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INVERSE LIMITS IN THE CATEGORY OF COMPACT HAUSDORFF SPACES AND UPPER SEMICONTINUOUS FUNCTIONS

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Abstract

We investigate inverse limits in the category $\mathcal{CHU}$ of compact Hausdorff spaces with upper semicontinuous functions. We introduce the notion of weak inverse limits in this category and show that the inverse limits with upper semicontinuous set-valued bonding functions (as they were defined by Ingram and Mahavier ['Inverse limits of upper semi-continuous set valued functions', *Houston J. Math.* **32** (2006), 119–130]) together with the projections are not necessarily inverse limits in $\mathcal{CHU}$ but they are always weak inverse limits in this category. This is a realisation of our categorical approach to solving a problem stated by Ingram [*An Introduction to Inverse Limits with Set-Valued Functions* (Springer, New York, 2012)].

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1. Introduction

Ingram in his book [13] states the following problem:

> **Problem 6.63.** What can be said about inverse limits with set-valued functions if the underlying directed set is not a sequence of integers?

In this paper we present a categorical approach to solving the above problem.

Consider an inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ of compact Hausdorff spaces and continuous bonding functions. It is a well-known fact that the space

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) = \Big\{(x_\gamma)_{\gamma\in A} \in \prod_{\gamma\in A} X_\alpha \mid \text{for all } \alpha, \beta \in A, \alpha < \beta, x_\alpha = f_{\alpha\beta}(x_\beta)\Big\},$$

together with the projection mappings

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma, \quad p_\gamma((x_\alpha)_{\alpha\in A}) = x_\gamma,$$

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is in fact an inverse limit in the category $\mathcal{CHC}$ of compact Hausdorff spaces with continuous functions.

In the present paper we extend the category $\mathcal{CHC}$ to the category $\mathcal{CHU}$ of compact Hausdorff spaces with upper semicontinuous (usc) functions in such a way that $\mathcal{CHC}$ is interpreted as a proper subcategory of $\mathcal{CHU}$. This can be done since every continuous function between compact Hausdorff spaces can be interpreted as a usc function.

As one of our main results we show that the inverse limits with upper semicontinuous set-valued bonding functions

$$\begin{aligned}&\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})\\ &\quad= \Big\{(x_\gamma)_{\gamma\in A} \in \prod_{\gamma\in A} X_\alpha \mid \text{ for all } \alpha,\beta\in A,\ \alpha<\beta,\ x_\alpha \in f_{\alpha\beta}(x_\beta)\Big\},\end{aligned}$$

together with the projections

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma, \quad p_\gamma((x_\alpha)_{\alpha\in A}) = \{x_\gamma\},$$

are not necessarily inverse limits in the category but they are always so-called weak inverse limits in $\mathcal{CHU}$.

In Section 2 we give the basic definitions that are used in the paper.

In Section 3 we give a detailed description of the category $\mathcal{CHU}$ of compact Hausdorff spaces with usc bonding functions.

In Section 4 we give results about inverse limits in the category $\mathcal{CHU}$.

In Section 5 we define objects in the category $\mathcal{CHU}$ that are called weak inverse limits in this category. We also show that for any inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ in $\mathcal{CHU}$, the corresponding inverse limit with usc set-valued bonding functions together with projections is always a weak inverse limit in category $\mathcal{CHU}$.

2. Definitions and notation

For any category $\mathcal{K}$ the class of objects of $\mathcal{K}$ will be denoted by $\mathrm{Ob}(\mathcal{K})$, the class of morphisms of $\mathcal{K}$ by $\mathrm{Mor}(\mathcal{K})$, and the partial binary associative operation (composition of morphisms) by $\circ$. For any $X \in \mathrm{Ob}(\mathcal{K})$ the identity morphism on X will be denoted by $1_X : X \to X$.

Given a directed set A (which is is nonempty and equipped with a reflexive and transitive binary relation $\leq$ with the property that every pair of elements has an upper bound), a family of objects $\{X_\alpha \mid \alpha \in A\}$ of $\mathcal{K}$, and a family of morphisms $\{f_{\alpha\beta} : X_\beta \to X_\alpha \mid \alpha,\beta \in A,\ \alpha \leq \beta\}$ of $\mathcal{K}$, such that:

(1) for each $\alpha \in A$, $f_{\alpha\alpha} = 1_{X_\alpha}$;
(2) for each $\alpha,\beta,\gamma \in A$, from $\alpha \leq \beta \leq \gamma$ it follows that $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$,

we call this an *inverse system* (in $\mathcal{K}$) and denote it by

$$(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}).$$

We assume throughout the paper that A is cofinite, that is, every $\alpha \in A$ has at most finitely many predecessors. For more details see [17].

