IDENTITIES AND CONGRUENCES OF THE RAMANUJAN TYPE

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1. Let $P(n)$ denote the number of unrestricted partitions of the positive integer n. Ramanujan[1] conjectured that

$$(1.1) \qquad P(n) \equiv 0 \pmod{5^{\alpha}7^{\beta}11^{\gamma}}$$

if $24n \equiv 1 \pmod{5^{\alpha}7^{\beta}11^{\gamma}}$. He also indicated that such congruences could be deduced from *identities* of the type

$$(1.2) \qquad P(4) + P(9)x + \ldots = 5 . \frac{[(1-x^5)(1-x^{10})\ldots]^5}{[(1-x)(1-x^2)\ldots]^6}.$$

G. N. Watson[2] proved that (1.1) is true for all α if $\beta = \gamma = 0$ and that it is false for $\beta > 2$ if $\gamma = 0$. However he established an alternative congruence for all powers of 7. It is not known if (1.1) is true for all γ. Watson, in the same paper, showed that identities of the type (1.2) exist for all powers of 5 and 7. Recently Rademacher[3] developed a method which enabled him to establish identities of the type (1.2) for all powers of 5, 7, and 13. He did not, however, use them to prove the congruences of the Watson-Ramanujan type.

In this paper we make use of a combination of the methods of Watson and Rademacher to establish identities and congruences analogous to (1.2) and (1.1) for the function

$$(1.3) \qquad \sum_{n=0}^{\infty} P_{\nu}(n)x^n = [(1-x)(1-x^2)\ldots]^{-\nu}, \qquad \nu > 0,$$

we prove that the coefficients $P_{\nu}(n)$ in (1.3) satisfy also identities of the type (1.2) for all powers of 5 and 7. The congruence properties of $P_{\nu}(n)$ are contained in the following two theorems:

THEOREM 1. *If* $24m \equiv \nu \pmod{5^{\alpha}}$ *then*

$$(1.4) \qquad P_{\nu}(m) \equiv 0 \ \left(\bmod \begin{Bmatrix} 5 \\ 5^{[\frac{\alpha-1}{2}]} \\ 5^{[\frac{\alpha}{2}]} \\ 5^{[\frac{\alpha+1}{2}]} \\ 5^{[\frac{\alpha+2}{2}]} \\ 5^{\alpha-1} \\ 5^{\alpha} \end{Bmatrix}\right) \ \textit{if}\ \nu \equiv \begin{Bmatrix} 12, 17, 22, 27 \\ 15, 20, 25 \\ 3, 4, 8, 9 \\ 16, 21, 26 \\ 2, 7 \\ 0, 5, 10 \\ 1, 6, 11 \end{Bmatrix} \pmod{30},$$

$[x]$ *being the largest integer in* x.

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[1]Ramanujan, S. *Collected papers*, Cambridge, 1927.

THEOREM 2. *If* $24m \equiv \nu \pmod{7^a}$ *then*

$$P_\nu(m) \equiv 0 \left(\bmod \begin{Bmatrix} 7 \\ 7^{\left[\frac{a-1}{2}\right]} \\ 7^{\left[\frac{a}{2}\right]} \\ 7^{\left[\frac{a+1}{2}\right]} \\ 7^{\left[\frac{a+2}{2}\right]} \\ 7^{a-1} \\ 7^a \end{Bmatrix}\right) \text{ if } \nu \equiv \begin{Bmatrix} 8, 15, 22 \\ 14, 21 \\ 2, 3, 5, 6 \\ 11, 18, 25 \\ 1 \\ 0, 7 \\ 4 \end{Bmatrix} \pmod{28}. \tag{1.5}$$

It may be noted that the Watson-Ramanujan congruences are included in Theorems 1 and 2. We may remark that by proving an analogue of Lemma 4 below for $p = 13$ we can deduce congruences of $P_\nu(n)$ for the modulus 13^δ. It will be seen that the proofs of Theorems 1 and 2 are simple and straightforward.

2. Throughout this paper $p = 5$ or 7, and by an integer we mean a rational integer.

Let[4] $\Gamma_0(p)$ be the group of unimodular substitutions

$$\tau' = \frac{a\tau + b}{c\tau + d}, \tag{2.1}$$

where a, b, c, d are integers such that $ad - bc = 1$ and $c \equiv 0 \pmod p$. It is known that $\Gamma_0(p)$ is a subgroup of index $p + 1$ in the whole modular group and its fundamental domain contains two parabolic points $\tau = 0$ and $\tau = i\infty$.

