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MONOCOREFLECTIVE SUBCATEGORIES IN GENERAL TOPOLOGY

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1. Introduction. Let $\mathscr{P}$ be a full subcategory of a category $\mathscr{C}$. $\mathscr{P}$ is said to be *coreflective* in $\mathscr{C}$ if for each object X in $\mathscr{C}$ there exists an object $\mathscr{P}X$ in $\mathscr{P}$ and a morphism $\sigma_{\mathscr{P}X} : \mathscr{P}X \rightarrow X$ such that for each object P in $\mathscr{P}$ and each morphism $f : P \rightarrow X$ there exists a unique morphism $g : P \rightarrow \mathscr{P}X$ such that $f = \sigma_{\mathscr{P}X} \circ g$. The morphism $\sigma_{\mathscr{P}X}$ is called the *coreflection morphism* from $\mathscr{P}X$ to X, and $\mathscr{P}X$ is called the *coreflection* of X (in $\mathscr{P}$). If each coreflection morphism is a monomorphism then $\mathscr{P}$ is said to be a *monocoreflective subcategory* of $\mathscr{C}$. We shall denote this by writing $\mathscr{P} < \mathscr{C}$. In this paper we study monocoreflective subcategories of the category $\mathscr{T}$ of topological spaces and the category $\mathscr{H}$ of Hausdorff spaces. Much is already known about such subcategories; see for instance the papers of Kennison [**5**] and Herrlich and Strecker [**4**]. Chapter 10 of Walker [**9**], especially Problems 10B and 10C, provides a succinct summary of the elementary properties of monocoreflective subcategories of $\mathscr{T}$. In contrast to this earlier work, however, our chief interest will be in the interaction of pairs of monocoreflective subcategories of $\mathscr{T}$ and of $\mathscr{H}$. Thus many of our results will be dual-like analogues of theorems appearing in [**10**], and for this reason some familiarity with the contents of [**10**] will be helpful (though not essential) to the reader.

The basic themes of this paper are as follows. First, we show that discrete spaces play a role in monocoreflective subcategories of $\mathscr{T}$ and $\mathscr{H}$ that is analogous to the role played by compact Hausdorff spaces in epireflective subcategories of the category of Tychonoff spaces and the category of zero-dimensional Hausdorff spaces. Second, we introduce the concept of *$\mathscr{P}$-pseudo-discreteness* that is analogous to the concept of $\mathscr{P}$-pseudocompactness defined in [**10**], and use it to analyze the relation between pairs of monocoreflective subcategories of $\mathscr{T}$ and of $\mathscr{H}$.

It should be noted that out "dual-like analogues" of theorems in [**10**] are not categorical duals in the strict technical sense described, for example, on page 31 of [**6**].

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We shall henceforth assume that all hypothesized subcategories of $\mathscr{T}$ are full and replete; thus if $\mathscr{C}$ is a subcategory of $\mathscr{T}$, the objects of $\mathscr{C}$ will be the class of all topological spaces possessing some given topological property and the morphisms of $\mathscr{C}$ will be the continuous functions between these objects. Hence a subcategory $\mathscr{C}$ of $\mathscr{T}$ will be specified by describing what class of topological spaces comprises its objects. A *map* will be a continuous function. If we wish to specify explicitly the topology τ of an object of $\mathscr{T}$, we shall write that object as (X, τ). A discussion of the categorical concepts used in this paper may be found in [**4**], [**6**], and in Chapter 10 of [**9**]. We make no assumptions that our topological spaces satisfy any separation axioms unless those axioms are explicitly stated.

We denote the category of zero-dimensional Hausdorff spaces by $\mathscr{H}_0$ (a space is *zero-dimensional* if its open-and-closed (clopen) subsets form a base for the open sets of the space). The category of Tychonoff (i.e. completely regular Hausdorff) spaces is denoted by $\mathscr{T}$ych. If $\mathscr{A}$ and $\mathscr{B}$ are topological properties we shall use the notation $\mathscr{A} \subseteq \mathscr{B}$, $\mathscr{A} \cap \mathscr{B}$, and $X \in \mathscr{A}$ to mean respectively: each space in $\mathscr{A}$ is in $\mathscr{B}$, the class of all spaces in both $\mathscr{A}$ and $\mathscr{B}$, and X is a space in the class $\mathscr{A}$. If $\mathscr{A}$ is a topological property, we denote by $\mathscr{A}_0$ the class of zero-dimensional spaces with $\mathscr{A}$.

In any topological category containing the space with one point a map is a monomorphism if and only if it is one-to-one; see Proposition 10.17 of [**9**]. In $\mathscr{T}$ a map is an epimorphism if and only if it is onto; however, in $\mathscr{H}$ a map is an epimorphism if and only if the image of the domain is a dense subspace of the range. An examination of the proof of this proposition (see page 255 of [**9**], for example) reveals that the following proposition is in fact proved:

1.1. PROPOSITION. *Let $\mathscr{C}$ be a subcategory of $\mathscr{H}$ with the following properties:*
(i) *If Y_1 and Y_2 are objects of $\mathscr{C}$, so is their free union $Y_1 \cup Y_2$.*
(ii) *If Y is an object of $\mathscr{C}$ and there is a closed finite-to-one mapping from Y onto a (necessarily Hausdorff) space Z, then Z is an object of $\mathscr{C}$.*
Then epimorphisms in $\mathscr{C}$ are mappings onto dense subspaces of the range.

It is straightforward to verify that the categories $\mathscr{H}_0$ and $\mathscr{T}$ych satisfy the hypotheses on $\mathscr{C}$ in 1.1.

To motivate the investigation we shall undertake, we give a brief discussion of the concept of an epireflective subcategory of a given category. A full subcategory $\mathscr{A}$ of a category $\mathscr{C}$ is said to be *reflective* in $\mathscr{C}$ if for each object X in $\mathscr{C}$ there exists an object $\mathscr{A}X$ in $\mathscr{A}$ and a morphism $e_{\mathscr{A}X} : X \rightarrow \mathscr{A}X$ such that for each object A of $\mathscr{A}$ and each morphism $f : X \rightarrow A$, there exists a unique morphism $f' : \mathscr{A}X \rightarrow A$ such that $f = f' \circ e_{\mathscr{A}X}$. This notion is the categorical dual to the concept of coreflection. If $e_{\mathscr{A}X}$ is an epimorphism for each object X of C then $\mathscr{A}$ is said to be an *epireflective subcategory* of $\mathscr{C}$. Epireflective subcategories of certain topological categories have been intensively studied. For example the category of compact Hausdorff spaces is an epireflective subcategory of the category of Tychonoff spaces; the epireflection morphism embeds each Tychonoff space X in its Stone-Čech compactification βX. Realcompact spaces, via the Hewitt realcompactification, form another epireflective subcategory of Tychonoff spaces. Many other examples are discussed in [**10**].