Next we define inverse limits in $\mathcal{K}$.

DEFINITION 2.1. An object $X \in \mathrm{Ob}(\mathcal{K})$, together with morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$, is an *inverse limit* of an inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ in the category $\mathcal{K}$, if:

(1) for all $\alpha, \beta \in A$, it follows from $\alpha \leq \beta$ that the diagram

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle p_\alpha} & \searrow^{p_\beta} & \\ X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta \end{array} \tag{2.1}$$

commutes;

(2) for any object $Y \in \mathcal{K}$ and any family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ it follows that if the diagram

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle \varphi_\alpha} & \searrow^{\varphi_\beta} & \\ X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta \end{array} \tag{2.2}$$

commutes, then there is a unique morphism $\varphi : Y \to X$ such that for each $\alpha \in A$ the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle \varphi_\alpha} & \swarrow_{p_\alpha} & \\ X_\alpha & & \end{array} \tag{2.3}$$

commutes.

A *map* or *mapping* is a continuous function.

If X is a compact Hausdorff space, then 2^X denotes the set of all nonempty closed subsets of X.

The *graph* $\Gamma(f)$ of a function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

A function $f : X \to 2^Y$ is *upper semicontinuous* if for each $x \in X$ and for each open set $U \subseteq Y$ such that $f(x) \subseteq U$ there is an open set V in X such that:

(1) $x \in V$;

(2) for all $v \in V$, $f(v) \subseteq U$.

The following is a well-known characterisation of usc functions between Hausdorff compacta (see [14, p. 120, Theorem 2.1]).

THEOREM 2.2. *Let X and Y be compact Hausdorff spaces and $f : X \to 2^Y$ a function. Then f is usc if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

To conclude this section we introduce the notion of inverse limits with usc set-valued bonding functions as introduced by Mahavier in [16] and Ingram and Mahavier

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in [14]. In the last section we use this notion as a motivation for defining inverse limits with usc set-valued bonding functions for arbitrary inverse systems.

An *inverse sequence* of compact Hausdorff spaces X_k with usc bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$ is usc for each k.

The *inverse limit with usc set-valued bonding functions* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ is defined to be the subspace of the product space $\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim\{X_k, f_k\}_{k=1}^{\infty}$.

Since the introduction of such inverse limits, there has been much interest in the subject and many papers have appeared [1–12, 15, 18–22].

3. The category $\mathcal{CHU}$

The category $\mathcal{CHU}$ of compact Hausdorff spaces and usc functions consists of the following objects and morphisms:

(1) Ob($\mathcal{CHU}$)—compact Hausdorff spaces;
(2) Mor($\mathcal{CHU}$)—the usc functions from X to Y are the set of morphisms from X to Y, denoted by Mor($\mathcal{CHU}$)(X, Y).

We also define the partial binary operation $\circ$ (composition) as follows. For each $f \in \mathrm{Mor}(\mathcal{CHU})(X, Y)$ and each $g \in \mathrm{Mor}(\mathcal{CHU})(Y, Z)$, $g \circ f \in \mathrm{Mor}(\mathcal{CHU})(X, Z)$ is defined by

$$(g \circ f)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$.

Theorem 3.1. *$\mathcal{CHU}$ is a category.*

Proof. First, we show that $\circ$ is well defined. Let $f : X \to Y$ and $g : Y \to Z$ be any morphisms. Let also $x \in X$ be arbitrary and let U be an open set in Z such that $(g \circ f)(x) \subseteq U$. Since g is usc and $f(x) \subseteq Y$, for each $y \in f(x)$ there is an open set W_y in Y such that:

(1) $y \in W_y$;
(2) for all $w \in W_y$, $g(w) \subseteq U$.

Let $W = \bigcup_{y \in f(x)} W_y$. Since W is open in Y, $f(x) \subseteq W$, and since f is usc, there is an open set V in X such that:

(1) $x \in V$;
(2) for all $v \in V$, $f(v) \subseteq W$.

Let $v \in V$ be arbitrary. Then

$$(g \circ f)(v) = g(f(v)) = \bigcup_{z \in f(v)} g(z) \subseteq U$$

since for each $z \in f(v)$, $g(z) \subseteq U$. Therefore $\circ$ is well defined.