All modular functions $\Gamma_0(p)$, that is meromorphic functions invariant under the substitutions of $\Gamma_0(p)$, are rational functions of[5]

$$\phi(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^g, \tag{2.2}$$

where $g = 24/p - 1$ and $\eta(\tau)$ is Dedekind's modular form

$$\eta(\tau) = e^{\pi i\tau/12} \prod_n (1 - e^{2n\pi i\tau}), \qquad \tau = x + iy, \quad y > 0. \tag{2.3}$$

The function $\phi(\tau)$, which is called the *Hauptmodul* of $\Gamma_0(p)$, has the following properties:

(i) It is single valued in the fundamental domain.

(ii) It is invariant under all transformations of $\Gamma_0(p)$.

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[2]Watson, G. N., *Ramanujans Vermutung über Zerfällungsanzahlen*, Jour. für Math., Bd. 179, (1938), p. 97.

[3]Rademacher, H. *The Ramanujan identities under modular substitutions*, Trans. Amer. Math. Soc., vol. 51, (1942) p. 609.

[4]All these are contained in Klein and Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunktionen*. Bd. 2, p. 64. Also see: L. J. Mordell, *Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers*. Proc. Lond. Math. Soc. (2), 20 (1922) p. 408.

[5]$\phi(\tau)$ depends on p. We omit this suffix p in general, but when explicit reference has to be made we write $\phi_p(\tau)$.

(iii) At $\tau = i\infty$ it has a zero of the first order and the Fourier expansion there is

$$\phi(\tau) = e^{2\pi i\tau} + \ldots$$

with integral coefficients.

(iv) It is regular at all points of $y > 0$ except at $\tau = 0$ where it has a pole of first order with residue $p^{-g/2}$ measured in terms of the uniformizing parameter $e^{-\frac{2\pi i}{p\tau}}$.

Let us call a modular function of $\Gamma_0(p)$ *entire modular*, if it is regular at all points of $y > 0$, except at $\tau = 0$ where it has a polar singularity; then we can prove easily that every entire modular function of $\Gamma_0(p)$ is a polynomial in $\phi(\tau)$, with rational coefficients. We can prove even more as is shown by

LEMMA 1. *If $f(\tau)$ is an entire modular function of $\Gamma_0(p)$, whose Fourier expansion at $\tau = i\infty$ has coefficients belonging to a module $\mathfrak{m}$, then $f(\tau)$ is a polynomial in $\phi(\tau)$ with coefficients belonging to the module.*

Proof. That $f(\tau)$ is a polynomial in $\phi(\tau)$ is obvious. Let

$$f(\tau) = \sum_{j=0}^{n} a_j \, \phi(\tau)^j; \tag{2.4}$$

since $\phi(\tau)$, at $\tau = i\infty$ has its first coefficient unity, we see on equating coefficients of the powers of the uniformizing parameter $e^{2\pi i\tau}$ in the expansion at $\tau = i\infty$, that

$$a_0, \ a_1, \ a_2 + I_1 a_1, \ a_3 + I_2 a_2 + I_3 a_1, \ldots, \quad a_n + I_k a_{n-1} + \ldots + I_r a_1, \tag{2.5}$$

are all in $\mathfrak{m}$ ($I_1, \ldots, I_r$ being integers). From the definition of a module we deduce that $a_1, \ldots, a_n$ are in $\mathfrak{m}$.

COROLLARY. If $\mathfrak{m}$ is the module of integers then $a_1, \ldots, a_n$ are integers.

3. We shall prove certain lemmas preliminary to the proof of the theorems.

LEMMA 2. *Let q be a prime > 3 and ν, μ two non-negative integers. Then*

$$h(\tau) = \left(\frac{\eta(q\tau)}{\eta(\tau)}\right)^{\mu} \sum_{\lambda=0}^{q-1} \left(\frac{\eta(\tau)}{\eta\left(\frac{\tau + 24\lambda}{q}\right)}\right)^{\nu}, \tag{3.1}$$

where

$$(\nu - \mu)(p - 1) \equiv 0 \pmod{24} \tag{3.2}$$

has the transformation equation

$$h(\tau') = \left(\frac{a}{q}\right)^{\nu - \mu} h(\tau), \tag{3.3}$$

where

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad c \equiv 0 \pmod{q}, \tag{3.4}$$

and (a/q) is the Legendre symbol.

This lemma includes, as particular cases, Theorems 1, 2 and 3 of Rademacher[3] and can be proved by using Lemmas 4 and 5 of his paper.