In [**10**] we studied epireflective subcategories $\mathscr{P}$ of topological categories $\mathscr{C}$ subject to the following conditions: (1) $\mathscr{C}$ is closed under the formation of product spaces and subspaces. (2) The epireflection $\mathscr{C}_{\mathscr{P}X} : X \rightarrow \mathscr{P}X$ embeds X as a dense subspace of $\mathscr{P}X$. (3) Each compact Hausdorff object in $\mathscr{C}$ is in $\mathscr{P}$ and each object in $\mathscr{C}$ has a Hausdorff compactification in $\mathscr{C}$. These conditions imply that each object of $\mathscr{C}$ be Tychonoff; in practice $\mathscr{C}$ was either $\mathscr{T}$ych of $\mathscr{H}_0$. We called an object X of $\mathscr{C}$ *$\mathscr{P}$-pseudocompact* if its $\mathscr{P}$-epireflection $\mathscr{P}X$ were compact. We systematically studied the relationship between two such epireflective subcategories of $\mathscr{C}$ by using the concept of $\mathscr{P}$-pseudocompactness. In this paper we investigate the analogous concept for monocoreflective subcategories. We are able to obtain dual-like analogues of many of the theorems in [**10**] although in some cases we pay a penalty for considering monocoreflective subcategories of $\mathscr{T}$ or $\mathscr{H}$ rather than just of $\mathscr{T}$ych or $\mathscr{H}_0$; namely our theorems are sometimes not as strong as their epireflective analogues.

We now discuss some well-known theorems and examples concerning monocoreflective subcategories. According to a theorem of Kennison, quoted as Theorem 4 of [**4**], all coreflective subcategories of $\mathscr{T}$ and of $\mathscr{H}$ are monocoreflective, so no greater generality is obtained by studying coreflective subcategories of $\mathscr{T}$ or $\mathscr{H}$.

The following is Theorem 12 of [**4**]; part of it also appears in [**5**].

1.2. THEOREM. *Let $\mathscr{P}$ be a topological property. Then $\mathscr{P}$ (respectively $\mathscr{P} \cap \mathscr{H}$) is a monocoreflective subcategory of $\mathscr{T}$ (respectively $\mathscr{H}$) if and only if $\mathscr{P}$ is closed under the formation of free unions and quotient images (respectively, Hausdorff quotient images).*

It follows that if $\mathscr{P}$ is *any* topological property, then the class $\mathscr{M}(\mathscr{P})$ of spaces that are quotient images of free unions of members of $\mathscr{P}$ (respectively $\mathscr{P} \cap \mathscr{H}$) forms a monocoreflective subcategory of $\mathscr{T}$ (respectively $\mathscr{H}$), the smallest monocoreflective subcategory containing $\mathscr{P}$. In special cases the monocoreflection in $\mathscr{M}(\mathscr{P})$ of an object X in $\mathscr{T}$ or $\mathscr{H}$ can be described explicitly. The following is Theorem 15 of [**4**].

1.3. THEOREM. *Let $\mathscr{P}$ be a topological property such that continuous images of spaces with $\mathscr{P}$ have $\mathscr{P}$. Then the monocoreflection in $\mathscr{M}(\mathscr{P})$ of a space (X, τ) is the space (X, τ'), where $V \in \tau'$ if and only if $V \cap P$ is open in P for each subspace P of (X, τ) such that P has $\mathscr{P}$. The identity map on the set X is the coreflection morphism from (X, τ') to (X, τ).*

As examples (mentioned in [**4**]), if $\mathscr{P}$ is the class of compact spaces, $\mathscr{M}(\mathscr{P})$ is the class of k-spaces; if $\mathscr{P}$ contains one space, namely the one-point compactification $\bar{N}^*$ of the countable discrete space $\bar{N}$, then $\mathscr{M}(\mathscr{P})$ is the class of

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sequential spaces (see 10B of [**9**]). The class of sequential spaces is also identical with $\mathscr{M}(\{\bar{2}^\omega\})$, where $\bar{2}^\omega$ is the Cantor space.

Another important example of a monocoreflective subcategory of $\mathscr{T}$ (respectively $\mathscr{H}$) is the category of (Hausdorff) P-spaces. Recall that a space (X, τ) is a *P-space* if its G_δ-sets are open (note that no separation axioms are assumed here). If $\mathscr{P}$ denotes the category of $\mathscr{P}$-spaces then $\mathscr{P} < \mathscr{T}$; $\mathscr{P}(X, \tau)$ is the topological space whose underlying set is X and whose topology τ' is the topology for which the G_δ-sets of (X, τ) form an open base. See [**3**] or [**9**] for a discussion of P-spaces. Similarly $\mathscr{P} \cap \mathscr{H} < \mathscr{H}$.

2. $\mathscr{P}$-pseudodiscrete spaces. As mentioned in Section 1 $\mathscr{P}$-pseudocompact spaces play an important role in studying the interrelation between pairs of epireflective subcategories of $\mathscr{H}_0$ or $\mathscr{T}$ych. To find a concept for monocoreflective subcategories of $\mathscr{T}$ or $\mathscr{H}$ that is analogous to $\mathscr{P}$-pseudocompactness in the category of (zero-dimensional) Tychonoff spaces, we first must find an analogue for the concept of compactness. To help us do this, we first give a category-theoretic characterization of the compact objects of the category of (zero-dimensional) Tychonoff spaces. Recall that in topological categories, "isomorphism" means "homeomorphism".

2.1. LEMMA. *Let $\mathscr{C}$ be either $\mathscr{T}$ych or $\mathscr{H}_0$. The following conditions on an object X of $\mathscr{C}$ are equivalent.*

(i) *X is compact.*

(ii) *If Y is an object of $\mathscr{C}$ and if $f : X \to Y$ is both a monomorphism and an epimorphism, then f is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii): By assumption $f[X]$ is a compact dense subspace of Y; thus as Y is Hausdorff, $f[X] = Y$. As X is compact, f is closed; as f is also one-to-one and continuous, f is a homeomorphism from X onto Y.