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It is obvious that the composition $\circ$ of usc functions is an associative operation.

All that is left to show is that for each $X \in \mathrm{Ob}(\mathcal{CHU})$ there is a morphism $1_X : X \to X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f : Y \to X$ and $g : X \to Z$. We can easily see that the identity map $1_X : X \to X$, defined by $1_X(x) = \{x\}$ for each $x \in X$, is the usc function satisfying the above conditions. □

4. Inverse limits in $\mathcal{CHU}$

In this section we show that if $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is an inverse system of compact Hausdorff spaces and usc set-valued bonding functions, then

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$$

(see Definition 4.1), together with the projections, is not necessarily an inverse limit in the category $\mathcal{CHU}$.

Motivated by [14, 16], we define in Definition 4.1 objects in $\mathcal{CHU}$ called inverse limits with usc set-valued bonding functions. Since such objects were first introduced by Mahavier in [16] and Ingram and Mahavier in [14], where they were called inverse limits with usc set-valued bonding functions, we continue to use the same name for them.

Definition 4.1. Let $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ be any inverse system in $\mathcal{CHU}$. We call the object

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) = \left\{x \in \prod_{\alpha\in A} X_\alpha \mid \text{for all } \alpha < \beta,\ x_\alpha \in f_{\alpha\beta}(x_\beta)\right\}$$

an *inverse limit with usc set-valued bonding functions*.

In the following theorem we prove that $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is really an object of $\mathcal{CHU}$.

Theorem 4.2. *Let* $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *be any inverse system in* $\mathcal{CHU}$. *Then the inverse limit with usc set-valued bonding functions*

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$$

is a compact Hausdorff space.

Proof. For each $\gamma \in A$, X_γ is a compact Hausdorff space, and therefore the product $\prod_{\gamma\in A} X_\gamma$ is a compact Hausdorff space. Since $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a subspace of the Hausdorff space, it is also a Hausdorff space. We show that $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a closed subset of the compact space $\prod_{\gamma\in A} X_\gamma$ to show that it is compact.

For all $\alpha, \beta \in A$, $\alpha < \beta$, let

$$G_{\alpha\beta} = \Gamma(f_{\alpha\beta}) \times \prod_{\gamma\in A\setminus\{\alpha,\beta\}} X_\gamma = \left\{x \in \prod_{\gamma\in A} X_\gamma \mid x_\alpha \in f_{\alpha\beta}(x_\beta)\right\}.$$

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Since the graph $\Gamma(f_{\alpha\beta})$ of $f_{\alpha\beta}$ is by Theorem 2.2 a closed subset of $X_\beta \times X_\alpha$, $G_{\alpha\beta}$ is also a closed subset of $\prod_{\gamma\in A} X_\gamma$. It is obvious that

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) = \bigcap_{\alpha,\beta\in A, \alpha<\beta} G_{\alpha\beta}$$

and hence $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ is a a closed subset of $\prod_{\gamma\in A} X_\gamma$. □

In the following example we construct an inverse limit with usc set-valued bonding functions that is not an inverse limit in $\mathcal{CHU}$ regardless of the choice of morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$.

Example 4.3. Let $A = \mathbb{N}$, $X_k = [0, 1]$, and let $f_{k(k+1)} = f$ for each $k \in \mathbb{N}$, where $f : [0, 1] \to 2^{[0,1]}$ is the function on $[0, 1]$ defined by its graph

$$\Gamma(f) = \{(t, t) \in [0, 1] \times [0, 1] \mid t \in [0, 1]\} \cup (\{1\} \times [0, 1]).$$

Also let $X = \varprojlim(\mathbb{N}, \{[0, 1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ and let $\{p_i : X \to X_i \mid i \in \mathbb{N}\}$ be any set of morphisms in $\mathcal{CHU}$, such that the diagrams (2.1) always commute. We show that X with $\{p_i : X \to X_i \mid i \in \mathbb{N}\}$ is not an inverse limit of $(\mathbb{N}, \{[0, 1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ in $\mathcal{CHU}$. Let $Y = [0, 1]$ be an object in $\mathcal{CHU}$ and let $\{\varphi_k : Y \to X_k \mid k \in \mathbb{N}\}$ be the family of morphisms where $\varphi_k(t) = [0, 1]$ for each k and each $t \in Y$. The diagram (2.2) always commutes. We distinguish the following two cases.

