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THE PERFECT SEPTENARY FORMS WITH $\Delta_4 = 2$

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The aim of this paper is to enumerate all equivalence classes of perfect septenary forms with $\Delta_4 = 2$. This is an important section of the complete enumeration of perfect septenary forms by the method which was outlined in Stacey (1975). There are nine equivalence classes of these forms, four of which were announced for the first time in Stacey (1975). This work forms part of the author's D. Phil. thesis and was done at Oxford under the supervision of Dr. B. J. Birch. Though many of the calculations required for the proof were first done by computer, most of them can be done by hand. For the calculations of §5, use of the computer was absolutely necessary; in contrast, in §4 the attempt to persuade the computer to do most of the work was best abandoned. The computing was done on the I.C.L. 1906A at Oxford University.

1. Definitions and notation

A *perfect form* is a positive definite quadratic form whose coefficients can be determined from its minimum value at nonzero integral points and from knowledge of the points where it attains this minimum. Positive definite n-ary forms f and g are *equivalent* if there exists an integral unimodular transformation T and a positive constant c such that $fT = cg$, i.e. for all points $\boldsymbol{x}, f(\boldsymbol{x}T) = cg(\boldsymbol{x})$. A nonzero integral n-vector where f attains its minimum value (always assumed to be one) is called a *minimal vector* of f and an $s \times n$ matrix $X(f)$ whose rows in any order are the minimal vectors is called a *minimal matrix* of f. We always require that only one of a minimal vector and its negative belong to $X(f)$ although they may be freely interchanged at any time. Clearly, if f is perfect, $s \geqq \frac{1}{2}n(n+1)$ since f has $\frac{1}{2}n(n+1)$ coefficients to be determined.

Let A be an $m \times n$ submatrix of a minimal matrix. If the n-vector $\boldsymbol{a}$ or $-\boldsymbol{a}$ is a row of A, we write $\boldsymbol{a} \in A$. If every row of A is a row of B then we write $A \subset B$, whilst if $A \subset B$ and $B \subset A$, then A and B are *equal as minimal submatrices*. A and B are *equivalent* if there exists an integral unimodular transformation T such that $AT = B$. If f is equivalent to g, $X(f)$ is equivalent to $X(g)$. Conversely

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if $X(f)$ is equivalent to $X(g)$ and f is perfect, f is equivalent to g. Aut(A) is the group of integral unimodular transformations T such that $AT = A$.

If $m > n$, we define $\mathrm{Det}(A) = 0$. Otherwise $\mathrm{Det}(A) = $ g.c.d.$\{|\mathrm{determinant}(A_\alpha)|: A_\alpha$ is any $m \times m$ submatrix of $A\}$. If all $|\mathrm{determinant}(A_\alpha)| = 0$ then $\mathrm{Det}(A) = 0$. We note that if $m = n$, $\mathrm{Det}(A) = |\mathrm{determinant}\, A|$. If $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$ are integral n-vectors, $\mathrm{Det}\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m\}$ is defined to be $\mathrm{Det}(V)$ where V is any $m \times n$ matrix with these vectors as its rows.

Let r be any positive nonzero integer. If $r > \min\{m, n\}$, $\Delta_r(A) = 0$. Otherwise $\Delta_r(A) = \max\{\mathrm{Det}(A_\alpha) \mid A_\alpha$ is any $r \times n$ submatrix of $A\}$. We define $\Delta_r(f) = \Delta_r(X(f))$.

The vector $\boldsymbol{v} = (v_1, \cdots, v_n)$ is *primitive* if $\boldsymbol{v} \neq \mathbf{0}$ and g.c.d. $\{v_1, \cdots, v_n\} = 1$. We define ${}^r[\boldsymbol{v}] = (v_1, \cdots, v_r)$ and $[\boldsymbol{v}]^r = (v_{n+1-r}, \cdots, v_n)$, for any $1 \leqq r \leqq 7$. $[\boldsymbol{v}]^r$ is called the *r-ending* of v. If z is an integral r-vector and A is a set (or matrix) of minimal vectors, $N_A(z) = N(z)$ is the number of vectors in A with r-ending congruent modulo 2 to z.

We denote by $\boldsymbol{e}_i$ the ith unit row vector. A matrix which contains $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_r$ and no row vector not in the linear space generated by these vectors is called an *r-array*. If B and C are $m_1 \times n$ and $m_2 \times n$ matrices respectively, $\{B, C\}$ is the $(m_1 + m_2) \times n$ matrix whose rows are the rows of B followed by the rows of C. The vectors of C are said to *combine with respect to* B if a linear combination of every pair of vectors of C is a vector of B. Unless it is specifically stated otherwise, vector always means row vector.

Two theorems are fundamental in all that follows.

THEOREM 1. *If f is a positive definite quadratic form then*

$$\Delta_1(f), \Delta_2(f), \Delta_3(f) \leqq 1$$

$$\Delta_4(f), \Delta_5(f) \leqq 2$$

$$\Delta_6(f) \leqq, 4\ \Delta_7(f) \leqq 8$$

PROOF. Watson (1971).

THEOREM 2. *Let f be an n-ary positive definite quadratic form with*

$$\Delta_n(f) = n_1.$$

Then

(i) *if $n = 4$ and $n_1 = 2$, f is equivalent to*

$$B_4(x_1, \cdots, x_4) = \textstyle\sum_{i=1}^4 x_i^2 - x_1(x_2 + x_3 + x_4).$$

(ii) *if $n = 6$ and $n_1 = 4$, f is equivalent to*

$$B_6(x_1, \cdots, x_6) = \textstyle\sum_{1 \leqq i \leqq j \leqq 6} x_i x_j - x_1 x_2.$$

(iii) *if $n = 7$ and $n_1 = 8$, f is equivalent to*

$$E_7(x_1, \cdots, x_7) = \textstyle\sum_{1 \leqq i \leqq j \leqq 7} x_i x_j - x_1(x_2 + x_3).$$

PROOF. Watson (1971).

COROLLARY 3. *If f is a septenary form and $\Delta_4(f) = 2$, then f is equivalent to a form whose minimal matrix contains the 12×7 matrix Y_4^{12} whose rows are e_i $(1 \leqq i \leqq 4)$, $e_i + e_4$ $(1 \leqq i \leqq 3)$, $z = \sum_{i=1}^4 e_i$, $z - e_i$ $(1 \leqq i \leqq 3)$ and $z + e_4$. If $X(f)$ contains four vectors y_1, y_2, y_3, y_4, such that* Det $\{y_1, y_2, y_3, y_4\}$ $= 2$ *then $X(f)$ also contains the eight vectors $\frac{1}{2}(y_1 \pm y_2 \pm y_3 \pm y_4)$.*

PROOF. Note that if $v \in Y_4^{12}$ then $[v]^3 = 0$ and ${}^4[v] \in X(B_4)$.

DEFINITION 4. Let A be an $a \times 7$ subset of a minimal matrix $X(f)$ containing e_n for some $2 \leqq n \leqq 6$ and all $(a-1)$ vectors of $X(f)$ which are in the subspace generated by $e_1, \cdots, e_{n-1}$. An integral vector v is a *fitting vector* of A if

(i) $v \neq e_n$ and $[v - e_n]^{8-n} = 0$,

(ii) for any $y \in A$, such that $v \equiv y + e_n$ (modulo 2) then $\{y, v, e_n\}$ is a dependent set of vectors,

(iii) for any $y_1, y_2 \in A$ such that $v \equiv y_1 + y_2 + e_n$ (modulo 2) either $\{v, y_1, y_2, e_n\}$ is dependent or the four vectors $\frac{1}{2}(v - e_n \pm y_1 \pm y_2)$ belong to A.

We denote by $F(A)$ the set of fitting vectors of A. Two fitting vectors v, $w \in F(A)$ are *compatible* if $v - w + e_n \in F(A)$. A set of m fitting vectors of A is *compatible* if its vectors are pairwise compatible.

LEMMA 5. *Let A be a matrix as in Definition 4 and let $\{w_1, \cdots, w_m\}$ be a set of m n-vectors with $[w_i - e_n]^{8-n} = 0$ for all $1 \leqq i \leqq m$. If $\{A, w_1, \cdots, w_m\}$ is a subset of a minimal matrix of a positive definite form then $w_i \in F(A)$ for all $1 \leqq i \leqq m$ and $\{w_1, \cdots, w_m\}$ is a compatible set of fitting vectors.*

PROOF. For each i, $\{A, w_i\}$ must satisfy (ii) and (iii) above because (ii) ensures that $\Delta_3(\{A, w_i\}) \neq 2$ and (iii) ensures that either $\Delta_4(\{A, w_i\}) \neq 2$ or $\Delta_4(\{A, w_i\}) = 2$ and Corollary 3 holds for A. Hence $w_i \in F(A)$. Given any pair i, j such that $1 \leqq i < j \leqq m$, $\{A, w_i, w_j\}$ is equivalent to $\{A, w_i - w_j + e_n, -w_j + 2e_n\}$. Hence, by the first part, $w_i - w_j + e_n \in F(A)$ so w_i and w_j are compatible. Consequently $\{w_1, \cdots, w_m\}$ is a compatible set of fitting vectors.