LEMMA 3. *If ν is any non-negative integer and μ the least non-negative residue of ν mod g then*

$$\psi_\nu(\tau) = \frac{1}{p}\left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^\mu \sum_{\lambda=0}^{p-1}\left(\frac{\eta(\tau)}{\eta\left(\dfrac{\tau+24\lambda}{p}\right)}\right)^\nu \tag{3.5}$$

is entire modular in $\Gamma_0(p)$ and is a polynomial in $\phi(\tau)$ with integral coefficients.

Proof. Since $p = 5$ or 7, g is even. If we put $q = p$ and $\nu \equiv \mu \pmod g$ in Lemma 2 we find that $\psi_\nu(\tau)$ is invariant under all substitutions of $\Gamma_0(p)$. $\psi_\nu(\tau)$ is evidently regular at $\tau = i\infty$. To investigate behaviour at $\tau = 0$ we make the transformation (following Rademacher[3]) $\tau \to -1/\tau$. We have

$$\psi_\nu(-\tau^{-1}) = \frac{1}{p}\left(\frac{\eta(-1/\tau)}{\eta(-1/\tau p)}\right)^\nu\left(\frac{\eta(-p/\tau)}{\eta(-1/\tau)}\right)^\mu + \frac{1}{p}\sum_{\lambda=1}^{p-1}. \tag{3.6}$$

We can prove that the quantity $\sum_{\lambda=1}^{p-1}$ is regular at $(-1/\tau) = 0$. The first term on the right of (3.6) could be transformed, by making use of the functional equation,

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \tag{3.7}$$

into

$$p^{-(\nu+\mu+2)/2}\,\eta(\tau)^{\nu-\mu}\,\eta(\tau/p)^\mu\,\eta(p\tau)^{-\nu}. \tag{3.8}$$

Using (2.3) we see that the order of the pole at $-1/\tau = 0$ measured in terms of $e^{2\pi i\tau/p}$ is $\dfrac{\nu(p^2-1)-(\nu-\mu)(p-1)}{24}$. Hence $\psi_\nu(\tau)$ is entire modular in $\Gamma_0(p)$. It is easily seen that the Fourier expansion of $\psi_\nu(\tau)$ at $\tau = i\infty$ has all its coefficients integral so that from Lemma 1 it follows that $\psi_\nu(\tau)$ is a polynomial in $\phi(\tau)$ with integral coefficients.

It may be noticed that the S_ν of Watson[6] and the $\psi_\nu(\tau)$ above are connected by the relation

$$\psi_\nu(\tau) = S_\nu\left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^\mu, \tag{3.9}$$

where μ is defined by Lemma 3. To calculate $\psi_\nu(\tau)$ we require only expressions for $\psi_1(\tau), \ldots, \psi_{p+1}(\tau)$ as polynomials[7] in $\phi(\tau)$. We obtain from Watson's table of S_ν the

LEMMA 4.

$$\psi_\nu(\tau) = \sum_{i>0} \alpha_i\, p^{i-1}\phi(\tau)^i, \tag{3.10}$$ [8]

where α_i are integers vanishing for a sufficiently large i. In fact $\alpha_i = 0$ if $24i > \nu(p^2-1) - (\nu-\mu)(p-1)$.

[6]Watson, loc. cit., pp. 106, 119, 120.

[7]There is another method of obtaining expressions for $\psi_\nu(\tau)$ as polynomials in $\phi(\tau)$. This will be published elsewhere.

[8]$\sum_{i>0}$ shall mean summation from $i = 1$ to a sufficiently large i.

It is evident that if we expand both sides of (3.10) at $\tau = 0$ and equate corresponding coefficients of the powers of the parameter $e^{2\pi i\tau/p}$ we find that the coefficients of the powers of $\phi(\tau)$ in (3.10) are of the above form if $i(g-2) \geqslant (\nu + \mu)$. It is only for obtaining this property for all i that we require a table of $\psi_1(\tau), \ldots, \psi_{p+1}(\tau)$ as polynomials in $\phi(\tau)$. It does not appear to be possible to obtain Lemma 4 without the said table.

It must be noticed that a_i's are integers depending on ν and p.

LEMMA 5.

$$f_\nu(\tau) = \frac{1}{p}\sum_{\lambda=0}^{p-1}\left(\frac{\eta(p\tau)}{\eta\left(\frac{\tau+24\lambda}{p}\right)}\right)^\nu \tag{3.11}$$

is entire modular in $\Gamma_0(p)$ *and*

$$f_\nu(\tau) = \sum_{i>0} a_i p^{i-1} \phi(\tau)^{i+\frac{\nu-\mu}{g}}, \tag{3.12}$$

where μ, a_i *are defined in Lemmas* 3 *and* 4.