(1) If there is a positive integer i_0, such that $1 \notin p_{i_0}(x)$ for each $x \in X$, then suppose that Φ is any morphism $Y \to X$. Then $\varphi_{i_0}(t) = [0, 1]$ but $1 \notin p_{i_0}(\Phi(t))$ for any $t \in Y$. Therefore the diagram (2.3) does not commute for $\alpha = i_0$.

(2) If for each positive integer i there is $x^i \in X$ such that $1 \in p_i(x^i)$, then let $s \in X$ be an accumulation point of the sequence $\{x^i\}_{i=1}^\infty$. We show first that $p_i(s) = [0, 1]$ for each i. Let k be any positive integer. Then for each $\ell > k$, it follows from

$$[0, 1] \supseteq p_k(x^\ell) = f_{k\ell}(p_\ell(x^\ell)) \supseteq f_{k\ell}(1) \supseteq [0, 1]$$

that $p_k(x^\ell) = [0, 1]$. Let $\{n_i\}_{i=1}^\infty$ be any increasing sequence of positive integers such that:

- $n_i > k$ for each i;
- $\lim_{i\to\infty} x^{n_i} = s$.

It follows from $p_k(x^{n_i}) = [0, 1]$ that $\{x^{n_i}\} \times [0, 1] \subseteq \Gamma(p_k)$ for each i. This means that for each $t \in [0, 1]$, the point $(x^{n_i}, t) \in \Gamma(p_k)$. Therefore $\lim_{i\to\infty}(x^{n_i}, t) = (s, t) \in \Gamma(p_k)$ for each t, since $\Gamma(p_k)$ is a closed subset of $X \times [0, 1]$. It follows that $\{s\} \times [0, 1] \subseteq \Gamma(p_k)$ and hence $p_k(s) = [0, 1]$.

Next, let $\Phi, \Psi : Y \to X$ be the morphisms in $\mathcal{CHU}$, defined by

$$\Phi(t) = X,$$
$$\Psi(t) = \{s\}$$

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for each $t \in Y$. It follows from

$$p_k(\Phi(t)) = p_k(X) = [0, 1] = \varphi_k(t)$$

and

$$p_k(\Psi(t)) = p_k(\{s\}) = [0, 1] = \varphi_k(t)$$

that the diagram (2.3) commutes for both $\varphi = \Phi$ and $\varphi = \Psi$. Therefore there is no unique morphism φ such that all diagrams (2.3) commute.

Note that in the second part of Example 4.3, $\Psi(t) \subseteq \Phi(t) = (\prod_{k=1}^{\infty} \varphi_k(t)) \cap X$ holds true for each $t \in Y$. The following lemma shows that such an inclusion is not accidental. It will be used in the proof of Theorem 5.5.

Lemma 4.4. *Let $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ be any inverse system in $\mathcal{CHU}$ and let $X = \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$. Suppose that for an object Y of $\mathcal{CHU}$ and a family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ the diagram (2.2) commutes for any α and β, $\alpha < \beta$. Then $\varphi : Y \to X$, defined by $\varphi(y) = (\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X$ for each $y \in Y$, is a morphism in $\mathcal{CHU}$ such that for each $\alpha \in A$ the diagram (2.3) commutes. Even more, for any morphism $\Psi : Y \to X$ such that $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$ for each $\alpha \in A$ and for each $y \in Y$, $\Psi(y) \subseteq \varphi(y)$ holds true for all $y \in Y$.*

Proof. We show that φ satisfies all the conditions in the following steps.

(1) The set $\prod_{\gamma\in A} \varphi_\gamma(y)$ is a closed subset of $\prod_{\alpha\in A} X_\alpha$, so that $\varphi(y)$ is a closed subset of X for any $y \in Y$.

(2) Next we show that for any $y \in Y$, the set $\varphi(y)$ is nonempty. Let $y \in Y$ be arbitrarily chosen. Next, for each positive integer n, let $A_n \subseteq A$ be the set of all elements $\alpha \in A$ that have exactly $n - 1$ predecessors. For any $\alpha \in A_1$ we arbitrarily choose $t_\alpha \in \varphi_\alpha(y)$. For any $\beta \in A_2$ there is an $\alpha \in A_1$ such that $\alpha < \beta$. For any such α and β it follows from $t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta \in \varphi_\beta(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta \in A_2$. Suppose that we have already constructed $t_\alpha \in \varphi_\alpha(y)$ for all $\alpha \in A_n$. Then for any $\beta \in A_{n+1}$ there is an $\alpha \in A_n$ such that $\alpha < \beta$. For any such α and β it follows from $t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta \in \varphi_\beta(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta \in A_{n+1}$.