(ii) $\Rightarrow$ (i): If X were not compact, the embedding of X in its Stone-Čech compactification (or its maximal zero-dimensional compactification if $\mathscr{C} = \mathscr{H}_0$; see [**10**]) is both a monomorphism and an epimorphism but not an isomorphism.

The "dual-like analogue" to 2.1, where $\mathscr{C}$ is now either $\mathscr{H}$ or $\mathscr{T}$, is given in 2.2 below. Note that statement 2.2 (ii) would be the categorical dual of 2.1 (ii) if $\mathscr{C}$ were the same category in 2.1 and 2.2. This is part of our justification for claiming that discrete spaces play a dual-like role in $\mathscr{H}$ or $\mathscr{T}$ to that which compact spaces play in $\mathscr{H}_0$ or $\mathscr{T}$ych.

2.2. LEMMA. *Let $\mathscr{C}$ be either $\mathscr{H}$ or $\mathscr{T}$. The following conditions on an object X of $\mathscr{C}$ are equivalent.*

(i) *X is discrete.*

(ii) *If Y is an object of $\mathscr{C}$ and if $f : Y \to X$ is both a monomorphism and an epimorphism, then f is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii). By assumption f is a one-to-one continuous function from

Y onto the discrete space X. It follows immediately that Y is discrete and that f is a homeomorphism.

(ii) $\Rightarrow$ (i). Suppose X were not discrete. Let Y be the discrete space of the same cardinality as X, and let f be any one-to-one function from Y onto X. Then f is continuous, a monomorphism and an epimorphism, but is not a homeomorphism.

Motivated by this analogy between compactness and discreteness, and by the definition of $\mathscr{P}$-pseudocompactness appearing in [**10**], we introduce the concept of $\mathscr{P}$-pseudodiscreteness as follows. Recall that $\mathscr{P} < \mathscr{C}$ means $\mathscr{P}$ is a monocoreflective subcategory of $\mathscr{C}$.

2.3. *Definition*. Let $\mathscr{C}$ be $\mathscr{T}$ or $\mathscr{H}$ and let $\mathscr{P} < \mathscr{C}$. An object X of $\mathscr{C}$ is *$\mathscr{P}$-pseudodiscrete* if $\mathscr{P}X$ is discrete. The class of $\mathscr{P}$-pseudodiscrete spaces will be denoted by $\mathscr{P}^*$.

We now develop the properties of $\mathscr{P}$-pseudodiscreteness. We begin with a preliminary lemma.

2.4. LEMMA. *Let $\mathscr{C}$ be $\mathscr{T}$ or $\mathscr{H}$, let $\mathscr{P} < \mathscr{C}$, and let X be an object* ${}_cf$ *$\mathscr{C}$. If A is a clopen subset of X then $\sigma_{\mathscr{P}X}{}^{\leftarrow}[A] = \mathscr{P}A$ and $\sigma_{\mathscr{P}X}|\sigma_{\mathscr{P}X}{}^{\leftarrow}[A] = \sigma_{\mathscr{P}A}$.*

Proof. By the corollary of Proposition 3 of [**4**] $\sigma_{\mathscr{P}X}{}^{\leftarrow}[A] \in \mathscr{P}$ and $\sigma_{\mathscr{P}X}|\sigma_{\mathscr{P}X}{}^{\leftarrow}[A]$ is a monomorphism from $\sigma_{\mathscr{P}X}{}^{\leftarrow}[A]$ onto A. Let $Y \in \mathscr{P}$ and let $f : Y \rightarrow A$ be continuous. Let i_A be the embedding of A in X. Then $i_A \circ f : Y \rightarrow X$ so there exists a unique map $g : Y \rightarrow \mathscr{P}X$ such that $\sigma_{\mathscr{P}X} \circ g = i_A \circ f$. If $y \in Y$ then $\sigma_{\mathscr{P}X}(g(y)) = f(y) \in A$ so g maps Y into $\sigma_{\mathscr{P}X}{}^{\leftarrow}[A]$. The lemma follows.

The elementary properties of the class $\mathscr{P}^*$ are summarized in the following lemma.

2.5. LEMMA. *Let $\mathscr{P} < \mathscr{T}$. Then*

(1) *If $Y \in \mathscr{P}^*$ and $f : X \rightarrow Y$ is a monomorphism then $X \in \mathscr{P}^*$.*

(2) *If $\mathscr{P} < \mathscr{T}$ and $\mathscr{Q} < \mathscr{T}$ and $\mathscr{P} \subseteq \mathscr{Q}$ then $\mathscr{Q}^* \subseteq \mathscr{P}^*$.*

(3) *The product of finitely many members of $\mathscr{P}^*$ is in $\mathscr{P}^*$.*

(4) *The free union of arbitrarily many members of $\mathscr{P}^*$ is in $\mathscr{P}^*$.*

Proof. (1) The map $f \circ \sigma_{\mathscr{P}X}$ is a monomorphism from $\mathscr{P}X$ to Y. By definition of $\mathscr{P}Y$ there is a unique map $g : \mathscr{P}X \rightarrow \mathscr{P}Y$ such that $\sigma_{\mathscr{P}Y} \circ g = f \circ \sigma_{\mathscr{P}X}$. As $f \circ \sigma_{\mathscr{P}X}$ is a monomorphism so is g. As $\mathscr{P}Y$ is discrete this implies that $\mathscr{P}X$ is discrete.

(2) If $X \in \mathscr{Q}^*$ then $\mathscr{Q}X$ is discrete. As $\mathscr{P}X \in \mathscr{Q}$ there is a unique map $f : \mathscr{P}X \rightarrow \mathscr{Q}X$ such that $\sigma_{\mathscr{P}X} = \sigma_{\mathscr{Q}X} \circ f$. Thus f is a monomorphism as $\sigma_{\mathscr{P}X}$ is; so $\mathscr{P}X$ is discrete, i.e. $X \in \mathscr{P}^*$.