Proof. This follows from Lemma 3 on using the fact that

$$f_\nu(\tau) = \psi_\nu(\tau)\left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^{\nu-\mu}. \tag{3.13}$$

Note that $\nu - \mu$ is divisible by g.

LEMMA 6. *Let* $k = \nu-\mu/g$, $t = \frac{1}{2}(1-(-1)^n)$ *and* $\epsilon = 1$ *or* p *according as* n *is odd or even, then*[9]

$$F_{n,\nu}(\tau) = \frac{1}{p^n}\sum_{\lambda=0}^{p^n-1}\left[\frac{\eta(\epsilon\tau)}{\eta\left(\frac{\tau+24\lambda}{p^n}\right)}\right]^\nu \tag{3.14}$$

is entire modular in $\Gamma_0(p)$ *and has the form* [10]

$$F_{n,\nu}(\tau) = \phi(\tau)^{kt}\sum_{i>0} a_i(n)\, p^{i-1}\phi(\tau)^i, \tag{3.15}$$

the $a_i(n)$ *being integers depending on* n, ν *and* p.

Proof. When $n = 1$,

$$F_{1,\nu}(\tau) = \psi_\nu(\tau)\phi(\tau)^k \tag{3.16}$$

and Lemma 5 shows that (3.14) and (3.15) are true. We can write

$$F_{1,\nu}(\tau) = \sum_{i>0} a_i(1)\, p^{i-1}\,\phi(\tau)^{i+k}. \tag{3.16'}$$

Now

$$\begin{aligned} F_{2,\nu}(\tau) &= \frac{1}{p}\sum_{\lambda=0}^{p-1} F_{1,\nu}\left(\frac{\tau+24\lambda}{p}\right) \\ &= \frac{1}{p}\sum_{i>0} a_i(1)\, p^{i-1}\sum_{\lambda=0}^{p-1}\left(\frac{\eta(\tau)}{\eta\left(\frac{\tau+24\lambda}{p}\right)}\right)^{(i+k)g}. \end{aligned} \tag{3.17}$$

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[9]The construction of the function $F_{n,\nu}(\tau)$ is suggested by the work of Rademacher, p. 622 and 624.

[10]To avoid complication we have not shown in $a_i(n)$, its dependence on ν and p.

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Applying Lemmas 3 and 4 to the inner sum, we get

$$(3.18)\qquad F_{2,\nu}(\tau) = \sum_{i>0} a_i(1)\, p^{i-1} \sum_{j>0} b_{ij}\, p^{j-1}\phi(\tau)^j = \sum_{j>0} a_j(2)\, p^{j-1}\phi(\tau)^j,$$

where

$$a_j(2) = \sum_{i>0} a_i(1)\, b_{ij}\, p^{i-1}$$

are integers. This proves that Lemma 6 is true for $n = 2$. We can now apply induction.

Let us assume that for a certain $n = 2m$

$$(3.19)\qquad F_{2m,\nu}(\tau) = \frac{1}{p^{2m}} \sum_{\lambda=0}^{p^{2m}-1} \left[\frac{\eta(\tau)}{\eta\left(\frac{\tau+24\lambda}{p^{2m}}\right)}\right]^{\nu} = \sum_{i>0} a_i(2m)\, p^{i-1}\phi(\tau)^i,$$

where $a_i(2m)$ are integers. Then

$$F_{2m+1,\nu}(\tau) = \frac{1}{p}\sum_{\lambda=0}^{p-1} F_{2m,\nu}\left(\frac{\tau+24\lambda}{p}\right)\left[\frac{\eta(p\tau)}{\eta\left(\frac{\tau+24\lambda}{p}\right)}\right]^{\nu}$$

$$= \frac{1}{p}\sum_{i>0} a_i(2m)\, p^{i-1} \sum_{\lambda=0}^{p-1}\left(\frac{\eta(\tau)}{\eta\left(\frac{\tau+24\lambda}{p}\right)}\right)^{gi}\left(\frac{\eta(p\tau)}{\eta\left(\frac{\tau+24\lambda}{p}\right)}\right)^{\nu}$$

that is

$$(3.20)\qquad F_{2m+1,\nu}(\tau) = \phi(\tau)^k \sum_{i>0} a_i(2m)\, p^{i-1}\psi_{gi+\nu}(\tau),$$

which shows that from the truth of Lemma 6 for $n = 2m$ we can deduce it for $n = 2m + 1$. In a similar way, if we assume for $n = 2m + 1$

$$(3.21)\qquad F_{2m+1,\nu}(\tau) = \sum_{i>0} a_i(2m+1)\, p^{i-1}\phi(\tau)^{i+k},$$

then we can deduce that

$$(3.22)\qquad F_{2m+1,\nu}(\tau) = \sum_{i>0} a_i(2m+1)\, p^{i-1}\psi_{gi+kg}(\tau).$$

The induction may now be completed easily.