    Then obviously $x = (t_\alpha)_{\alpha\in A} \in \varphi(y)$ and therefore $\varphi(y)$ is nonempty.

(3) We show that φ is a usc function. Let $y \in Y$ be an arbitrary point and let

$$U = (U_{\gamma_1} \times U_{\gamma_2} \times U_{\gamma_3} \times \cdots \times U_{\gamma_n}) \times \prod_{\gamma\in A\setminus\{\gamma_1,\gamma_2,\ldots,\gamma_n\}} X_\gamma$$

be an open set in X such that $\varphi(y) \subseteq U$, where for each $i = 1, 2, 3, \ldots, n$, U_{γ_i} is an open set in X_{γ_i}. It follows from the definitions of φ and U that $\varphi_{\gamma_i}(y) \subseteq U_{\gamma_i}$

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for each $i = 1, 2, 3, \ldots, n$. Since each φ_{γ_i} is usc, there is an open set V_i in Y such that:

(a) $y \in V_i$;
(b) for each $x \in V_i$, $\varphi_{\gamma_i}(x) \subseteq U_{\gamma_i}$,

for each i. We define $V = \bigcap_{i=1}^{n} V_i$. Then V is an open set in Y for which:

(a) $y \in V$;
(b) for each $x \in V$, $\varphi(x) = \prod_{\gamma\in A} \varphi_\gamma(x) \subseteq U$

hold true. Therefore φ is a usc function and so it is a morphism from Y to X.

(4) Next we show that the diagram (2.3) commutes, that is, for any $\alpha \in A$ and any $y \in Y$, $\varphi_\alpha(y) = (p_\alpha \circ \varphi)(y)$ holds true. Choose any $\alpha \in A$ and any $y \in Y$. Obviously

$$p_\alpha(\varphi(y)) = p_\alpha\Big(\Big(\prod_{\gamma\in A} \varphi_\gamma(y)\Big) \cap X\Big) \subseteq p_\alpha\Big(\prod_{\gamma\in A} \varphi_\gamma(y)\Big) = \varphi_\alpha(y).$$

Next we show that $\varphi_\alpha(y) \subseteq p_\alpha(\varphi(y))$. Let $z \in \varphi_\alpha(y)$ be arbitrarily chosen. We show that $z \in p_\alpha(\varphi(y))$ by showing that there is a point $x \in \varphi(y)$ such that $z \in p_\alpha(x)$. Let k be the positive integer such that $\alpha \in A_k$. For each $\gamma \in A_k \setminus \{\alpha\}$ let $t_\gamma \in \varphi_\gamma(y)$ be arbitrary and let $t_\alpha = z$. For each $\gamma \in A_{k-1}$ we choose $t_\gamma \in \varphi_\gamma(y)$ such that if $\alpha \in A_{k-1}$, $\beta \in A_k$, and $\alpha < \beta$, then $t_\alpha \in f_{\alpha\beta}(t_\beta)$. This can be done since $f_{\alpha\beta}(\varphi_\beta(y)) = \varphi_\alpha(y)$ and therefore $f_{\alpha\beta}(t_\beta) \subseteq \varphi_\alpha(y)$.

Continuing in the same fashion, we choose for each $i = 1, 2, 3, \ldots, k-1$ and each $\gamma \in A_i$ an element $t_\gamma \in \varphi_\gamma(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, $\beta \in A_{i+1}$, $\alpha < \beta$.

Next, for each $\beta \in A_{k+1}$ and for each $\alpha \in A_k$ such that $\beta > \alpha$, since $t_\alpha \in \varphi_\alpha(y) = f_{\alpha\beta}(\varphi_\beta(y))$, there is $t_\beta \in \varphi_\beta(y)$, such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$.

We continue inductively in the same fashion and choose for each $i = k+1, k+2, k+3, \ldots$ and each $\beta \in A_{i+1}$ an element $t_\beta \in \varphi_\alpha(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, such that $\alpha < \beta$.

Let $x \in \prod_{\gamma\in A} X_\gamma$ be such an element that $p_\gamma(x) = \{t_\gamma\}$ for each $\gamma \in A$. It follows from the construction of x that $x \in \varphi(y)$ and $z \in p_\alpha(x)$.