The concepts of fitting vectors and compatibility are basic to our method of enumerating perfect forms. We will find all inequivalent minimal matrices which obey Theorem 1 and contain Y_4^{12} (necessary, by Corollary 3) by extending Y_4^{12}, one dimension at a time, by all inequivalent suitably sized compatible sets of fitting vectors. To eliminate equivalent compatible sets, the following observations are useful.

LEMMA 6. *Let A be as in Definition 4 and let w be an integral n-vector.*

(i) *If $[w]^{8-n}$ is primitive, $\{A \backslash \{e_n\}, w\}$ is equivalent to A.*

(ii) *If $v \in F(A)$, $T \in$* Aut(A) *and $e_n T = e_n$ then $\{A, v\}$ is equivalent to $\{A, vT\}$.*

(iii) *If $v, w \in F(A)$ then $\{A, v, w\}$ is equivalent to $\{A, -v + 2e_n, w - v + e_n\}$.*

(iv) *If $v, w \in F(A)$ then $\{A, v\}$ is equivalent to $\{A, -v + 2e_n\}$.*

PROOF. This is obvious.

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2. The 5-arrays

Henceforth, let f denote a perfect septenary form with $\Delta_4(f) = 2$. We will carry out the enumeration of forms by finding all inequivalent minimal matrices which obey Theorem 1, contain Y_4^{12} and are large enough to be minimal matrices of perfect forms, i.e. have at least thirty rows. The matrix requires thirty rows because Y_4^{12} has twelve rows but defines only ten coefficients. We first describe the 5-arrays which must be contained in the minimal matrices.

LEMMA 7. *If* $X(f) \supset Y_4{}^{12}$, $\Delta_6(f) < 4$ *and* $\boldsymbol{v}, \boldsymbol{w} \in X(f) \backslash Y_4{}^{12}$, *then* $[\boldsymbol{v}]^3$ *is primitive and if* $[\boldsymbol{v}]^3 \equiv [\boldsymbol{w}]^3$ (modulo 2) *then* $[\boldsymbol{v}]^3 = \pm [\boldsymbol{w}]^3$.

PROOF. Let $\boldsymbol{v} = (v_1, \cdots, v_7)$. Since $\boldsymbol{v} \in X(f) \backslash Y_4{}^{12}$, $[\boldsymbol{v}]^3 \neq 0$. If a positive integer k divides v_5, v_6, v_7 then $2k$ divides $D = \mathrm{Det}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + 2\boldsymbol{e}_4, \boldsymbol{v}\}$. As $\Delta_5(f) \leqq 2$ and $D \neq 0$, $k = 1$ so $[\boldsymbol{v}]^3$ is primitive.

If $\Delta_6(f) < 4$, then $D = \mathrm{Det}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + 2\boldsymbol{e}_4, \boldsymbol{v}, \boldsymbol{w}\} \leqq 3$ for all $\boldsymbol{v}, \boldsymbol{w} \in X(f)$. Hence if $[\boldsymbol{v}]^3 \equiv [\boldsymbol{w}]^3$ (mod 2) then the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + 2\boldsymbol{e}_4, \boldsymbol{v}, \boldsymbol{w}$ must be dependent otherwise $D \geqq 4$. Consequently $[\boldsymbol{v}]^3 = \gamma[\boldsymbol{w}]^3$ and since $[\boldsymbol{v}]^3$ is primitive, $\gamma = \pm 1$.

DEFINITION 8.

$$Y_5^{15} = \{Y_4{}^{12}, \boldsymbol{e}_5, \boldsymbol{e}_1 + \boldsymbol{e}_5, \boldsymbol{e}_4 + \boldsymbol{e}_5\}$$

$$Y_5^{16} = \{Y_5^{15}, -\boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_4 + \boldsymbol{e}_5\}$$

$$Y_5^{20} = \{\boldsymbol{v} \colon [\boldsymbol{v}]^2 = 0 \text{ and } {}^5[\boldsymbol{v}] \text{ is a}$$

minimal vector of the perfect quinary form $\sum_{1 \leqq i < j \leqq 5} x_i x_j - x_1(x_2 + x_3)\}$.

LEMMA 9. *The fitting vectors of* $\{Y_4^{12}, \boldsymbol{e}_5\}$ *are*

(i) $\boldsymbol{y} + \boldsymbol{e}_5$, $-\boldsymbol{y} + \boldsymbol{e}_5$ *for all* $\boldsymbol{y} \in Y_4^{12}$

(ii) $\pm(\boldsymbol{e}_i + \boldsymbol{e}_j) + \boldsymbol{e}_5$ *for all* $1 \leqq i < j \leqq 3$

$\pm(\boldsymbol{e}_i + \boldsymbol{e}_j + 2\boldsymbol{e}_4) + \boldsymbol{e}_5$ *for all* $1 \leqq i < j \leqq 3$

$\pm(\boldsymbol{e}_i + \boldsymbol{e}_j + 2\boldsymbol{e}_k + 2\boldsymbol{e}_4) + \boldsymbol{e}_5$ *for all* $1 \leqq i, j, k \leqq 3$, $i < j$ *and* $k \neq i$, $k \neq j$.

If $\boldsymbol{v}$ *is any fitting vector of* (ii) *and* $X(f) \supset \{Y_4{}^{12}, \boldsymbol{e}_5, \boldsymbol{v}\}$ *then there exists an integral unimodular transformation* S *such that* $X(f)S \supset Y_5{}^{20}$.

PROOF. The calculation of fitting vectors is straightforward from Definition 4. If $\boldsymbol{v}$ is any fitting vector of (ii) above, transformations as in Lemma 6(ii) and (iv), can be found to show that $\{Y_4{}^{12}, \boldsymbol{e}_5, \boldsymbol{v}\}$ is equivalent to $\{Y_4{}^{12}, \boldsymbol{e}_5, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_5\} = B$. By applying Corollary 3 to B we see that if $X(g) \supset B$ then $X(g)$ also contains

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$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and an integral unimodular S exists such that $\{B, C\}S = Y_5^{20}$.

PROPOSITION 10. *f is equivalent to a perfect form whose minimal matrix contains Y_5^{15}, Y_5^{16} or Y_5^{20}*

PROOF. If $\Delta_6(f) = 4$, by Theorem 2, f is equivalent to a form $g(x_1, \cdots, x_7)$ such that $g(x_1, \cdots, x_6, 0) = B_6(x_1, \cdots, x_6)$. Since there exists a 6×6 unimodular transformation T such that $Y_5^{20} \subset X(B_6)T$ the result follows.

Hence we may assume $\Delta_6(f) \leqq 3$ and $X(f) \supset Y_4^{12}$. By Lemma 7, vectors in $X(f) \backslash Y_4^{12}$ have at most seven different 3-endings, yet, for f to be perfect, $X(f) \backslash Y_4^{12}$ contains at least eighteen vectors. Consequently one nonzero 3-ending occurs at least 3 times so we may choose $\boldsymbol{u} \in X(f) \backslash Y_4^{12}$ such that $N([\boldsymbol{u}]^3) \geqq N([\boldsymbol{w}]^3)$ for all $\boldsymbol{w} \in X(f) \backslash Y_4^{12}$ and $N([\boldsymbol{u}]^3) \geqq 3$. By Lemma 7 and Lemma 6(i), there exists an integral unimodular transformation T such that $\{Y_4^{12}, \boldsymbol{u}\}T = \{Y_4^{12}, \boldsymbol{e}_5\}$ and $X(f)T$ contains at least two vectors $\boldsymbol{v}$, $\boldsymbol{w}$ with $[\boldsymbol{v}]^3 = [\boldsymbol{w}]^3 = [\boldsymbol{e}_5]^3$ $\boldsymbol{v}, \boldsymbol{w} \neq \boldsymbol{e}_5$. Now to satisfy Theorem 1, $\boldsymbol{v}$ and $\boldsymbol{w}$ must belong to $F(\{Y_4^{12}, \boldsymbol{e}_5\})$. By Lemma 9 and Lemma 6(iii), if $\{\boldsymbol{v} - \boldsymbol{e}_5, \boldsymbol{w} - \boldsymbol{e}_5, \boldsymbol{v} - \boldsymbol{w}\} \not\subset Y_4^{12}$, there exists an integral unimodular S such that $X(fTS) \supset Y_5^{20}$. On the other hand, it is fairly easy to see that, using the equivalences of Lemma 6, there are just two inequivalent sets of compatible fitting vectors, producing Y_5^{15} and Y_5^{16}, which have the property that $\{\boldsymbol{v} - \boldsymbol{e}_5, \boldsymbol{w} - \boldsymbol{e}_5, \boldsymbol{v} - \boldsymbol{w}\} \subset Y_4^{12}$ for all suitable pairs $\boldsymbol{v}$, $\boldsymbol{w}$. The proposition is thereby proved.