LEMMA 7.

$$(3.23)\qquad F_{n,\nu}(\tau) = x^{\delta}\prod_{m=1}^{\infty}(1 - x^{\epsilon m})\sum_{l=0}^{\infty} P_{\nu}(p^n l + \rho)x^l,$$

where

$$(3.24)\qquad x = e^{2\pi i\tau},$$

(3.25) *ρ is the least positive solution of* $24\rho \equiv \nu \pmod{p^n}$,

$$(3.26)\qquad \delta = \frac{24\rho - \nu + \epsilon\nu p^n}{24p^n}$$

is an integer and $\epsilon = 1$ *or* p *according as* n *is even or odd.*

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Proof. Using (2.3) we get

$$F_{n,\nu}(\tau) = e^{\frac{2\pi i\tau\nu(\epsilon p^n - 1)}{24p^n}} \prod_{m=1}^{\infty} (1 - e^{2m\pi i\epsilon\tau})^\nu \sum_{r=0}^{\infty} P_\nu(r) A_{p^n}(r)\, e^{2\pi i\tau r/p^n}, \tag{3.27}$$

where

$$A_{p^n}(r) = \frac{1}{p^n} \sum_{\lambda=1}^{p^n} \exp\left(2\pi i\lambda \frac{(24r - \nu)}{p^n}\right). \tag{3.28}$$

$A_{p^n}(r)$ vanishes unless $24r - \nu \equiv 0 \pmod{p^n}$ when its value is unity.

It can be seen that δ is an integer. For from the definition of ρ, $24\rho - \nu + \nu\epsilon p^n$ is divisible by p^n; furthermore $\epsilon p^n - 1$ is divisible by $p^2 - 1$ and hence by 24.

LEMMA 8. *If c is any positive integer and all the $a_i(n)$ in* (3.15), *$i > 0$ are divisible by p^c then $P_\nu(p^n l + \rho)$ in* (3.23) *for all $l \geqslant 0$ are divisible by p^c.*

Proof. This follows immediately on writing down

$$F_{n,\nu}(\tau) = \phi(\tau)^{kt} \sum_{i>0} a_i(n) p^{i-1} \phi(\tau)^i \tag{3.29}$$

$$= x^\delta \sum_{m=0}^{\infty} b_m x^m \sum_{l=0}^{\infty} P_\nu(p^n l + \rho) x^l,$$

where $b_0 = 1$ and b_m are integers, and using the fact that $\phi(\tau)$ at $\tau = i\infty$ has a Fourier expansion in $x = e^{2\pi i\tau}$ beginning with x with the coefficient unity. We have merely to compare coefficients on both sides.

Lemma 8 shows us that for investigating the congruence properties of $P_\nu(p^n l + \rho)$ it is enough if we confine ourselves to the $a_i(n)$. We must only remember that if all the $a_i(n)$, $i > 0$ for a given n, ν are divisible by p^c then $P_\nu(p^n l + \rho)$ for all $l \geqslant 0$ are divisible by p^c.

4. Lemmas 6 and 7 establish the existence of identities of the Ramanujan type for the function $P_\nu(n)$. For from (3.15) and (3.23) we get

$$x^\delta \prod_{m=1}^{\infty} (1 - x^{\epsilon m})^\nu \sum_{l=0}^{\infty} P_\nu(p^n l + \rho) x^l = \phi(\tau)^{kt} \sum_{i>0} a_i(n) p^{i-1} \phi(\tau)^i \tag{4.1}$$

$$= \frac{x^{kt} \prod (1 - x^{pm})^{ktg}}{\prod (1 - x^m)^{ktg}} \sum_{i>0} \frac{a_i(n) x^i \prod (1 - x^{pm})^{ig}}{\prod (1 - x^m)^{ig}},$$

where $a_i(n)$ are integers.