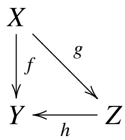
(5) Suppose that $\psi : Y \to X$ is a morphism in $\mathcal{CHU}$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$. Let $y \in Y$ be arbitrary and let $z \in \psi(y)$. Obviously $z \in X$ since ψ is a morphism from Y to X. It follows from $p_\alpha(z) \subseteq p_\alpha(\psi(y)) = \varphi_\alpha(y)$ (for each α) that $z \in \prod_{\gamma\in A} \varphi_\gamma(y)$. Therefore $z \in \varphi(y)$ and hence $\psi(y) \subseteq \varphi(y)$. □

5. Weak inverse limits in $\mathcal{CHU}$

In this section we introduce the notion of weak inverse limits in $\mathcal{CHU}$ and show that $\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$, together with the projections, is always a weak inverse limit in $\mathcal{CHU}$.

In Definition 5.1 we define a weak commutation of a diagram in the category $\mathcal{CHU}$.

DEFINITION 5.1. Let $X, Y, Z \in \mathrm{Ob}(\mathcal{CHU})$ and let $f : X \to Y$, $g : X \to Z$ and $h : Z \to Y$ be any morphisms in $\mathcal{CHU}$. The diagram

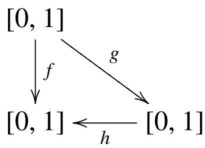


weakly commutes if, for any $x \in X$, $f(x) \subseteq (h \circ g)(x)$.

EXAMPLE 5.2. Let $f : [0, 1] \to 2^{[0,1]}$, $g : [0, 1] \to 2^{[0,1]}$ be identity functions on $[0, 1]$ and let $h : [0, 1] \to 2^{[0,1]}$ be defined by

$$h(x) = [0, 1]$$

for all $x \in [0, 1]$. Then the diagram



weakly commutes but does not commute.

In the following definition we generalise the notion of inverse limits in $\mathcal{CHU}$.

DEFINITION 5.3. An object $X \in \mathrm{Ob}(\mathcal{CHU})$, together with morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$, is a *weak inverse limit* of an inverse system

$$(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$$

in $\mathcal{CHU}$, if:

(1) for all $\alpha, \beta \in A$, it follows from $\alpha \leq \beta$ that the diagram (2.1) weakly commutes;
(2) for any object $Y \in \mathcal{CHU}$ and any family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ it follows that if the diagram (2.2) commutes, then for any morphism $\Psi : Y \to X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$.

Note that each inverse limit in $\mathcal{CHU}$ is always a weak inverse limit in $\mathcal{CHU}$.

EXAMPLE 5.4. Let $X = \varprojlim(\mathbb{N}, \{[0, 1]\}_{k\in\mathbb{N}}, \{f_{k\ell}\}_{k,\ell\in\mathbb{N}})$ be the inverse limit with usc set-valued bonding functions that we defined in Example 4.3. Then X, together with the projection mappings, is obviously not an inverse limit but it is a weak inverse limit in $\mathcal{CHU}$.

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We show in the following theorem that the inverse limits with upper semicontinuous set-valued bonding functions together with projections are always weak inverse limits in $\mathcal{CHU}$.

THEOREM 5.5. *Let* $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *be any inverse system in* $\mathcal{CHU}$. *Then the inverse limit with usc set-valued bonding functions*

$$\varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}),$$

together with projections

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A}) \to X_\gamma, \quad p_\gamma((x_\alpha)_{\alpha\in A}) = \{x_\gamma\},$$

is a weak inverse limit of the inverse system $(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$ *in* $\mathcal{CHU}$.

PROOF. Let $X = \varprojlim(A, \{X_\alpha\}_{\alpha\in A}, \{f_{\alpha\beta}\}_{\alpha,\beta\in A})$. First, we prove that the diagram (2.1) weakly commutes. Choose any $x \in X$ and let $\alpha < \beta$. Then

$$p_\alpha(x) = \{x_\alpha\} \subseteq f_{\alpha\beta}(\{x_\beta\}) = (f_{\alpha\beta} \circ p_\beta)(x).$$

Next, suppose that for an object $Y \in \mathcal{CHU}$ and a family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ the diagram (2.2) commutes. By Lemma 4.4, for any morphism $\Psi : Y \to X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma\in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$. □

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