(3) Let $X_1, \ldots, X_n \in \mathscr{P}^*$, put $X = \prod_{j=1}^n X_j$, and let $p_j : X \rightarrow X_j$ be the jth projection map. Then $p_j \circ \sigma_{\mathscr{P}X}$ maps $\mathscr{P}X$ to X_j so there exists a unique map $k_j : \mathscr{P}X \rightarrow \mathscr{P}X_j$ such that $\sigma_{\mathscr{P}X_j} \circ k_j = p_j \circ \sigma_{\mathscr{P}X} (j = 1 \text{ to } n)$. Define $k : \mathscr{P}X \rightarrow \prod_{j=1}^n \mathscr{P}X_j$ by $q_j \circ k = k_j$, where $q_j : \prod_{i=1}^n \mathscr{P}X_i \rightarrow \mathscr{P}X_j$ is the jth

projection map. Then k is continuous. Also k is one-to-one for if $k(x) = k(y)$ then $k_j(x) = k_j(y)$ for $j = 1$ to n; thus $p_j \circ \sigma_{\mathscr{P}X}(x) = p_j \circ \sigma_{\mathscr{P}X}(y)$ for $j = 1$ to n, so $\sigma_{\mathscr{P}X}(x) = \sigma_{\mathscr{P}X}(y)$. As $\sigma_{\mathscr{P}X}$ is one-to-one, $x = y$ so k is one-to-one. By hypothesis $\prod_{j=1}^{n} \mathscr{P}X_j$ is discrete so $\mathscr{P}X$ is discrete, i.e. $X \in \mathscr{P}^*$.

(4) Let $X = \bigcup_{\alpha \in \Sigma} X_\alpha$ where $X_\alpha \in \mathscr{P}^*$. By Lemma 2.4 $\sigma_{\mathscr{P}X}^{\leftarrow}[X_\alpha] = \mathscr{P}X_\alpha$ for each $\alpha \in \Sigma$. Thus $\mathscr{P}X$ is a free union of discrete spaces, and so $X \in \mathscr{P}^*$.

It is worth noting that while the analogues of (1), (2) and (3) of 2.5–respectively 2.2 (a) and (c), 2.2 (e), and 2.2 (b) of [**10**]–are true for $\mathscr{P}$-pseudocompact spaces, the analogue of 2.5 (4) is not true, for if $\mathscr{P}$ is realcompactness, the product of just two $\mathscr{P}$-pseudocompact (i.e. pseudocompact) spaces need not be pseudocompact (see 9.15 of [**3**]).

We can now prove the first of our main theorems.

2.6. THEOREM. *Let $\mathscr{P} < \mathscr{T}$ and $\mathscr{Q} < \mathscr{T}$. If $X \in \mathscr{T}$ put*

$$S_{\mathscr{P}}X = \{p \in X : \sigma_{\mathscr{P}X}^{\leftarrow}(p) \text{ is not isolated in } \mathscr{P}X\}.$$

The following are equivalent.

(a) $\mathscr{P}^* \cap \mathscr{T}_0 = \mathscr{Q}^* \cap \mathscr{T}_0$.

(b) *For each $X \in \mathscr{T}_0$ each of $S_{\mathscr{P}}X$ and $S_{\mathscr{Q}}X$ is dense in $S_{\mathscr{P}}X \cup S_{\mathscr{Q}}X$.*

Proof. (a) $\Rightarrow$ (b): Let $X \in \mathscr{T}_0$, let $p \in S_{\mathscr{P}}X$, and let A be a clopen set of X with $p \in A$. As $p \in S_{\mathscr{P}}X$, $\sigma_{\mathscr{P}X}^{\leftarrow}[A]$ is not discrete, so by 2.4 $A \notin \mathscr{P}^*$. As $A \in \mathscr{T}_0$ it follows that $A \notin \mathscr{Q}^*$. Thus by 2.4 $\sigma_{\mathscr{Q}X}^{\leftarrow}[A]$ is not discrete, so $A \cap S_{\mathscr{Q}}X \neq \emptyset$.

(b) $\Rightarrow$ (a). Suppose (a) fails and $X \in (\mathscr{P}^* \cap \mathscr{T}_0) - (\mathscr{Q}^* \cap \mathscr{T}_0)$. Thus $\mathscr{P}X$ is discrete and $\mathscr{Q}X$ is not discrete. In other words $S_{\mathscr{P}}X = \emptyset$ and $S_{\mathscr{Q}}X \neq \emptyset$, so (b) fails.

The above theorem is analogous to Theorem 2.6 of [**9**]. To prove that there exist distinct monocoreflective subcategories $\mathscr{P}$ and $\mathscr{Q}$ of $\mathscr{T}$ such that $\mathscr{P}^* = \mathscr{Q}^*$, we next prove an analogue of Theorem 3.9 of [**9**] (Theorem 2.10 below). Roughly speaking this will show that given a monocoreflective subcategory $\mathscr{P}$ of $\mathscr{T}$, there exists a "largest" monocoreflective subcategory $\mathscr{Q}$ of $\mathscr{T}$ such that $\mathscr{P}^* = \mathscr{Q}^*$. Some preliminaries are necessary. Theorem 2.7 below is analogous to (a generalization of) Theorem 3.6 of [**9**].

2.7 THEOREM. *Let $\mathscr{A}$ be any topological property. Let $\hat{\mathscr{A}}$ be the class of (Hausdorff) topological spaces X satisfying the following condition: if $f : X \to Y$ is continuous and $Y \in \mathscr{A}$ (respectively $Y \in \mathscr{A} \cap \mathscr{H}$) then $f[X]$, equipped with the quotient topology induced on it by f, is discrete. Then $\hat{\mathscr{A}} < \mathscr{T}$ (respectively $\hat{\mathscr{A}} < \mathscr{H}$).*

Proof. By 1.2 it suffices to show that $\hat{\mathscr{A}}$ is closed under the formation of free unions and quotient images.

Let $(X_\alpha)_{\alpha \in \Sigma}$ be a set of spaces in $\hat{\mathscr{A}}$ and let X be their free union. Let $k_\alpha : X_\alpha \to X$ be the canonical embedding of X_α into X. Let $f : X \to Y$ be continuous,

$Y \in \mathscr{A}$, and let Z denote the space $f[X]$ with the quotient topology induced on it by f. Thus f can be factored as $j \circ q$ where q is a quotient map from X onto Z and j is a one-to-one onto map from Z onto $f[X]$. Similarly for each $\alpha \in \Sigma$ let Z_α denote the space $f \circ k_\alpha[X_\alpha]$ with the quotient topology induced on it by $f \circ k_\alpha$. Then $f \circ k_\alpha$ can be written as $j_\alpha \circ q_\alpha$ where q_α is $f \circ k_\alpha$ regarded as a quotient map from X_α onto Z_α and j_α is a one-to-one map from Z_α into $f[X]$ (see Figure 1).