THEOREM 11. *A perfect septenary form with $\Delta_4 = 2$ is equivalent to a form f with*

(i) $\Delta_7(f) = 8$

or (ii) $\Delta_7(f) = 4 = \Delta_6(f)$

or (iii) $\Delta_7(f) \leqq 4$, $\Delta_6(f) \leqq 3$, $X(f) \supset Y_6^{36}$ *where Y_6^{36} is the 36×7 matrix with rows $\boldsymbol{v}$ such that $[\boldsymbol{v}]^1 = 0$ and ${}^6[\boldsymbol{v}]$ is a minimal vector of the perfect senary form* $E_6(\boldsymbol{x}) = \sum_{1 \leqq i \leqq j \leqq 6} x_i x_j - x_1(x_2 + x_3)$

or (iv) $\Delta_7(f) \leqq 4$, $\Delta_6(f) \leqq 3$, $X(f) \supset Y_5^{20}$, $X(f) \not\supset Y_6^{36}T$ *for any integral unimodular T*

or (v) $\Delta_7(f) \leqq 4$, $\Delta_6(f) \leqq 3$, $X(f) \supset Y_4^{12}$ *but* $X(f) \not\supset Y_5^{20}T$ *for any integral unimodular* T. *In this case we may also insist that either* $X(f) \supset Y_5^{15}$ *and* $N(z) \leqq 3$ *for all nonzero integral* z *or* $X(f) \supset Y_5^{16}$.

PROOF. The preceding theorems, lemmas and propositions. Note that $Y_6^{36} \supset Y_5^{20}$.

In the remaining sections we will treat in turn each of the above five categories, finding all equivalent classes of perfect forms which fall in each.

3. Categories (i), (ii) and (iii) of Theorem 11

PROPOSITION 12. *If* $\Delta_7(f) = 8$, f *is equivalent to* E_7, *the absolutely extreme form in seven variables.*

PROOF. Theorem 2.

PROPOSITION 13. *There is no perfect form with* $\Delta_7(f) \leqq 4$ *and* $X(f) \supset Y_6^{36}$.

PROOF. If $X(f) \supset Y_6^{36}$, thirty-six vectors determine twenty-one coefficients and so at least seven more vectors are required for $X(f)$. Consequently $X(f)$ has at least forty-three minimal vectors and so by Watson (1971a), f is equivalent to E_7 and $\Delta_7(f) = 8$.

We now turn to consider category (ii). By Theorem 2 we may assume that $X(f) \supset Y_6^{30}$, the 30×7 minimal matrix whose rows $\boldsymbol{v}$ are such that $[\boldsymbol{v}]^1 = 0$ and ${}^6[\boldsymbol{v}]$ is a minimal vector of B_6. Since $\Delta_6(f) = \Delta_7(f)$, for all $\boldsymbol{v} \in X(f) \setminus Y_6^{30}$, $|[\boldsymbol{v}]^1| = 1$ so that, for any such v, there exists an integral unimodular T such that $\{Y_6^{30}, \boldsymbol{v}\}T = \{Y_6^{30}, \boldsymbol{e}_7\}$; hence we may assume that $X(f) \supset \{Y_6^{30}, \boldsymbol{e}_7\} = A$.

Following Barnes (1957), p. 469 we display the symmetry of A by using the transformation U with determinant -2,

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$Y_6^{30}U$ has thirty rows $\boldsymbol{e}_i \pm \boldsymbol{e}_j$ $(1 \leqq i < j \leqq 6)$ and $\mathrm{Aut}(Y_6^{30})$ contains the group $U\mathscr{S}U^{-1}$, where $\mathscr{S}$ is the group of transformations which permute and arbitrarily change the signs of the first six components of the 7-vectors.

LEMMA 14. *$F(A)U$ has the following members*:

(i) $\sum_{i,j} \varepsilon_i e_i + e_7$ *for all* $|\varepsilon_i| = |\varepsilon_j| = 1,\ 1 \leqq i < j \leqq 6$,

(ii) $e_7 + \sum_{i,j,k,h} \varepsilon_i e_i$ *for all* $|\varepsilon_i| = |\varepsilon_j| = |\varepsilon_k| = |\varepsilon_h| = 1,\ 1 \leqq i < j < k < h \leqq 6$,

(iii) $2\varepsilon_i e_i + e_7$ *for* $|\varepsilon_i| = 1,\ 1 \leqq i \leqq 6$,

(iv) *any vector congruent modulo* 2 *to* $\sum_{i=1}^{7} e_i$ *with seventh component equal to* $+1$.

If $X(f)$ contains $\{A, v\}$ then, by repeated use of Corollary 3, *$X(f)$ also contains the submatrix $I(v)$ defined by*

(v) $I((\sum_{i,j} \varepsilon_i e_i + e_7)U^{-1}) = \phi$

(vi) $I((\sum_{i,j\,k,h} \varepsilon_i e_i + e_7)U^{-1})U = \{\varepsilon_m e_m + \varepsilon_p e_p + e_7 \mid m < p; m, p \in \{i,j,k,h\}\}$

(vii) $I((2\varepsilon_i e_i + e_7)U^{-1})U = \{\varepsilon_i e_i \pm e_j + e_7 \mid 1 \leqq j \leqq 6,\ i \neq j\}$,

(viii) *if* $v \equiv \sum_{i=1}^{7} e_i$ (modulo 2) then $I(vU^{-1}) = \phi$.

PROOF. A straightforward calculation. Although the results have been given in transformed coordinates, the calculations must be performed in the original coordinates.

PROPOSITION 15. *If $X(f)$ contains a submatrix equivalent to $\{A, (2\varepsilon_i e_i + e_7)U^{-1}\}$, then f is equivalent to the perfect septenary form with forty-two minimal vectors, $B_7(x) = \sum_{1 \leqq i \leqq j \leqq 7} x_i x_j - x_1 x_2$.*

PROOF. $X(f)$ also contains the ten vectors of $I((2\varepsilon_i e_i + e_7)U^{-1})$ and this complete matrix defines the form.

PROPOSITION 16. *The minimal matrix of every perfect form of category* (ii) *contains a submatrix equivalent to $\{A, (2\varepsilon_i e_i + e_7)U^{-1}\}$.*

PROOF. Let us assume $X(f)$ contains no submatrix equivalent to $\{A, (2\varepsilon_i e_i + e_7)U^{-1}\}$. Consequently no two different vectors of $X(f)U \setminus AU$ are congruent modulo 2.

If $X(f)$ contains any vector $v = (\sum_{i,j,h,k} \varepsilon_i e_i + e_7)U^{-1}$, it also contains $I(v)$ and $\{A, v, I(v)\}$ is equivalent to $B = \{A, w = (\sum_3^7 e_i)U^{-1}, I(w)\}$. B contains thirty-eight minimal vectors and determines a form with two parameters. If $z \in F(A)$, z is compatible with $B \setminus A$ and $\{B, z\}$ satisfies the hypothesis of this proposition then it can readily be seen that $\{B, z\}$ is equivalent to $\{B, (e_1 + e_2 + e_7)U^{-1}\}$, $\{B, (e_1 + e_3 + e_7)U^{-1}\}$ or $\{B, (\sum_{i=1}^{7} e_i)U^{-1}\}$. On applying Corollary 3 to each of these arrays, it is seen that each implies the presence of at least four more fitting vectors in $X(f)$. Consequently $X(f)$ has at least forty-three minimal vectors and by Watson (1971a), f is equivalent to E_7.

Thus it remains to assume that $X(f)$ contains no subset equivalent to $\{A, (\sum_3^7 e_i)U^{-1}\}$ or to $\{A, (2e_1 + e_7)U^{-1}\}$. The vectors of $X(f)U \setminus AU$ are thus of the form (i) and (iv) of Lemma 14, and for all pairs $v, w \in X(f)U \setminus AU$, $v - w + e_7$ is also of form (i) or (iv). But it is not hard to see that it is impossible to obtain a compatible set of more than five such vectors, whilst $X(f) \in A$ requires at least six vectors. The result follows.

Hence we have shown by Propositions 16 and 15 that B_7 is the only perfect form belonging to category (ii). Thus categories (i), (ii) and (iii) contain only two forms E_7 and B_7.