As an example we take

$$\sum_{n=0}^{\infty} P_3(n) x^n = [1 - 3x + 5x^3 - 7x^6 + \ldots]^{-1}. \tag{4.2}$$

Here $\nu = 3$. Let $p = 5$ so that $g = 6$, $\epsilon = 5$, $n = 1$, $\rho = 2$, $t = 1$, $\delta = 1$ and $k = 0$. Then

$$P_3(2) + P_3(7)\, x + \ldots = \frac{9 \prod (1 - x^{5m})^3}{\prod (1 - x^m)^6} \tag{4.3}$$

$$+ \frac{75 \cdot 5x \prod (1 - x^{5m})^9}{\prod (1 - x^m)^{12}} + \frac{125 \cdot 5^2 x^2 \prod (1 - x^{5m})^{15}}{\prod (1 - x^m)^{18}}.$$

Similarly an identity for the modulus 7 may be derived.

5. We shall now prove congruence properties of $P_\nu(n)$. Because of Lemma 8 it is enough if we find congruence properties of $a_i(n)$.

Let now ν be a fixed integer. Let p^{λ_n} for a given n be the highest power of p which divides *all* $a_i(n)$ for $i > 0$. We shall study now the sequence $\lambda_1, \lambda_2, \ldots$. Consider (3.20);

$$F_{2m+1,\nu}(\tau) = \phi(\tau)^k \sum_{i>0} a_i(2m)\, p^{i-1} \psi_{gi+\nu}(\tau)$$

(5.1)
$$= \phi(\tau)^k \sum_{j>0} a_j(2m+1)\, p^{j-1} \phi(\tau)^j.$$

Let

(5.1′)
$$\psi_{gi+\nu}(\tau) = \sum_{j>0} b_{ij}(i)\, p^{j-1} \phi(\tau)^j.$$

Substituting in (5.1) we get

(5.2)
$$a_j(2m+1) = \sum_{i>0} b_{ij}(i)\, p^{i-1} a_i(2m).$$

In a similar way from (3.22), if

(5.1″)
$$\psi_{gi+kg}(\tau) = \sum_{j>0} b'_{ij}(i)\, p^{j-1} \phi(\tau)^j,$$

then

(5.3)
$$a_j(2m+2) = \sum_{i>0} a_i(2m+1)\, b'_{ij}(i)\, p^{i-1}.$$

All the quantities involved in (5.2) and (5.3) are integers and hence $p^{\lambda_{2m}}$ divides all $a_j(2m+1)$ and $p^{\lambda_{2m+1}}$ divides all $a_j(2m+2)$ so that

$$\lambda_{2m+2} \geqslant \lambda_{2m+1} \geqslant \lambda_{2m}.$$

Hence

(5.4)
$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots.$$

Now (5.2) may be written

(5.5)
$$a_j(2m+1) = b_{1j}(1)\, a_1(2m) + p \sum_{i>1} b_{ij}(i)\, p^{i-1} a_i(2m).$$

The second term on the right is divisible by $p \cdot p^{\lambda_{2m}} = p^{1+\lambda_{2m}}$. The first term $b_{1j}(1) a_1(2m)$ is *in general* only divisible by $p^{\lambda_{2m}}$. We cannot say anything regarding $a_1(2m)$. But if $b_{1j}(1)$ is divisible by p for all $j > 0$ then $a_j(2m+1)$ is divisible by $p^{1+\lambda_{2m}}$ for all $j > 0$. That is,

$$\lambda_{2m+1} \geqslant 1 + \lambda_{2m}.$$

From (5.1′), $b_{1j}(1)$ are coefficients in $\psi_{g+\nu}(\tau)$, considered as a polynomial in $\phi(\tau)$. Thus

(5.6) *If all the coefficients* $b_{1j}(1), j > 0$ *in*

$$\psi_{g+\nu}(\tau) = \sum_{j>0} b_{1j}(1)\, p^{j-1} \phi(\tau)^j$$

are divisible by p, *then* $\lambda_{2m+1} \geqslant 1 + \lambda_{2m}$.

In a similar way we derive from (5.3) and (5.1″) that:

(5.7) *If all the coefficients* $b'_{1j}(1), j > 0$ *in*

$$\psi_{g+\nu-\mu}(\tau) = \sum_{j>0} b'_{1j}(1)\, p^{j-1} \phi(\tau)^j$$

are divisible by p, *then* $\lambda_{2m+2} \geqslant 1 + \lambda_{2m+1}$.

It must be noticed that the conditions (5.6) and (5.7) are only *sufficient;* they are by no means necessary. It can be seen also that the conditions (5.6) and (5.7) depend only on the parity, and not on the actual value of n.

The conditions (5.6) and (5.7) enable us to divide the sequence (5.4) into four categories.