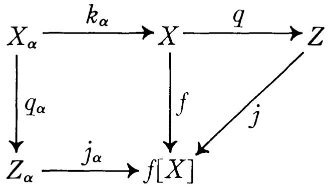

FIGURE 1

Let $p \in Z$. Let $\Lambda = \{\alpha \in \Sigma : j(p) \in f \circ k_\alpha[X_\alpha]\}$; note that $\Lambda \neq \emptyset$. If $\alpha \in \Lambda$ then since $X_\alpha \in \hat{\mathscr{A}}$, Z_α is discrete so $j_\alpha^{\leftarrow}(j(p))$ is open in Z_α. Thus $q_\alpha^{\leftarrow}[j_\alpha^{\leftarrow}(j(p))]$ is open in X_α, i.e. $k_\alpha^{\leftarrow}[f^{\leftarrow}(j(p))]$ is open in X_α. As each such k_α embeds X_α as an open subset of X, $\bigcup \{f^{\leftarrow}(j(p)) \cap k_\alpha[X_\alpha] : \alpha \in \Lambda)$ is open in X, i.e. $f^{\leftarrow}(j(p))$ is open in X. As j is one-to-one, $f^{\leftarrow}(j(p)) = q^{\leftarrow}(p)$. As q is quotient, this implies that $\{p\}$ is open in Z. Thus Z is discrete and $X \in \hat{\mathscr{A}}$.

If $X \in \hat{\mathscr{A}}$ and $q : X \to Y$ is quotient map, let $f : Y \to Z$ be continuous and $Z \in \mathscr{A}$. If S denotes $f[Y]$ equipped with the quotient topology induced by f, $f \circ q$ is a quotient map from X onto S, and S maps into Z. Thus as $X \in \hat{\mathscr{A}}$, S is discrete. Hence $Y \in \hat{\mathscr{A}}$ and the theorem follows.

The following properties of the ^ operator are stated without proof.

2.8. PROPOSITION. *Let $\mathscr{A}$ and $\mathscr{B}$ be topological properties. If $\mathscr{A} \subset \mathscr{B}$ then $\hat{\mathscr{B}} \subset \hat{\mathscr{A}}$.*

2.9. PROPOSITION. *Let $\mathscr{A}$ be a topological property such that if $Y \in \mathscr{A}$ and $j : X \to Y$ is a monomorphism then $X \in \mathscr{A}$. Then $X \in \hat{\mathscr{A}}$ if and only if quotient images of X in $\mathscr{A}$ are discrete.*

2.10. THEOREM. *Let $\mathscr{P} < \mathscr{T}$. Then*:

(1) $\widehat{(\mathscr{P}^*)} < \mathscr{T}$.

(2) $(\widehat{(\mathscr{P}^*)})^* = \mathscr{P}^*$.

(3) *If $\mathscr{Q} < \mathscr{T}$ and $\mathscr{P}^* = \mathscr{Q}^*$ then $\mathscr{Q} \subset \widehat{(\mathscr{P}^*)}$.*

Proof. (1) is a special case of 2.7.

To prove (3) let $X \in \mathscr{Q}$ and let $f : X \to Y$ be continuous, where $Y \in \mathscr{P}^*$. As in 2.7 factor f into $j \circ q$ where q is quotient and j is one-to-one. Let Z denote the quotient image of X under q. As $Y \in \mathscr{P}^*$ by 2.5 (1) $Z \in \mathscr{P}^*$. Thus by hypothesis $Z \in \mathscr{Q}^*$. As $X \in \mathscr{Q}$ by 1.2 $Z \in \mathscr{Q}$. Thus $\mathscr{Q}Z = Z$ so Z is discrete. Thus $X \in \widehat{(\mathscr{P}^*)}$.

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To prove (2) note that by (3) and 2.5 (2) $(\widehat{(\mathscr{P}^*)})^* \subset \mathscr{P}^*$. Conversely let $X \in \mathscr{P}^*$. By 2.5 (1) $\widehat{(\mathscr{P}^*)}X \in \mathscr{P}^*$. Let Z denote the set X with the quotient topology induced on it by $\sigma_{\widehat{(\mathscr{P}^*)}X}$. As $X \in \mathscr{P}^*$, Z is discrete. As $\sigma_{\widehat{(\mathscr{P}^*)}X}$ is one-to-one it follows that $\widehat{(\mathscr{P}^*)}X$ is discrete so $X \in (\widehat{(\mathscr{P}^*)})^*$.

We now consider some examples.

2.11. *Examples.* (a) Let $\mathscr{P}$ denote the class of P-spaces (described in Section 1); then $\mathscr{P}^*$ is the class of spaces whose singleton sets are G_δ-sets. By 2.9 a space X is in $\widehat{(\mathscr{P}^*)}$ if and only if, whenever Y is a quotient image of X and each singleton set of Y is a G_δ-set, Y is discrete. Obviously all first countable T_1 spaces are in $\mathscr{P}^*$, while $\beta\bar{N}\text{-}\bar{N}$ is a compact space that is not in $\mathscr{P}^*$. It is easy to verify that $X \in \widehat{(\mathscr{P}^*)}$ if and only if given a partition $\mathscr{D}$ of X such that each member of $\mathscr{D}$ is the intersection of countably many $\mathscr{D}$-saturated open subsets of X, then each member of $\mathscr{D}$ is open in X. If (X, τ) is a topological space and τ when ordered by inclusion is a chain, then $(X, \tau) \in \widehat{(\mathscr{P}^*)}$. In particular let ω denote the least infinite ordinal, and consider the set $\omega + 1$ of ordinals no greater than ω with the topology $\tau = \{\omega + 1 - \{0, \ldots, n\} : n < \omega\}$. Then $(\omega + 1, \tau) \in \widehat{(\mathscr{P}^*)}$ but $(\omega + 1, \tau) \notin \mathscr{P}$; thus $\mathscr{P} \neq \widehat{(\mathscr{P}^*)}$.