4. Category (iv) of Theorem 11

In this section it is assumed that $X(f) \supset Y_5^{20}$, but $X(f) \not\supset Y_6^{36}T, X(f) \not\supset Y_6^{36}T$ for any integral unimodular T. Following Barnes (1957), p. 482 we define

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which has $\det P = 3$ and which transforms Y_5^{20} to the matrix Y with rows $e_i - e_j$ $(1 \leqq i < j \leqq 5)$ and $e_i + e_j + e_k$ $(1 \leqq i < j < k \leqq 5)$. The following lemma shows that when calculations involve only tests of equality and congruence modulo 2 (in particular, calculation of fitting vectors and testing compatibility) we can work with Y instead of Y_5^{20} to take advantage of its obvious symmetry. Results can be transformed back to Y_5^{20} without loss; in particular $F(\{Y_5^{20}, e_6\}P) = F(\{Y_5^{20}, e_6\})P$. We also note that $\operatorname{Aut}(Y_5^{20})$ contains $P\mathscr{S}P^{-1}$ where $\mathscr{S}$ is the group of transformations permuting the first five coordinates.

Lemma 17. *If $v = (v_1, \cdots, v_7)$ is an integral vector, so is vP; whilst if $\sum_{i=1}^{5} v_i \equiv 0$ (modulo 3), vP^{-1} is also integral. If $p \not\equiv 0$ (modulo 3) $\operatorname{Det}\{w_1, w_2, \cdots, w_n\} \equiv 0 \pmod p$ if and only if $\operatorname{Det}\{w_1P, \cdots, w_nP\} \equiv 0 \pmod p$ and $\operatorname{Det}\{w_1, \cdots, w_n\} = 0$ if and only if $\operatorname{Det}\{w_1P, \cdots, w_nP\} = 0$.*

Proof. This is obvious after calculating P^{-1} and noting that, provided g.c.d. $\{p, \det P\} = 1$, $vP \equiv \mathbf{0}$ (modulo p) implies $v \equiv \mathbf{0}$ (modulo p).

Lemma 18. *$F(\{Y_5^{20}, e_6\})P$ consists of:*

(i) $\pm y + e_6$ *for all* $y \in Y_5^{20}$,

(ii) $\pm(\sum_{i=1}^{5} e_i - 2e_j) + e_6$ *for all* $1 \leqq j \leqq 5$,

(iii) $\pm(\sum_{i=1}^{5} e_i + e_j) + e_6$ *for all* $1 \leqq j \leqq 5$,

(iv) $\pm(2e_i + e_j) + e_6$ *for all* $1 \leqq i, j \leqq 5$; $i \neq j$,

(v) $e_i + e_j + e_k - e_h + e_6$ *for all* $1 \leqq i,j,k,h \leqq 5$; i,j,k,h *distinct.*

If $X(f)$ *contains* $\{Y_5^{20}, e_6, v\}$ *then* $X(f)$ *also contains the submatrix* $I(v)$ *defined by*:

(vi) $I((\pm y + e_6)P^{-1})P = \phi$ *for all* $y \in Y_5^{20}$,

(vii) $I((\sum_{i=1}^5 e_i - 2e_j + e_6)P^{-1})P = \{e_r + e_s + e_t + e_6,\ e_r - e_j + e_6 \mid 1 \leqq r, s, t \leqq 5, r, s, t$ *distinct*, $j \notin \{r,s,t\}\}$,

(viii) $I((\sum_{i=1}^5 e_i + e_j + e_6)P^{-1})P = \{e_r + e_s + e_j + e_6 \mid 1 \leqq r < s \leqq 5, j \notin \{r,s\}\}$,

(ix) $I(2e_i + e_j + e_6)P^{-1})P = \{e_i + e_j + e_r + e_6,\ e_i - e_r + e_6 \mid 1 \leqq r \leqq 5, r \notin \{i,j\}\}$,

(x) $I(e_i + e_j - e_k - e_h + e_6)P^{-1})P = \{e_r - e_s + e_6 \mid r \in \{i,j\}, s \in \{k,h\}\} \cup \{e_i + e_j + e_t + e_6,\ -e_k - e_h - e_t + e_6 \mid t \in \{1,\cdots,5\} \backslash \{i,j,k,h\}\}$.

PROOF. Calculation from the definition of fitting vectors. The matrices $I(v)$ are calculated by repeated use of Corollary 3. By Lemma 17, Y_5^{20} can be used throughout. Its obvious symmetry simplifies the calculations a great deal

PROPOSITION 19. *If* $X(f)P$ *contains a subset equivalent to* $\{Y, e_6, \sum_{i=1}^5 e_i - 2e_j + e_6\}$ $(1 \leqq j \leqq 5)$ *then* $\Delta_6(f) \geqq 4$.

PROOF. Any such $X(f)$ is equivalent to a minimal matrix $X(g)$ containing $\{Y_5^{20}, e_6,\ v = (1,1,1,1,-1,1,0)P^{-1}, I(v)\}$. These thirty minimal vectors completely determine a perfect senary form equivalent to B_6 and $\Delta_6(B_6) = 4$.

LEMMA 20. *If* v *is a fitting vector of* Y_5^{20} *of type* (iii), (iv) *or* (v) *of Lemma* 18, *then* $C(v) = \{Y_5^{20}, e_6, v, I(v)\}$ *is equivalent to* $C((2,1,1,1,1,1,0)P^{-1})$.

PROOF. As $\mathscr{S} \subset \mathrm{Aut}(Y_5^{20}P)$ and $P\mathscr{S}P^{-1} \subset \mathrm{Aut}(Y_5^{20})$, we need consider only $vP \in \{w_1P = (1,2,0,0,0,1,0),\ w_2P = (0,1,1,-1,-1,1,0),\ w_3P = (2,1,1,1,1,1,0)$ and

$$C(w_1).\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = C(w_2).\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = C(w_3)$$

PROPOSITION 21. *If* $v, w \in X(f)\backslash Y_5^{20}$ *and* $[w]^2 \equiv [v]^2$ (modulo 2) *then* $[w]^2 = \pm[v]^2$.

PROOF. Let us assume that $[w]^2 \equiv [v]^2$ (modulo 2) but $[v]^2 \neq \pm[w]^2$. $X(f)$ may be transformed in such a way that $X(f)T \supset \{Y_5^{20}, w = -e_6 + e_7, v\}$ where $[v]^2 = (1,1)$. If $v_0 \equiv v \pmod 2$ and $[v_0]^2 = (1,1)$ then there exists an integral unimodular S such that $\{Y_5^{20}, w, v\}S = \{Y_5^{20}, w, v_0\}$. Hence we may assume that $vP \in \{(1,-1,0,0,0,1,1), (1,1,-1,-1,0,1,1), (1,1,1,1,0,0,1,1), (-2,-1,0,0,0,1,1), (1,1,1,1,-1,1,1)\}$. Since $\Delta_3(f) \neq 2$, $vP = (1,-1,0,0,0,1,1)$ and $vP = (1,1,1,0,0,1,1)$ are immediately excluded. Repeated application of Corollary 3 shows that with the four remaining choices of v,

$$X(f) \supset B = \{Y_5^{20}, w, v, e_6, e_7, v-e_6, v-e_7, I(v-e_6), I(v-e_6)\sigma\}$$

where $e_i\sigma = e_i$ for $1 \leqq i \leqq 5$ and $e_6\sigma = e_7$, $e_7\sigma = e_6$. Now the case $vP = (1,1,1,1,-1,1,1)$ may be excluded by Proposition 19. Thus it remains to consider only $vP \in \{(-2,-1,0,0,0,1,1), (-2,-1,-1,-1,-1,1,1)\}$. By Lemma 20, these cases are equivalent so we take $vP = (-2,-1,0,0,0,1,1)$. B then defines a two parameter form:

$$b(x_1, \cdots, x_7) = E_7(x_1, \cdots, x_7) - x_2((1-\varepsilon)x_6 + (1-\eta)x_7),$$

so at least two more vectors u_1 and u_2 are required for $X(f)$. If $[u_1]^2 \equiv (1,0)$ (modulo 2), then by using $\Delta_7(f) < 4$, it is easy to see that $\pm[u_1]^2 \in \{(1,0), (1,2), (1,-2)\}$. But if $[u_1]^2 = \pm(1,0)$ then $X(f)$ contains more than twenty-eight minimal vectors in the 6-space defined by $x_7 = 0$; hence by Watson (1971a) $X(f)$ belongs to category (i), (ii) or (iii). Otherwise we have $\pm[u_1]^2 \in \{(1,2), (1,-2)\}$ but in these cases either $\det([u_1]^2, (1,-1)) = 3$ or $\det([v_1]^2, (1,1)) = 3$ and then, since $\Delta_5(Y_5^{20}) = 2$, $\Delta_7(f) \geqq 6$ so f belongs to category (i). A similar argument shows that $[u_1]^2 \equiv (0,1)$ (modulo 2) is also impossible, thus $[u_1]^2 \equiv [u_2]^2 \equiv (1,1)$ (modulo 2). In order that the two parameters ε and η of $b(x)$ can be determined, $[u_1]^2 \neq [u_2]^2$ and since $\Delta_7(f) < 4$, we may take $[u_2]^2 = (1,1)$ and $[u_2]^2 = (-1,1)$.