(i) $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ ((5.6) and (5.7) both hold good).
(ii) $\lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \dots$ ((5.6) but not (5.7) holds).
(iii) $\lambda_1 < \lambda_2 \leqslant \lambda_3 < \dots$ ((5.7) but not (5.6) holds).
(iv) $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots$ ((5.6) and (5.7) don't hold).

We shall examine these cases and see what consequences they lead to with regard to congruence properties of the $a_i(n)$.

Case (i). Since (5.6) and (5.7) hold good,

(5.8) $$\lambda_n \geqslant \lambda_1 + (n-1)$$

and $p | b_{1j}$ and $p | b'_{1j}$, $j > 0$.

Let $p = 5$ then $g = 6$ and Watson's table[5] shows that if

(5.9) $$s \equiv 1, 2, \pmod 5,$$

then $$a_{1j}(1) \equiv 0 \pmod 5,$$

where $$\psi_s(\tau) = \sum_{j>0} a_{1j}(1)\, 5^{j-1}\, \phi_5(\tau)^j.$$

Hence for our conditions (5.8) we obtain

(5.10) $$\begin{cases} \nu \equiv 0, 1 \pmod 5, \\ \nu - \mu \equiv 0, 1 \pmod 5, \\ 0 \leqslant \mu < 6, \\ \nu \equiv \mu \pmod 6. \end{cases}$$

The ν satisfying these conditions are $\nu \equiv 0, 1, 5, 6, 10, 11 \pmod{30}$. Using (5.9) we see that $\lambda_1 \geqslant 1$, if $\nu \equiv 1, 2 \pmod 5$ and therefore if $\nu \equiv 1, 6, 11 \pmod{30}$; otherwise $\lambda_1 \geqslant 0$. Now $a_i(n)$, $i > 0$ are divisible by $p^{\lambda_n} (= 5^{\lambda_n})$ and hence by (5.8) they are divisible by $5^{n-1+\lambda_1}$. We may summarize these in the conclusion, for $i > 0$:

(5.11′) $$a_i(n) \equiv 0 \pmod{5^n}, \qquad \nu \equiv 1, 6, 11 \pmod{30};$$

(5.11″) $$a_i(n) \equiv 0 \pmod{5^{n-1}}, \qquad \nu \equiv 0, 5, 10 \pmod{30}.$$

In exactly a similar way if $p = 7$; $g = 4$ and Watson's table shows that if

(5.12) $$r \equiv 1, 4 \pmod 7,$$

then $$a'_{1j}(1) \equiv 0 \pmod 7,$$

where $$\psi_s(\tau) = \sum_{j>0} a'_{1j}(1)\, 7^{j-1}\, \phi_7(\tau)^j.$$

Hence, for $i > 0$:

(5.13′) $$a_i(n) \equiv 0 \pmod{7^n}, \qquad \nu \equiv 4 \pmod{28};$$

(5.13″) $$a_i(n) \equiv 0 \pmod{7^{n-1}}, \qquad \nu \equiv 0, 7 \pmod{28}.$$

Case (ii). (5.6) holds but not (5.7). Hence

(5.14) $$\begin{cases} \lambda_{2n+1} \geqslant n + \lambda_1, \\ \lambda_{2n} \geqslant (n-1) + \lambda_2. \end{cases}$$

Using (5.9) we see that (5.14) holds good if

$$(5.15)\qquad \begin{cases} \nu \equiv 0,\ 1 \pmod 5, \\ \nu - \mu \not\equiv 0,\ 1 \pmod 5, \\ \nu \equiv \mu \pmod 6, \\ 0 \leqslant \mu < 6. \end{cases}$$

$\lambda_2 \geqslant \lambda_1$, for otherwise we will be in Case (i). It can be easily seen that $\lambda_1 \geqslant 1$ if $\nu \equiv 16, 21, 26 \pmod{30}$ and $\lambda_1 \geqslant 0$ if $\nu \equiv 15, 20, 25 \pmod{30}$. Hence

$$(5.16')\qquad a_i(n) \equiv 0 \left(\bmod\ 5^{\left[\frac{n+1}{2}\right]}\right), \quad \nu \equiv 16, 21, 26 \pmod{30};$$

$$(5.16'')\qquad a_i(n) \equiv 0 \left(\bmod\ 5^{\left[\frac{n-1}{2}\right]}\right), \quad \nu \equiv 15, 20, 25 \pmod{30}.$$