(b) Let $\mathscr{C}$ be a topological property closed under the formation of continuous images. Using 1.3 it is easily seen that $X \in [\mathscr{M}(\mathscr{C})]^*$ if and only if the only subspaces of X having $\mathscr{C}$ are discrete; one direction is obvious, and if C is a non-discrete subspace of X with $\mathscr{C}$, find $p \in C$ such that $\{p\}$ is not open in $\mathscr{C}$; then by 1.3 $\sigma_{\mathscr{M}(\mathscr{C})X}{}^{\leftarrow}(p)$ is not open in $\mathscr{M}(\mathscr{C})X$ and so $X \notin [\mathscr{M}(\mathscr{C})]^*$. In particular if $\mathscr{C}$ is the class of compact spaces, then $\mathscr{M}(\mathscr{C})$ is the category $\mathscr{K}$ of k-spaces and $\mathscr{K}^*$ is the class of spaces whose compact subspaces are finite. It is known that all Tychonoff P-spaces are in $\mathscr{K}^*$; see 4K of [**3**] or 1.65 of [**9**]. More generally, it is easy to see that each countable subset of a T_1 P-space is closed and discrete, so if $\mathscr{T}_1$ denotes the category of T_1-spaces then $\mathscr{P} \cap \mathscr{T}_1 \subseteq \mathscr{K}^* \cap \mathscr{T}_1$. An example of a Tychonoff member of $\mathscr{K}^*$ that is not a P-space may be found in 3.5 of [**7**].

As another example if $\mathscr{C} = \{\bar{N}^*\}$ then $\mathscr{M}(\mathscr{C})$ is the class of sequential spaces (as remarked earlier) and $[\mathscr{M}(\mathscr{C})]^*$ is the class of spaces containing no convergent sequences.

We now give an analogue of Theorem 2.3 of [**10**]. Let us call a topological space X *fully disconnected* if each singleton set is the intersection of the clopen sets that contain it. Let $\mathscr{T}_F$ denote the category of fully disconnected spaces.

2.12. THEOREM. *Let $\mathscr{P} < \mathscr{T}$ and $\mathscr{Q} < \mathscr{T}$. If x is a point of a space X, put $\mathscr{N}_{\mathscr{P}}(x) = \{V \subseteq X : \sigma_{\mathscr{P}X}{}^{\leftarrow}[V]$ is a $\mathscr{P}X$-neighborhood of $\sigma_{\mathscr{P}X}{}^{\leftarrow}(x)\}$. If $\mathscr{Q} \cap \mathscr{T}_F \subseteq \mathscr{P}^* \cap \mathscr{T}_F$, then for each $X \in \mathscr{T}_0 \cap \mathscr{T}_F$, $\{x \in X : \mathscr{N}_{\mathscr{Q}}(x) - \mathscr{N}_{\mathscr{P}}(x) \neq \emptyset\}$ is dense in $\{x \in X : \mathscr{N}_{\mathscr{Q}}(x) - \mathscr{N}_{\mathscr{P}}(x) \neq \emptyset\} \cup \{x \in X : \sigma_{\mathscr{P}X}{}^{\leftarrow}(x)$ is not isolated in $\mathscr{P}X\}$.*

Proof. We prove the contrapositive. Suppose $X \in \mathscr{T}_0 \cap \mathscr{T}_F$ and there is a clopen subset A of X such that $A \subseteq \{x \in X : \mathscr{N}_{\mathscr{Q}}(x) \subseteq \mathscr{N}_{\mathscr{P}}(x)\}$ and there exists $x_0 \in A$ such that $\sigma_{\mathscr{P}X}^{\leftarrow}(x_0)$ is not isolated in $\mathscr{P}X$. Now $\sigma_{\mathscr{P}X}^{\leftarrow}[A] = \mathscr{P}A$ by 2.4 so $\mathscr{P}A \notin \mathscr{P}^*$. As $A \subseteq \{x \in X : \mathscr{N}_{\mathscr{Q}}(x) \subseteq \mathscr{N}_{\mathscr{P}}(x)\}$ the function $\sigma_{\mathscr{Q}X}^{\leftarrow} \circ \sigma_{\mathscr{P}A}$ from $\mathscr{P}A$ onto $\sigma_{\mathscr{Q}X}^{\leftarrow}[A]$ $(= \mathscr{Q}A)$ is continuous and one-to-one. As $X \in \mathscr{T}_F$, $\mathscr{Q}A \in \mathscr{Q} \cap \mathscr{T}_F$. Also $\mathscr{Q}A \notin \mathscr{P}^*$, for if $\mathscr{Q}A \in \mathscr{P}^*$ by 2.5(1) $\mathscr{P}A \in \mathscr{P}^*$. The theorem follows.

2.12. *Example.* Let $\mathscr{P}$ be the category of P-spaces and let $\mathscr{K}$ be the category of k-spaces. Since $\mathscr{T}_F \subseteq \mathscr{T}_1$, it follows from 2.11(a) that $\mathscr{P} \cap \mathscr{T}_F \subseteq \mathscr{K}^* \cap \mathscr{T}_F$. Hence 2.12 says that if X is a fully disconnected zero-dimensional space (e.g. a zero-dimensional Hausdorff space) and x_0 is a limit point of some compact subset of X, then if V is a neighborhood of x_0 there exists a point $y \in V$, a G_δ-set G, and a compact subset K of X such that $y \in G$ and $G \cap K$ is not open in K.

We next derive an analogue of Theorem 2.8 of [**10**] (see Theorem 2.16 below). Some preliminaries are necessary.

2.14. LEMMA. *Let $(\tau_\alpha)_{\alpha\in\Sigma}$ be a set of topologies on a set X and let $\mathscr{P} < \mathscr{T}$. If $\tau = \bigcap_{\alpha\in\Sigma}\tau_\alpha$ and $(X, \tau_\alpha) \in \mathscr{P}$ for each α, then $(X, \tau) \in \mathscr{P}$.*

Proof. Let Y be the free union of the spaces $\{(X, \tau_\alpha) : \alpha \in \Sigma\}$ and let $k_\alpha : (X, \tau_\alpha) \to Y$ be the canonical embedding. Let $q : Y \to (X, \tau)$ be the map induced by the identity functions $(X, \tau_\alpha) \to (X, \tau)$. If $V \subset X$ then $q^{\leftarrow}[V]$ is open in Y if and only if $q^{\leftarrow}[V] \cap (X, \tau_\alpha) \in \tau_\alpha$ for $\alpha \in \Sigma$, i.e. if and only if $V \in \bigcap_{\alpha\in\Sigma} \tau_\alpha = \tau$. Thus q is a quotient map and $(X, \tau) \in \mathscr{P}$.

Let us call a topology τ on a set X *almost discrete* if the only topology on X that properly contains τ is the discrete topology (this is the dual-like analogue to the concept of *almost compact* spaces discussed in Problem 6J of [**3**]). The almost discrete spaces are identical to the "ultraspaces" discussed by Steiner in [**8**]. In [**2**] Fröhlich proves Theorem 2.15 below; see Steiner [**8**] for a discussion of these results.