As B has thirty-eight minimal vectors, $({}^5[u_1], 0, 0)$ must belong to $\pm Y_5{}^{20}$; otherwise $X(f)$ has at least six more vectors ending in (1,1) and this is forbidden by Watson (1971a). If $u_1 = (y_1, \cdots, y_5, 0, 0)$, $f(u_1) = 1 = f(y_1, \cdots, y_5, 0, 0) + 1 + (\varepsilon-\eta)y_2 = 2 + (\varepsilon-\eta)y_2$. Since $|y_2| \leqq 1$, for any $(y_1, \cdots, y_5, 0, 0) \in Y_5{}^{20}$, we immediately have $|\varepsilon-\eta| = 1$ and so either $(0,1,0,0,0,-1,1)$ or $(0,-1,0,0 0,-1,1)$ belongs to $X(f)$. But on applying Corollary 3 again, it is seen that $X(f)$ contains a subset equivalent to Y_6^{36}. The result follows.

LEMMA 22. *The largest set of vectors from* Y_5^{20} *which combine with respect to* Y_5^{20} *has four members. Under* $\mathrm{Aut}(Y_5^{20})$ *there are two inequivalent sets of three combining vectors and one set of four combining vectors.*

PROOF. The transformation

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

takes Y_5^{20} to the matrix with rows $e_1 \pm e_j$ $(1 \leqq i \leqq j \leqq 5)$. $Y_5^{20}S$ is stabilized by $\mathscr{S}$, the group of all permutations and arbitrary changes of sign of the first five components; indeed by Barnes (1957, p. 469) $\mathrm{Aut}(Y_5^{20}) = S\mathscr{S}S^{-1}$. It is now easy to find the inequivalent combining sets of $Y_5^{20}S$. When transformed to become subsets of $Y_5^{20}P$, the inequivalent combining sets are:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

LEMMA 23. *If $X(f)$ belongs to category* (iv) *it is equivalent to a minimal matrix with* $N(1,0) \geqq N(z)$ *for all integral 2-vectors* $z \not\equiv 0$ (modulo 2) *which contains*

(i) $A = \{Y_5^{20}, e_6, v = (-2,-1,0,0,0,1,0)P^{-1}, I(v)\}$ *if* $N_{X(f)}(z) > 5$ *for some integral 2-vectors* $z \not\equiv 0$ (modulo 2),
or (ii) $B = \{Y_5^{20}, e_6\} \cup \{(e_1 - e_j + e_6)P^{-1} \mid 2 \leqq j \leqq 5\}$ *if* $N_{X(f)}(z) \leqq 5$ *for all integral 2-vectors* $z \not\equiv 0$ (modulo 2).

A defines the eight parameter form $a(x_1, \cdots, x_7) = E_6(x_1, \cdots, x_6) - (1-\varepsilon)x_2x_6 + \sum_{i=1}^{7} \gamma_i x_i x_7$, *whilst B defines the form* $b(x_1, \cdots, x_7) = E(x_1, \cdots, x_6) - (1-\varepsilon)x_1x_6 + \sum_{i=1}^{7} \gamma_i x_i x_7$. E_6 *was defined in Theorem* 11.

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PROOF. Choose a vector $v \in X(f)\backslash Y_5^{20}$ with $N([v]^2)$ as large as possible. If a positive integer k divides $[v]^2$, then

$$k \cdot \Delta_5(Y_5^{20}) = 2k < \Delta_6(f) < 4$$

so $k = 1$. Hence $[v]^2$ is primitive and there exists an integral unimodular T such that $\{Y_5^{20}, v\}T = \{Y_5^{20}, e_6\}$. As Y_5^{20} determines fifteen coefficients, $X(f)T$ requires at least thirteen vectors not in Y_5^{20}: hence $N_{X(f)T}(1,0) \geqq 5$. Consequently by Proposition 21, there are at least five vectors with 2-ending $(1,0)$. Let $X(f)T \supset \{Y_5^{20}, e_6, v\}$. By Proposition 19, vP is not of the type (ii) of Lemma 18 whilst if vP is of the type (iii), (iv) or (v) of Lemma 18, $X(f)T$ contains a subset equivalent to A and $N(1,0) \geqq 8$. Note that, by Watson (1971a), $N_{X(f)T}(1,0) = 8$.

The case remains where all vectors of $X(f)T\backslash\{Y_5^{20}, e_6\}$ which end in $(1,0)$ are of the type (i) of Lemma 18. In order to ensure that $X(f)T$ does not contain a subset equivalent to A, we also insist that if $v, w \in X(f)T\backslash\{Y_5^{20}, e_6\}$ then $v - w + e_6$ is also of type (i) (by Lemma 6(iii)). This means that the set of vectors of $X(f)T$ ending in $(1,0)$ is a combining set with respect to Y_5^{20}. Lemma 22 now immediately shows that all suitable such arrays are equivalent to B.

PROPOSITION 24. *$X(f)$ is equivalent to a minimal matrix which has $N(0,1) \geqq N(1,1)$ and contains one of*

(i) $G_1 = A \cup \{e_7, v = (-2,-1,0,0,0,0,1)P^{-1}, I(v)\}$,

(ii) $G_2 = A \cup \{e_7, v = (-1,-2,0,0,0,0,1)P^{-1}, I(v)\}$,

(iii) $G_3 = A \cup \{e_7, v = (-2,-1,-1,-1,-1,0,1)P^{-1}, I(v)\}$,

(iv) $G_4 = A \cup \{e_7, v = (-1,-2,-1,-1,-1,0,1)P^{-1}, I(v)\}$,

(v) *A and two sets of vectors ending respectively in $(0,1)$ and $(1,1)$ or $(-1,1)$ which combine with respect to Y_5^{20},*

(vi) *B and two sets of vectors ending respectively in $(0,1)$ and $(1,1)$ or $(-1,1)$, which combine with respect to Y_5^{20}.*

PROOF. By Lemma 23, $X(f)$ is equivalent to a minimal matrix $X(g)$ containing $C \in \{A, B\}$ and having $N(1,0) \geqq N(z)$ for all integral 2-vectors $z \not\equiv 0$ (modulo 2). If $X(g) \supset \{C, v\}$ then as $\Delta_5(Y_5^{20}) = 2$, $2\left|[v]^1\right| \leqq \Delta_7(g) \leqq 4$ so $\left|[v]^1\right| \leqq 2$. Since $[v]^1 \equiv 1$ (modulo 2) for $v \notin C$, we deduce $[v]^1 = \pm 1$. Choose $v \in X(g)\backslash C$ such that $N([v]^2) \geqq N([w]^2)$ for all $w \in X(g)\backslash C$. Then there exists T such that $\{C, v\}T = \{C, e_7\}$ and $N_{X(g)T}(0,1) \geqq N_{X(g)T}(1,1)$. The remaining part of the proposition is proved by considering $F(Y_5^{20} \cup e_7) = F(Y_5^{20} \cup e_6)\sigma$ where $e_i\sigma = e_i$ for $1 \leqq i \leqq 5$, $e_6\sigma = e_7$ and $e_7\sigma = e_6$. On considering $\mathrm{Aut}(C)$, the result follows similarly to Lemma 23.

LEMMA 25. *There is no perfect form of category* (iv) *which contains a subset equivalent to G_1 or G_4.*

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PROOF. The coefficients of $a(x)$ defined by G are $\gamma_1 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_7 = 1$ and by G_4 are $1 = \gamma_1 = \gamma_7$, $\gamma_3 = \gamma_4 = \gamma_5 = 0$. As all vectors in $X(f)\backslash G_1$ and $X(f)\backslash G_4$ have the same 2-ending (by Proposition 21) in neither case can the parameters ε and γ_2 be determined separately.

LEMMA 26. *There is no perfect form of category* (iv) *whose minimal matrix contains a subset equivalent to G_2 or G_3.*

PROOF. G determines the form $a(x)$ with the relations $\gamma_1 = \gamma_7 = 1$, $1 + \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5$ and G_3 determines the form $a(x)$ with the relations $\gamma_1 = \gamma_7 = 1$, $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5$. Hence if we assume $X(f) \supset G_i$ ($i = 2$ or 3), $X(f)$ requires at least three minimal vectors with 2-ending congruent modulo 2 to $(1,1)$. Since $\Delta_5(f) = 2$ and $\Delta_7(f) \leqq 4$ we see that if $\boldsymbol{v} \in X(f)\backslash G_i$ then $[\boldsymbol{v}]^2 \in \pm\{(1,1), (-1,1)\}$. If the vectors of $X(f)\backslash G_i$ do not combine with respect to Y_5^{20}, then by Lemmas 6 and 18, $N_{X(f)}(1,1) \geqq 8$ and $X(f)$ contains at least forty-four minimal vectors. But, by Watson (1971a), f is equivalent to E_7 and does not belong to category (iv). Hence the vectors $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_k$ of $X(f)\backslash G_i$ combine with respect to Y_5^{20}.

