1)^{i+1} f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+3})$ is an injective resolution of the $\mathcal{R}$-bimodule $\mathcal{R}_*$.

For any Banach $\mathcal{R}$-bimodules Y and Z, we denote the Banach space of all morphisms from Y to Z by ${}_{\mathcal{R}}h_{\mathcal{R}}(Y, Z)$.

Lemma. *For every $n \geqslant 2$, there is a topological isomorphism of Banach spaces*

$$J_n\colon {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^n(\mathcal{R}, \mathbb{C})) \to C_\sigma^{n-2}(\mathcal{R}, \mathcal{R}_*),$$

which sends a morphism $\varphi \in {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^n(\mathcal{R}, \mathbb{C}))$ to $J_n\varphi$ where the latter is defined by $\langle J_n\varphi(a_1, \dots, a_{n-2}), a\rangle = \varphi(e)(a, a_1, \dots, a_{n-2}, e)$. ($e$ is the identity in $\mathcal{R}$).

It is easy to see that J_n is well-defined, that is belongs to $C_\sigma^{n-2}(\mathcal{R}, \mathcal{R}_*)$. We consider a map

$$J_n^{-1}\colon C_\sigma^{n-2}(\mathcal{R}, \mathcal{R}_*) \to {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^n(\mathcal{R}, \mathbb{C}))$$

where, for $F \in C_\sigma^{n-2}(\mathcal{R}, \mathcal{R}_*)$,

$$[J_n^{-1}F(a)](a_1, \dots, a_n) = \langle F(a_2, \dots, a_{n-1}), a_n a a_1\rangle.$$

In virtue of separately weakly* continuity of the multiplication in $\mathcal{R}$, $J_n^{-1}F(a)$ belongs to $C_\sigma^n(\mathcal{R}, \mathbb{C})$. A routine calculation shows that $\phi = J_n^{-1}F$ is a morphism of R-bimodules, and J_n^{-1} is indeed the inverse map of J_n. It remains to notice that operators J_n and J_n^{-1} are obviously bounded.

Remark. Instead of explicitly displaying the inverse to J_n, which we believe is more convenient for the reader, one could show that J_n, after several identifications, turned out to be a special case of the isomorphism which was constructed in Lemma 7 of [**3**] (if we put now $X = \mathcal{R}^*$).

End of the proof of the theorem. Now it is a routine thing to check that a diagram

$$(1)\qquad \begin{array}{ccccccc}
0 & \longrightarrow & {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^2(\mathcal{R}, \mathbb{C})) & \xrightarrow{h(\varepsilon_*^0)} & {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^3(\mathcal{R}, \mathbb{C})) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle J_2} & & \downarrow{\scriptstyle J_3} & & \\
0 & \longrightarrow & C_\sigma^0(\mathcal{R}, \mathcal{R}_*) & \xrightarrow{\delta^0} & C_\sigma^1(\mathcal{R}, \mathcal{R}_*) & \longrightarrow & \dots
\end{array}$$

where $h(\varepsilon_*^n)$ is the operator defined by $[h(\varepsilon_*^n)](\varphi) = \varepsilon_*^n \circ \varphi$ for $\varphi \in {}_{\mathcal{R}}h_{\mathcal{R}}(\mathcal{R}, C_\sigma^{n+2}(\mathcal{R}, \mathbb{C}))$, is commutative. Computing $\mathrm{Ext}^n_{\mathcal{R}-\mathcal{R}}(\mathcal{R}, \mathcal{R}_*)$ by means of the injective resolution $St_*\mathcal{R}_*$ of $\mathcal{R}_*$ [**3**], we obtain that these spaces are just the cohomology of the upper complex in the Diagram (1). At the same time, the cohomology of the lower complex in this Diagram is, by definition, $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*)$. Thus up to a topological isomorphism, we have $\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*) = \mathrm{Ext}^n_{\mathcal{R}-\mathcal{R}}(\mathcal{R}, \mathcal{R}_*)$ for any $n \geqslant 0$. We only need to remind the reader that, from the standard expression of the cohomology in terms of Ext [**4**, Chapter III, Theorem 4.9], we have $\mathcal{H}^n(\mathcal{R}, \mathcal{R}_*) = \mathrm{Ext}^n_{\mathcal{R}-\mathcal{R}}(\mathcal{R}, \mathcal{R}_*)$; $n \geqslant 0$.

Corollary 1. *Let $\mathcal{R}$ be an injective (in the sense of Connes) von Neumann algebra. Then*

$$\mathcal{H}_\sigma^n(\mathcal{R}, \mathcal{R}_*) = \mathcal{H}^n(\mathcal{R}, \mathcal{R}_*) = 0$$

for all $n \geqslant 1$.

By virtue of [**3**, Theorem 3], the assumption on $\mathcal{R}$ implies that $\mathcal{R}_*$ is injective as a Banach bimodule over $\mathcal{R}$. Hence $\mathrm{Ext}^n_{\mathcal{R}-\mathcal{R}}(\mathcal{R}, \mathcal{R}_*) = 0$.

Here we should remind the reader that the vanishing of the cohomology $\mathcal{H}^n_\sigma(\mathcal{R}, \mathcal{R}_*)$; $n \geqslant 1$, for an injective von Neumann algebra $\mathcal{R}$ was originally proved by means of a different approach in [**1**, Theorem 3.2]

COROLLARY 2. *Let* $\mathcal{R}$ *be a properly infinite von Neumann algebra. Then* $\mathcal{H}^n(\mathcal{R}, \mathcal{R}_*) = 0$ *for all* $n \geqslant 1$.

The indicated spaces coincide, as we have proved, with $\mathcal{H}^n_\sigma(\mathcal{R}, \mathcal{R}_*)$, and the latter vanish by virtue of [1, Theorem 3.2].

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