2.15. THEOREM. *Let X be a set, $p \in X$, and $\mathscr{U}$ an ultrafilter on X. Put $\mathscr{G}(p, \mathscr{U}) = \{A \subset X : p \notin A\} \cup \mathscr{U}$. Then*

(a) *$\mathscr{G}(p, \mathscr{U})$ is a topology on X.*

(b) *If τ is a topology on X then (X, τ) is almost discrete if and only if $\tau = \mathscr{G}(p, \mathscr{U})$ for some choice of p and $\mathscr{U}$ as described above.*

(c) *If τ is any topology on X then τ is the intersection of all topologies on X containing τ and of the form $\mathscr{G}(p, \mathscr{U})$.*

2.16. THEOREM. *Let $\mathscr{P} < \mathscr{T}$ and $\mathscr{Q} < \mathscr{T}$. If $\mathscr{P}^* \subset \mathscr{Q}$ then either $\mathscr{P} = \mathscr{T}$ or $\mathscr{Q} = \mathscr{T}$.*

Proof. Suppose $\mathscr{P} \neq \mathscr{T}$ and $\mathscr{Q} \neq \mathscr{T}$. By 2.15(c) and 2.14 there exist almost discrete spaces X_1 and X_2 such that $X_1 \notin \mathscr{P}$ and $X_2 \notin \mathscr{Q}$. By 2.15(b) the

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topology on X_i is of the form $\mathscr{G}(x_i, \mathscr{U}_i)$ $(i = 1, 2)$. Put

$$\mathscr{F} = \{A_1 \times A_2 : A_i \in \mathscr{U}_i\}.$$

Then $\mathscr{F}$ is a filterbase on $X_1 \times X_2$. Let $\mathscr{U}$ be any ultrafilter on $X_1 \times X_2$ such that $\mathscr{F} \subset \mathscr{U}$ and let X denote the topological space $(X_1 \times X_2, \mathscr{G}((x_1, x_2), \mathscr{U}))$. Consider the projection map $q : X \to X_1$. We claim this is a quotient map. If $V \subset X_1$ then $x_1 \notin V$ if and only if $(x_1, x_2) \notin q^{\leftarrow}[V]$. Further, if $V \in \mathscr{U}_1$ then $q^{\leftarrow}[V] = V \times X_2 \in \mathscr{F} \subset \mathscr{U}$, while if $V \notin \mathscr{U}_1$ then $X_1 - V \in \mathscr{U}_1$ and so $(X_1 - V) \times X_2 \in \mathscr{F} \subset \mathscr{U}$, so $V \times X_2 \notin \mathscr{U}$. Thus q is a quotient map.

As X is almost discrete, either $X \in \mathscr{P}$ or $X \in \mathscr{P}^*$. If $X \in \mathscr{P}$ by 1.2 its quotient image $X_1 \in \mathscr{P}$, in contradiction to hypothesis. Thus $X \in \mathscr{P}^*$. But X_2 is also a quotient image of X, so $X \notin \mathscr{Q}$ as $X_2 \notin \mathscr{Q}$. Thus $X \in \mathscr{P}^* - \mathscr{Q}$.

2.17. *Example.* No proper monocoreflective subcategory of $\mathscr{T}$ contains all spaces whose singletons are G_δ-sets.

In 2.9 of [**10**] we divided topological extension properties into two classes; those that contain spaces containing a closed copy of the countably infinite discrete space $\bar{N}$ (and hence contain all $\bar{N}$-compact spaces), and those that do not (and hence are contained in the class of countably compact spaces). We showed in 4.5 of [**10**] that any extension property in the latter class must be contained in α-compactness for some free ultrafilter α on $\bar{N}$ (if we regard α as a point of $\beta\bar{N}$-$\bar{N}$, a space X is α-compact if each map from $\bar{N}$ into X can be continuously extended to $\bar{N} \cup \{\alpha\}$; see [**1**] and [**10**] for details). It is known (see 3.3-3.5 of [**1**]) that a completely regular Hausdorff space X is α-compact for each $\alpha \in \beta\bar{N}$-$\bar{N}$ if and only if it is ω-bounded (i.e. each countable subset of X has compact X-closure). We derive analogues of these results for monocoreflective subcategories of $\mathscr{T}$ and $\mathscr{H}$ although in this case the analogues differ greatly from their epireflective models.

If we regard $\bar{N}$ as the smallest "non-trivial" free union (i.e. coproduct in $\mathscr{T}$ or $\mathscr{H}$) of topological spaces, it is evident that its dual-like analogue should be the smallest "non-trivial" product of topological spaces in $\mathscr{T}$ or $\mathscr{H}$, namely the Cantor space $\bar{2}^\omega$. A space X contains no closed copy of $\bar{N}$ if and only if there is no extremal monomorphism in $\mathscr{H}$ or $\mathscr{T}$ych (i.e. no closed embedding; see 10.19 of [**9**]) from $\bar{N}$ into X; a space X satisfies the analogue of this property if and only if there is no extremal epimorphism in $\mathscr{H}$ or $\mathscr{T}$ from X to $\bar{2}^\omega$; i.e. if there is no quotient map from X onto $\bar{2}^\omega$ (see 10C.2 of [**9**]). The class $\mathscr{C}$ defined in 2.18 below is, by 2.19, precisely this class.

Let Σ denote the set of subspaces of $\bar{2}^\omega$ that are homeomorphic to the one-point compactification of $\bar{N}$. If $\sigma \in \Sigma$ let $p(\sigma)$ denote the nonisolated point of σ. If τ is a topology on $\bar{2}^\omega$ that strictly contains the product topology, it is easily seen that there exists $\sigma \in \Sigma$ such that $V(\sigma) = (\bar{2}^\omega - \sigma) \cup \{p(\sigma)\} \in \tau$. Let $C(\sigma)$ denote $\bar{2}^\omega$ equipped with the topology generated by the product topology together with $V(\sigma)$, and let i_σ be the obvious canonical map from $C(\sigma)$ to $\bar{2}^\omega$.