It can be easily seen that if the minimum value of $a(\boldsymbol{x})$ is one, $0 \leqq \varepsilon \leqq 1$, whilst if $\varepsilon = 0$ or $\varepsilon = 1$, $X(f) \supset Y_6^{36}T$ for some integral unimodular T. Consequently, the value of ε must not be integral. Let $\boldsymbol{w}_i = (w_{i1}, \cdots, w_{i7})$ where $|w_{i6}| = w_{i7} = 1$. For both G_2 and G_3, ε is determined from a system of three equations: $\varepsilon w_{j2}w_{j6} + \gamma_3 w_{j7}(w_{j2} + w_{j3} + w_{j4} + w_{j5}) + \gamma_6 w_{j6}w_{j7} =$ integer for suitable $j \in \{1, \cdots, k\}$. Assume, without loss of generality that $j \in \{1,2,3\}$. Consider the determinant of the set of equations

$$D = \begin{vmatrix} w_{12}w_{16} & (w_{12}+w_{13}+w_{14}+w_{15})w_{17} & w_{17}w_{16} \\ w_{22}w_{26} & \cdot & \cdot \\ w_{32}w_{36} & \cdot & w_{37}w_{36} \end{vmatrix}$$

Since $w_{17} = w_{27} = w_{37} = 1$, $w_{16} = w_{26} + w_{36} \in \{+1, -1\}$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ combine with respect to Y_5^{20},

$$\pm D = \begin{vmatrix} * & * & 1 \\ y_{12} & y_{12}+y_{13}+y_{14}+y_{15} & 0 \\ y_{22} & y_{22}+y_{23}+y_{24}+y_{25} & 0 \end{vmatrix} = \begin{vmatrix} y_{12} & y_{13}+y_{14}+y_{15} \\ y_{22} & y_{23}+y_{24}+y_{25} \end{vmatrix}$$

where $\boldsymbol{y}_1 = (y_{11}, \cdots, y_{17})$ and $\boldsymbol{y}_2 = (y_{21}, \cdots, y_{27})$ are 2 combining vectors of Y_5^{20}. But it is easily seen that $\{(z_2, z_3 + z_4 + z_5) \mid (z_1, z_2, \cdots, z_7) \in Y_5^{20}\} =$

$\{\pm(1,0),\ \pm(0,1),\ \pm(1,-2),\ \pm(0,0)\}$. Since ε is not an integer, $|D|$ must not be 0 or 1. Consequently we must have $(y_{12}, y_{13}+y_{14}+y_{15}) \in \{(+1,0),(-1,0)\}$ and $(y_{22}, y_{23}+y_{24}+y_{25}) \in \{(1,-2),(-1,2)\}$. But then y_1 and y_2 do not combine with respect to Y_5^{20}. The lemma is thereby proved.

It remains now to consider cases (v) and (vi) of Proposition 24. We require first the following lemma.

LEMMA 27. *If* $y_1, \cdots, y_5 \in Y_5^{20}$ *and* $\{y_1, y_2, y_3, y_4\}$ *is a combining set with respect to* Y_5^{20} *then* $D = \det\{y_1, \cdots, y_5\} < 2$. *If* $\{y_1, y_2, y_3\}$ *combines with respect to* Y_5^{20} *then* $D < 2$ *unless* $\{y_1, y_2, y_3\}$ *is equivalent to*

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} P^{-1}.$$

PROOF. $D \leqq \Delta_5(Y_5^{20}) = 2$. If $\{y_1, \cdots, y_4\}$ is a combining set, by Lemma 22, it can be transformed to $\{e_1 - e_j \mid j = 2,3,4,5\}P^{-1}$ and as no vector of $Y_5^{20}P$ is congruent modulo 2 to $e_2 + e_3 + e_4 + e_5$, clearly $D \neq 2$. If $\{y_1, y_2, y_3\}$ is a combining set we can assume it to be $\{e_1 - e_j \mid j = 2,3,4\}$ or $\{e_1 - e_2,\ e_1 - e_3,\ e_1 + e_4 + e_5\}$ and it is easy to check that D can be two only in the latter case.

PROPOSITION 28. *If* $X(f)$ *belongs to cases* (*v*) *or* (*vi*) *of Proposition* 24, *then* $X(f)$ *is equivalent to a minimal matrix containing* $A_1 = \{A, E_1\}$, $A_2 = \{A, E_2\}$, $B_1 = \{B, E_2\}$ *where* $E_1 = \{e_7, e_1 - e_2 + e_7,\ e_1 - e_3 + e_7,\ e_1 + e_4 + e_5 + e_7\}P^{-1}$ *and* $E_2 = \{e_7, e_1 + e_2 + e_3 + e_7,\ e_3 - e_4 + e_7,\ e_3 - e_5 + e_7\}P^{-1}$.

PROOF. Since $X(f)$ belongs to cases (v) or (vi) of Proposition 24, we assume $X(f)$ contains A or B. As has been noted before $X(f) \not\supset Y_6^{36}T$ for any integral unimodular T implies that the coefficient ε of $a(x)$ or $b(x)$ is not an integer. It may also again be assumed that $e_7 \in X(f)$ and that $N_{X(f)}(1,0) \geqq N_{X(f)}(0,1) \geqq N_{X(f)}(1,1)$. The coefficients $\varepsilon, \gamma_1, \cdots, \gamma_6$ of $a(x)$ or $b(x)$ are determined by seven vectors $w_1, \cdots, w_r$ with $[w_i]^2 = (0,1)$ and $z_1, \cdots, z_t$ with $[z_i]^2 = [z_j]^2 \in \{(1,1),(-1,1)\}$ for all $1 \leqq i < j \leqq t$. Here $t = 7 - r$ and $r + 1 \geqq t$. On writing $w_i = (w_{i1}, \cdots, w_{i7})$ and $z_i = (z_{i1}, \cdots, z_{i7})$ the equations which determine the coefficients are

$$\begin{cases} \sum_{i=1}^{6} \gamma_i w_{ji} w_{j7} = \text{integer} & \text{for } j = 1, \cdots r \\ \varepsilon z_{jk} z_{j6} + \sum_{i=1}^{6} \gamma_i z_{ji} z_{j7} = \text{integer} & \text{for } j = 1, \cdots t \end{cases}$$

where $k = 1$ or $k = 2$ according as $X(f)$ contains B or A. Since ε is not an integer the determinant D of these equations is not 0, 1 or -1. The sets $\{w_1, \cdots, w_r\}$ and $\{z_1, \cdots, z_t\}$ combine with respect to Y_5^{20}, so by Lemma 22, $r \leqq 4$. As $1 + r \geqq 7 - r$ we need consider only $r = 4$ and $r = 3$.

Let us first assume $r = 4$. On simplifying we have

$$\pm D = \begin{vmatrix} 0 & w_{11} w_{12} & \cdots & w_{15} & 0 \\ 0 & . & & . & 0 \\ 0 & . & & . & 0 \\ 0 & w_{41} & \cdots & w_{45} & 0 \\ z_{1k} & z_{11} & \cdots & z_{15} & 1 \\ z_{2k} & . & & . & 1 \\ z_{3k} & z_{31} & \cdots & z_{35} & 1 \end{vmatrix} .$$

$$= \begin{vmatrix} 0 & w_{11} & w_{12} & \cdots & w_{15} \\ 0 & . & . & . & . \\ 0 & . & . & . & . \\ 0 & . & . & . & w_{45} \\ y_{1k} & y_{11} & . & . & y_{15} \\ y_{2k} & y_{21} & . & . & y_{25} \end{vmatrix} \quad \begin{array}{ll} \text{where} & z_1 - z_3 = y_1 \in Y_5^{20} \\ \text{and} & z_1 - z_2 = y_2 \in Y_5^{20} \\ \text{and} & y_1 - y_2 \in Y_5^{20}. \end{array}$$

As $D \neq 0$, $(y_{1k}, y_{2k}) \neq (0,0)$. Define y_0 by $y_0 = {}^5[y_1]$ if $y_{1k} = 0$, $y_0 = {}^5[y_2]$ if $y_{2k} = 0$ and $y_0 = {}^5[y_1 - y_2]$ otherwise. Then, since $|z_i| \leqq 1$ for all $1 \leqq i \leqq 5$ and $(z_1, \cdots, z_5, 0, 0) \in Y_5^{20}$ we have $|D| = \mathrm{Det}({}^5[w_1], {}^5[w_2], {}^5[w_3], {}^5w[_4], y_0)$. But by Lemma 27, $|D| \in \{0, 1\}$ so there is no suitable minimal matrix for $r = 4$ and we assume the other possibility, $r = 3$. Treating D similarly we arrive at $|D| = \mathrm{Det}\{{}^5[w_1], {}^5[w_2], {}^5[w_3], y_0, y_1\}$ where $({}^5[w_i], 0, 0)$, $(y_i, 0, 0) \in Y_5^{20}$; $\{w_1, w_2, w_3\}$ is a combining set with respect to $Y_5{}^{20}$ and $(y_0, 0, 0)$, $(y_1, 0, 0) \in Y_5{}^{20}$. By Lemma 27, $D \notin \{0, 1, -1\}$ only if $\{{}^5[w_1], {}^5[w_2], {}^5[w_3]\}$ is equivalent under $\mathrm{Aut}(Y_5^{20})$ to $\{(1, -1, 0, 0, 0, 0, 0), (1, 0, -1, 0, 0, 0, 0), (1, 0, 0, 1, 1, 0, 0\}P^{-1}$. On examining such combining sets inequivalent on A and B, the proposition follows.