Case (iii). Since (5.7) but not (5.6) holds

$$(5.17)\qquad \begin{cases} \lambda_{2n+1} \geqslant n + \lambda_1 = \left[\dfrac{2n+1}{2}\right] + \lambda_1, \\ \lambda_{2n} \geqslant n + \lambda_1 = \left[\dfrac{2n}{2}\right] + \lambda_1, \end{cases}$$

since $\lambda_2 \geqslant \lambda_1 + 1$. Also

$$(5.18)\qquad \begin{aligned} \nu &\not\equiv 0,\ 1 \pmod 5, \\ \nu - \mu &\equiv 0,\ 1 \pmod 5. \end{aligned}$$

Exactly as before $\lambda_1 \geqslant 1$ if $\nu \equiv 2, 7 \pmod{30}$ and $\lambda_1 \geqslant 0$ if $\nu \equiv 3, 4, 8, 9 \pmod{30}$. Hence

$$(5.19')\qquad a_i(n) \equiv 0 \left(\bmod\ 5^{\left[\frac{n+2}{2}\right]}\right), \quad \nu \equiv 2, 7 \pmod{30};$$

$$(5.19'')\qquad a_i(n) \equiv 0 \left(\bmod\ 5^{\left[\frac{n}{2}\right]}\right), \quad \nu \equiv 3, 4, 8, 9 \pmod{30}.$$

Case (iv). This merely asserts

$$(5.20)\qquad \lambda_n \geqslant \lambda_1.$$

The only interesting case is $\lambda_1 \geqslant 1$ which happens when $\nu \equiv 12, 17, 22, 27, \pmod{30}$. Therefore

$$(5.21)\qquad a_i(n) \equiv 0 \pmod 5, \quad \nu \equiv 12, 17, 22, 27 \pmod{30}.$$

We have omitted consideration of $p = 7$ since it runs exactly parallel to the above and can be easily completed. In case $\nu = 1$ we get the Watson-Ramanujan congruence properties of the partition function. It is seen that for $p = 5$, (5.6) as well as (5.7) hold good so that

$$(5.22')\qquad P(m) \equiv 0 \pmod{5^a}, \quad 24m \equiv 1 \pmod{5^a},$$

as then we are in Case (i). But if $p = 7$ (5.6) does not hold good as $g = 4$, $\nu = 1$ and $4 + 1 \not\equiv 1$ or $4 \pmod 7$; whereas (5.7) holds good. We are therefore in Case (iii) and hence

$$(5.22'')\qquad P(m) \equiv 0 \left(\bmod\ 7^{\left[\frac{a+2}{2}\right]}\right), \quad 24m \equiv 1 \pmod{7^a}.$$

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In (5.21) we did not consider $\lambda_1 = 0$. This will merely mean that for $\nu \equiv 13$, 14, 18, 19, 23, 24, 28, 29 (mod 30), in general,

$$a_i(n) \not\equiv 0 \pmod 5. \tag{5.23}$$

Our results (5.22′) and (5.22″) for $\nu = 1$ are *Hauptsätze* 1, 3, 4 and results (5.45) and (5.46) of Watson.[11] It is not difficult to obtain analogues of Watson's *Hauptsätze* 2 and 5 for $P_\nu(n)$ from our foregoing results.

6. We may finally make a remark about the case $p = 13$. The group $\Gamma_0(13)$ has all the properties of $\Gamma_0(5)$ and $\Gamma_0(7)$, and Klein and Fricke have shown that $(\eta(13\tau)/\eta(\tau))^2$ is a *Hauptmodul* of $\Gamma_0(13)$. Further Lemmas 1, 2, and 3 hold good. We have only to prove an analogue of Lemma 4 by constructing a table of $\psi_1(\tau), \ldots \psi_{14}(\tau)$ as polynomials in $(\eta(13\tau)/\eta(\tau))^2$. Here $\psi_\nu(\tau)$ is given by

$$13\psi_\nu(\tau) = \left(\frac{\eta(13\tau)}{\eta(\tau)}\right)^\mu \sum_{\lambda=0}^{12} \left(\frac{\eta(\tau)}{\eta\left(\dfrac{\tau + 24\lambda}{13}\right)}\right)^\nu, \tag{6.1}$$

$0 \leqslant \mu < 2, \nu \equiv \mu \pmod 2$. Zuckerman[12] has found an expression for $\psi_1(\tau)$ as a polynomial in $(\eta(13\tau)/\eta(\tau))^2$. The general case of $\psi_\nu(\tau)$ offers no particular difficulty except that of computation.

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[11]Watson, *loc. cit.* p. 98, 99 and 124.

[12]Zuckerman, H. S., *Identities analogous to Ramanujan's identities involving the partition function*, Duke Math. Jour., vol. 5 (1939) p. 98.

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