The set Σ plays a role analogous to that played by the set $\beta\bar{N}\text{-}\bar{N}$ of free ultrafilters on $\bar{N}$, and $C(\sigma)$ is the analogue of the space $\bar{N} \cup \{\alpha\}$ (for $\alpha \in \beta\bar{N}\text{-}\bar{N}$). The analogy breaks down in one important way: since $|\beta\bar{N}\text{-}\bar{N}| = 2^{2^\omega}$ and there are only 2^ω maps from $\bar{N}$ to itself, there is a set of 2^{2^ω} pairwise non-homeomorphic spaces of the form $\bar{N} \cup \{\alpha\}$. However:

2.18. LEMMA. *If σ and δ are in Σ then $C(\sigma)$ and $C(\delta)$ are homeomorphic.*

Proof. It obviously suffices to exhibit a homeomorphism h from $\bar{2}^\omega$ to itself such that $h[\sigma] = \delta$. Let $U(\sigma) = \bar{2}^\omega - \sigma$. Evidently the one-point compactification $U(\sigma)^*$ of $U(\sigma)$ is a compact totally disconnected metric space without isolated points and hence is homeomorphic to $\bar{2}^\omega$. As $\bar{2}^\omega$ is homogeneous it follows that there is a homeomorphism g from $U(\sigma)$ onto $U(\delta)$. Let k be a homeomorphism from σ onto δ. Then $g \cup k$ is the desired h.

2.19. *Definition.* A topological space X is *countably discrete* if given a map $f : X \rightarrow \bar{2}^\omega$ there is $\sigma \in \Sigma$ and a map $f_\sigma : X \rightarrow C(\sigma)$ such that $i_\sigma \circ f_\sigma = f$. Let $\mathscr{C}$ denote the class of countably discrete spaces.

By 2.18 it does not matter which σ we use in definition 2.19; either no member of Σ satisfies the condition therein, or else they all do. Theorem 2.20 (b) below demonstrates that $\mathscr{C}$ is analogous (in the sense described above) to the class of countably compact spaces; hence our choice of terminology.

2.20. THEOREM. (a) $\mathscr{C} < \mathscr{T}$.

(b) *$\mathscr{C}$ is the largest monocoreflective subcategory of $\mathscr{T}$ that does not contain the class of sequential spaces.*

(c) *$X \in \mathscr{C}$ if and only if each countable collection of clopen sets of X has an open intersection.*

(d) *A zero-dimensional Hausdorff space is in $\mathscr{C}$ if and only if it is a P-space.*

Proof. (a) Let $(X_\alpha)_{\alpha \in A}$ be a set of spaces in $\mathscr{C}$ and let X be their free union. Let $f : X \rightarrow \bar{2}^\omega$ be a map, and let $\sigma \in \Sigma$. By hypothesis for each $\alpha \in A$ there exists a map $g_\alpha : X_\alpha \rightarrow C(\sigma)$ such that $i_\sigma \circ g_\alpha = f|X_\alpha$. Let $g = \bigcup_{\alpha \in A} g_\alpha$; then g maps X to $C(\sigma)$ and $i_\sigma \circ g = f$. Thus $X \in \mathscr{C}$.

If $X \in \mathscr{C}$ and $g : X \rightarrow Y$ is a quotient map onto Y, let f map Y to $\bar{2}^\omega$. As $X \in \mathscr{C}$ there is a map $k : X \rightarrow C(\sigma)$ such that $i_\sigma \circ k = f \circ q$. As i_σ is one-to-one, k is constant on preimages under q of points of Y. Thus one can unambiguously define $j : Y \rightarrow C(\sigma)$ such that $j \circ q = k$. As q is quotient j is continuous and $i_\sigma \circ j = f$. Thus $Y \in \mathscr{C}$. By 1.2 $\mathscr{C} < \mathscr{T}$.

(b) Obviously $\bar{2}^\omega \notin \mathscr{C}$. Conversely if $X \notin \mathscr{C}$ then for each $\sigma \in \Sigma$ there is a map $g_\sigma : X \rightarrow \bar{2}^\omega$ such that g_σ cannot be factored through $C(\sigma)$. Let X_σ be a homeomorphic copy of X (for each $\sigma \in \Sigma$), put $Y = \dot{\bigcup}_{\alpha \in \Sigma} X_\alpha$, and define $g : Y \rightarrow \bar{2}^\omega$ by $g|X_\sigma = g_\sigma$. Obviously g is continuous. If V is not open in the product topology on $\bar{2}^\omega$ there exists $\sigma \in \Sigma$ such that the topology on $C(\sigma)$ is contained in the topology generated on $\bar{2}^\omega$ by V and the product topology. Thus $g_\sigma^{\leftarrow}[V]$ is not open in X_σ and so $g^{\leftarrow}[V]$ is not open in Y. Thus g is a quotient map.

Since $\mathscr{M}(\{\bar{2}^\omega\})$ is the class of sequential spaces, it follows from 1.2 that any monocoreflective subcategory of $\mathscr{T}$ not contained in $\mathscr{C}$ will contain the class of sequential spaces.

(c) Suppose $X \in \mathscr{C}$. Let $(A_n)_{n<\omega}$ be a countable set of clopen subsets of X. For $n < \omega$ define $j_n : X \to \bar{2}$ (the two-point discrete space) by $j_n^{\leftarrow}(1) = A_n$. Define $j : X \to \bar{2}^\omega$ by putting $\pi_n \circ j = j_n$. For each $n \leqq \omega$ let $q_n \in \bar{2}^\omega$ be defined by $\pi_j(q_n) = 1$ if and only if $j < n$. Thus $(q_n)_{n\leqq\omega} \in \Sigma$; let $(q_n)_{n\leqq\omega} = \sigma$. By hypothesis j factors through $C(\sigma)$, so $j^{\leftarrow}\{q_n : n < \omega\}$ is closed in X. But $j^{\leftarrow}\{q_n : n < \omega\} = X - \bigcap_{n<\omega} A_n$.

Conversely suppose each countable collection of clopen sets of X has an open intersection and let f map X to $\bar{2}^\omega$. If $\sigma \in \Sigma$ then $\{p(\sigma)\}$ is the intersection of countably many clopen subsets of $\bar{2}^\omega$ so $f^{\leftarrow}(p(\sigma)$ is open in X. It follows that f factors through $C(\sigma)$ and so $X \in \mathscr{C}$.

(d) This follows immediately from (c).

Since $\mathscr{C}$ is the analogue of the class of countably compact (Hausdorff) spaces, 2.20 (d) leads us to regard the zero-dimensional Hausdorff P-spaces as the analogue of the completely regular Hausdorff ω-bounded spaces.

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