Proposition 29. *If $X(f)$ belongs to cases* (v) *or* (vi) *of Proposition* 24, *f is equivalent to the form* $\phi_6(x_1, \cdots, x_7) = \sum_{1 \leqq i \leqq j \leqq 7} x_i x_j - \frac{1}{2}(x_1 x_2 + x_1 x_4 + x_1 x_6 + x_2 x_5 + x_2 x_6 + 2x_3 x_4 + 2x_3 x_5 + 2x_3 x_6 + x_4 x_5 + x_6 x_7)$.

Proof. On calculating the coefficients for the arrays of Proposition 28 we have for A_1 and B_1 $\gamma_1 = 1 + \alpha + \beta$, $\gamma_2 = \gamma_3 = -1$, $\gamma_4 = \alpha$, $\gamma_5 = \beta$, $\gamma_6 = \gamma$ whilst for A_2 and B_2 $\gamma_1 = -1$, $\gamma_2 = \beta$, $\gamma_3 = \alpha$, $\gamma_4 = \alpha - 1$, $\gamma_5 = \alpha - 1$, $\gamma_6 = \gamma$. Consequently no minimal matrix containing A_2 which satisfies Proposition 21 can determine separately the parameters ε and β, so A_2 is not a suitable subset of $X(f)$. Once again if the minimum value of f is to be one and f belongs to category (iv) we require $0 < \varepsilon < 1$.

If $X(f) \supset B$, the determinant of the definining equations is

$$\pm D = \begin{vmatrix} z_{11}z_{16} & (z_{11}+z_{14})z_{17} & (z_{11}+z_{15})z_{17} & z_{16}z_{17} \\ z_{21}z_{26} & \cdot & \cdot & \cdot \\ z_{31}z_{36} & \cdot & \cdot & \cdot \\ z_{41}z_{46} & \cdot & \cdot & z_{46}z_{47} \end{vmatrix}$$

$$= \begin{vmatrix} y_{11} & y_{14} & y_{15} \\ y_{21} & y_{24} & y_{25} \\ y_{31} & y_{34} & y_{35} \end{vmatrix} \quad \text{for combining set } \{y_1, y_2, y_3\} \text{ of vectors in } Y_5^{20}.$$

But it is simple to check for such a set $\{y_1, y_2, y_3\}$, $D \in \{0, \pm 1\}$ so that ε is an integer. Consequently, B_1 is not a suitable subset of $X(f)$. Similarly for B_2 we require

$$\pm D = \begin{vmatrix} y_{11} & y_{12} & y_{13}+y_{14}+y_{15} \\ y_{21} & \cdot & \cdot \\ y_{31} & \cdot & y_{33}+y_{34}+y_{35} \end{vmatrix} \notin \{0, 1, -1\}$$

and again it is easy to check that this never happens for a combining set $\{y_1, y_2, y_3\}$.

With A_1 we require

$$\pm D = \begin{vmatrix} y_{12} & y_{11}+y_{14} & y_{11}+y_{15} \\ y_{22} & \cdot & \cdot \\ y_{32} & \cdot & y_{31}+y_{45} \end{vmatrix} \notin \{0, +1\}.$$

As $\{(w_2, w_1+w_4, w_1+w_5) \mid w \in \pm Y_5^{20}\} = \pm\{(0,1,1), (0,0,1), (0,1,0), (1,0,1), (1,1,0), (1,0,0), (1,-1,0), (1,0,-1), (0,1,-1)\}$ there are combining sets for which $|D| = 2$, although not for which $|D| > 2$. Since $0 < \varepsilon < 1$, we have in consequence, $\varepsilon = \frac{1}{2}$. Also if α or β is integral, the minimal matrix contains further minima with $x_6 = 0$, so we have $\alpha, \beta \in \{\pm\frac{1}{2}\}$. Since $|1+\alpha+\beta| < 1$ (otherwise we have another unsuitable minimal vector), we therefore have $\alpha = \beta = -\frac{1}{2}$. Then since $f(0,0,1,0,0,-1,1) = 1-\gamma \geqq 1$ we have $\gamma \leqq 0$, hence $\gamma = 0$, $\gamma = -\frac{1}{2}$ or $\gamma = -1$. But the form with $\gamma = -\frac{1}{2}$ is not perfect whilst if $\gamma = 0$ or $\gamma = -1$ we obtain perfect forms as given in the proposition.

Thus, the above lemmas, and propositions have proved that there is only one class of perfect forms belonging to category (iv), namely those equivalent to $\phi_6(x_1, \cdots, x_7)$ which has thirty-six minimal vectors and determinants $4/2^7$.

5. Category (v) of Theorem 11

In this section we search for all perfect forms f such that $\Delta_4(f) = 2$ and $X(f)$ contains no set of twenty minimal vectors in a 5-space. By Proposition 10, it can be assumed that $X(f) \supset Y_5^{16}$ or, if $N(z) \leqq 3$ for all nonzero 3-vectors z, that $X(f) \supset Y_5^{15}$. Further, in the proof of Proposition 10, it is shown that if $v, w \in X(f) \backslash Y_4^{12}$ and $[v]^3 \equiv [w]^3$ (modulo 2) then v and w combine with respect to Y_4^{12}.

LEMMA 30. *If* $X(f) \supset Y_5^r$ *where* $r = 15$ *or* 16 *and* $v, w \in X(f) \backslash Y_5^r$ *then* $[v]^2$ *is primitive (or maybe zero if* $r = 15$*) and* $[v]^2 \equiv [w]^2$ (modulo 2) *implies that* $[v]^2 = \pm [w]^2$.

PROOF. If $[v]^2 \neq \mathbf{0}$ and k divides $[v]^2$ then $k \cdot \Delta_5(Y_5^r) = 2k \leqq \Delta_6(f) \leqq 3$ so $k = 1$ and $[v]^2$ is primitive. Now assume that $[v]^2 \equiv [w]^2 \not\equiv 0$ (modulo 2) and $[v]^2 \neq \pm [w]^2$. $X(f)$ may then be transformed by an integral unimodular transformation T so that $X(f)T$ contains $\{Y_5^r, wT = e_6, vT\}$ where $[vT]^2 = (1,2)$. Also if $v_0 T \equiv vT$(modulo 2) and $[v_0 T]^2 = \pm(1,2)$ then $\{Y_5^r, e_6, v_0 T\}$ is equivalent to $\{Y_5^r, e_6, vT\}$. Using this transformation and $\mathrm{Aut}(Y_5^r)$ it can be readily seen that if $r = 16$, we must consider $vT \in \{(1,0,0,0,1,1,2), (0,1,0,0,1,1,2), (0,0,0,1,1,1,2)\}$ and if $r = 15$, $vT \in \{(1,0,0,0,1,1,2), (0,1,0,0,1,1,2), (1,1,0,1,1,1,2), (0,0,0,1,1,1,2)\}$. As the case of $r = 15$ is similar, we consider only $r = 16$. If $vT = (1,0,0,0,1,1,2)$ then clearly $\Delta_3(X(f)) = 2$. If $vT = (0,1,0,0,1,1,2)$ then, by Corollary 3 another sixteen minimal vectors belong to $X(fT)$. After calculating the coefficients defined by this array, it can be seen that there are even more minimal vectors and twenty vectors belong to the 5-space $x_6 = x_7 = 0$. If $vT = (0,0,0,1,1,1,2)$, by Corollary 3 twelve more vectors belong to $X(fT)$ and there are twenty vectors in the 5-space $x_2 = x_3 = 0$. Hence if $r = 16$, $[v]^2 \equiv [w]^2$ (modulo 2) implies that $[v]^2 = \pm [w]^2$. The proof for $r = 15$ is done by a similar examination of cases.

The procedure for completing Y_5^r ($r = 15, 16$) to a minimal matrix is now established. Since equations from Y_5^r determine $(r-2)$ coefficients, there exists an integral 2-vector $z \not\equiv 0$ (mod 2) such that $Y_5^r \cup \{v \mid v \in X(f), [v]^2 \equiv z \text{ (mod 2)}\}$ determines at least $(r+3)$ coefficients. On transforming one such vector v to e_6, it can be shown that $X(f)$ contains one of the fourteen inequivalent double subscripted 6-arrays listed below. Define

$$A_2 = \{Y_5^{15}, e_6, e_1 + e_6, \ e_1 + e_2 + e_3 + e_4 + e_6\}$$

$$A_3 = \{Y_5^{15}, \ e_6, \ e_2 + e_6, \ -e_3 - e_4 + e_6\}$$

$$D_2 = \{Y_5^{16}, e_6, e_1 + e_6, \ - e_2 - e_4 + e_6\}$$

$$C_1 = \{Y_5^{16}, e_6, \ e_1 + e_6, e_1 + e_4 + e_6, \ -e_2 - e_3 - e_4 + e_6\}$$

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$$C_2 = \{Y_5^{16}, e_6, e_1 + e_4 + e_6, e_1 + e_6, e_1 + e_3 + e_4 + e_6\}$$

$$C_3 = \{Y_5^{16}, e_6, e_1 + e_6, -e_2 - e_4 + e_6, e_1 + e_3 + e_4 + e_6\}.$$

Let $w = -e_5 + e_6$ and $u = e_5 + e_6$. The fourteen inequivalent 6-arrays are:

$$\begin{aligned}
A_{21} &= \{A_2, w, e_2 + w\} \\
A_{22} &= \{A_2, e_1 + u, 2e_1 + e_2 + e_3 + 2e_4 + u\} \\
A_{31} &= \{A_3, e_1 + u, e_1 + e_2 + e_4 + u, -e_3 - e_4 + u\} \\
A_{32} &= \{A_3, e_1 + u, e_1 + e_2 + e_4 + u\} \\
D_{22} &= \{D_2, w, e_3 + w\} \\
C_{31} &= \{C_3, w, e_3 + w\} \\
C_{32} &= \{C_3, w\} \\
C_{33} &= \{C_3, e_3 + w\} \\
C_{11} &= \{C_1, w\} \\
C_{12} &= \{C_1, e_1 - e_2 + w\} \\
C_{21} &= \{C_2, w\} \\
C_{22} &= \{C_2, e_1 - e_2 + u\} \\
C_{23} &= \{C_1, e_1 + u, 2e_1 + e_4 + u, e_1 + e_4 + u\} \\
C_{24} &= \{C_{23}, e_1 - e_2 + u\}.
\end{aligned}$$

PROPOSITION 31. *There are six equivalence classes of perfect forms with* $\lambda_4 = 2$ *which have no set of twenty minimal vectors in a 5-space. They are represented by*:

$$\phi_{19}(x) = x \begin{bmatrix} 4 & 0 & 0 & -2 & -2 & -2 & -2 \\ 0 & 4 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 4 & -2 & 1 & 0 & 0 \\ -2 & -2 & -2 & 4 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 & 4 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 4 & 2 \\ -2 & 1 & 0 & 1 & 0 & 2 & 4 \end{bmatrix} x^{tr}$$

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$$\phi_9(x) = x \begin{bmatrix} 4 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 4 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 4 & 1 & 2 \\ 2 & 2 & 1 & 2 & 1 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{bmatrix} x^{tr},$$

$$K_5(x) = x \begin{bmatrix} 6 & 3 & 3 & 3 & -3 & -2 & -3 \\ 3 & 6 & 3 & 3 & -3 & 0 & -2 \\ 3 & 3 & 6 & 2 & -3 & -3 & -3 \\ 3 & 3 & 2 & 6 & 0 & 0 & 0 \\ -3 & -3 & -3 & 0 & 6 & 1 & 2 \\ -2 & 0 & -3 & 0 & 1 & 6 & 1 \\ -3 & -2 & -3 & 0 & 2 & 1 & 6 \end{bmatrix} x^{tr},$$

$$K_6(x) = x \begin{bmatrix} 6 & 0 & 0 & -3 & -3 & -1 & -1 \\ 0 & 6 & 0 & -3 & 1 & -3 & -3 \\ 0 & 0 & 6 & -3 & 1 & 1 & 1 \\ -3 & -3 & -3 & 6 & 0 & 2 & 2 \\ -3 & 1 & 1 & 0 & 6 & -2 & -2 \\ -1 & -3 & 1 & 2 & -2 & 6 & 2 \\ -1 & -3 & 1 & 2 & -2 & 2 & 6 \end{bmatrix} x^{tr},$$

$$K_7(x) = x \begin{bmatrix} 4 & 0 & 0 & -2 & 0 & 0 & -2 \\ 0 & 4 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 4 & -2 & 0 & -2 & 1 \\ -2 & -2 & -2 & 4 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 4 & 0 & -2 \\ 0 & 0 & -2 & 1 & 0 & 4 & -2 \\ 2 & 1 & 1 & 1 & -2 & -2 & 4 \end{bmatrix} x^{tr},$$

$$K_8(x) = x \begin{bmatrix} 4 & 0 & 0 & -2 & -2 & -2 & 1 \\ 0 & 4 & 0 & -2 & 1 & 1 & -2 \\ 0 & 0 & 4 & -2 & 1 & -1 & 0 \\ -2 & -2 & -2 & 4 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 & 4 & 2 & 0 \\ -2 & 1 & -1 & 1 & 2 & 4 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 & 4 \end{bmatrix} x^{tr},$$

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The first two forms above are "well-known" and have thirty-four and thirty-two pairs of minimal vectors respectively. K_5, K_6 and K_8 each have thirty minimal vectors and K_7 has thirty two. These perfect forms were announced for the first time in Stacey (1975).

PROOF. Unfortunately the calculations involved in extending all the 6-arrays are long and a computer is required. The computation required approximately fifteen minutes on the Oxford Computing Centre's 1906A after considerable experimentation to find a workable method.

The success of the computation depends largely on having an efficient algorithm for implementing Corollary 3. For the computation the notion of compatible fitting vectors is extended so that if A is one of the 6-arrays listed above v, $w \in F(\{A, e_7\})$, v and w are compatible if and only if

(i) $v - w + e_7 \in F(\{A, e_7\})$,

(ii) $[v]^3 \equiv [w]^3$ (modulo 2) implies v and w combine with respect to $Y_4{}^{12}$,

(iii) whenever $\mathrm{Det}\{w_1, w_2, w_3, w_4\} = 2$ for $w_i \in \{A, v, w\}$ implies, by Corollary 3, the existence of v_0 with $[v_0]^1 = \pm 1$ in $X(f)$, then $v_0 \in F\{(A, e_7\})$ and he pairs $\{v, v_0\}$ and $\{w, v_0\}$ satisfy (i) and (ii) above.

The 6-arrays are extended in the following way. If $v \in X(f) \backslash A$ then $[v]^1 \,|\, .\Delta_5(Y_5') = 2 \,|\, [v]^1 \,| \leqq \Delta_6(f) < 4$, so that $[v]^1 = \pm 1$. Choose a $v \in X(f) \backslash A$ uch that $N([v]^2) \geqq N(0,1), N(1,1)$. There exists an integral unimodular transformation T such that $vT = e_7$, $AT = A$ and $X(f)T$ has the property that $N(1,0) \geqq N(0,1) \geqq N(1,1)$. The usual counting argument on the number of coefficients to be determined gives a bound on $N(0,1)$ and all suitably sized compatible subsets of fitting vectors with 2-ending $(0,1)$ can then be found. There are about 400 of these. The computer programme made elementary use of the equivalences of Lemma 6 to eliminate redundant arrays and always made full use of Corollary 3. The arrays can then be completed by searching for compatible subsets of $F(\{A, e_7\})$ with 2-ending congruent to (1,1) modulo 2. If a sufficiently large array is produced, it is tested to see if it defines a perfect form. The perfect forms are then divided into equivalence classes by hand.

PROPOSTION 32. *$K_8(x)$ is not equivalent to either of the perfect forms L_7^2 of Barnes* (1959) *or R_7* (5,2) *of Scott* (1964) *although all three have thirty minimal vectors and, when the minimum value is one, determinant* $3^4 \cdot 2^{-11}$.

PROOF. $\Delta_4(L_7^2) = \Delta_4(R_7(5,2)) = 1$ but $\Delta_4(K_8) = 2$.

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